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Aggregation, Non-Contradiction and Excluded-Middle

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Abstract

This paper investigates the satisfaction of the Non-Contradiction (NC) and Excluded-Middle (EM) laws within the domain of aggregation operators. It provides characterizations both for those aggregation operators that satisfy NC/EM with respect to (w.r.t.) some given strong negation, as well as for those satisfying them w.r.t. any strong negation. The results obtained are applied to some of the most important known classes of aggregation operators.

Keywords: Non-Contradiction and Excluded-Middle laws, Aggregation Operators, Strong Negations.

1 Introduction

Information aggregation is a crucial issue in the construction of many intelligent systems, and it is used in different application domains, such as medicine, economics, engineering, statistics or decision-making processes. It is particularly useful in situations presenting some degree of uncertainty or imprecision, a feature that explains the great development that this discipline has experimented in recent years within the field of Fuzzy Logic. It is a well assorted research field, whose topics of interest range from theoretical aspects to the use of the different aggregation methods and techniques in practical situations. A large collection of distinguished classes of aggregation operators and construction methods is nowadays available, and different potential application fields have been explored (see for example [4], [1] or the recent overview on aggregation theory given in [2]).

When using aggregation techniques in practical situations, one of the first problems that one has to face up is the election, among all the aggregation operators that are available, of the most suited one. Clearly, there is not a universal answer to this problem, since the decision is largely context-dependent. Notwithstanding, there are several criteria that may help in making this decision, such as the achievement of empirical experiments, the analysis of the expected operator's behavior (tolerant, intolerant, compensatory) or the need of some mathematical/logical properties (e.g. idempotency, symmetry, associativity, the existence of neutral or annihilator elements, etc).

This paper deals with the satisfaction of two of these mathematical properties, namely the well-known *Non-Contradiction* (NC) and *Excluded-Middle* (EM) laws ($p \wedge \neg p = 0$ and $p \vee \neg p = 1$, respectively), once both laws have been appropriately translated into the aggregation operators field. This is performed as follows: if the binary connectives $\wedge$ and $\vee$ are represented by means of *binary aggregation operators* acting on $[0,1]$ (i.e., operators of the form $A : [0,1]^2 \to [0,1]$ fulfilling some basic properties that will be recalled later), and the logical negation $\neg$ is represented by means of a *strong negation function* $N : [0,1] \to [0,1]$ (see Sec. 2 for details on strong negations) then the NC and EM laws can be interpreted, respectively, as "$A(x, N(x)) = 0$ for any $x \in [0,1]$" and "$A(x, N(x)) = 1$ for any $x \in [0,1]$".

Taking into account that the satisfaction of these inequalities seems to have only been considered partially and for just some specific kinds of aggregation operators, this paper tries to address the problem from a general point of view. It is organized as follows. After a brief remainder of the main issues related to aggregation operators and strong negations, sections 3 and 4 provide different conditions and characterizations regarding, respectively, the satisfaction of the NC and the EM law. Then section 5 applies the results obtained to some concrete families of aggregation operators, and, finally, the paper ends with some conclusions and pointers to related works.

2 Preliminaries

Although aggregation operators are defined for the general multidimensional case ([2]), in this paper we will only deal with *binary aggregation operators*, i.e., non-decreasing operators $A : [0,1]^2 \to [0,1]$ verifying the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$. Aggregation operators may be compared pointwise, that is, given two operators A_1 and A_2, it is said that A_1 is *weaker* than A_2 (or A_2 is *stronger* than A_1), and it is denoted $A_1 \leq A_2$, when it is $A_1(x,y) \leq A_2(x,y)$ for any $x, y \in [0,1]$. The relation $\leq$ is clearly a partial order, i.e., there are couples of aggregation operators which are non-comparable. Aggregation operators may be classified, by means of the relation $\leq$ and the distinguished operators Min (minimum) and Max (maximum), in the following four categories:

- *Conjunctive* operators, which are those verifying $A \leq Min$. This class includes the well-known *triangular norms* (*t-norms*) as well as *copulas* (see, respectively, [8] and [9]).

- *Disjunctive* operators, verifying $Max \leq A$, such as *triangular conorms (t-conorms)* and *dual copulas*.

- *Averaging* operators (or *mean* operators), which verify $Min \leq A \leq Max$. These operators are always idempotent (i.e., $A(x,x) = x$ holds for any $x \in [0,1]$), and some distinguished ones in this class are those based on the arithmetic mean, such as *quasi-linear means* or *OWA operators*, as well as those based on integrals, such as *Lebesgue, Choquet* or *Sugeno* integral-based aggregations.

- Finally, the class of *hybrid* aggregation operators contains all the operators that do not belong to any of the three previous categories, i.e., operators that are not comparable with Min and/or are not comparable with Max. This class includes different aggregation operators related to t-norms and t-conorms (such as *uninorms*, *nullnorms* or *compensatory operators*) as well as *symmetric sums*.

In order to translate the Non-Contradiction and Excluded-Middle laws to the aggregation operators' field, a way for representing the logical negation is needed. The latter is usually done by means of the so-called *strong negations* ([13]), i.e., non-increasing functions $N : [0,1] \to [0,1]$ which are involutive, that is, verify $N(N(x)) = x$ for any $x \in [0,1]$. Due to their definition, strong negations are continuous and strictly decreasing functions, they satisfy the boundary conditions $N(0) = 1$ and $N(1) = 0$, and they have a unique fixed point in $]0,1[$, that we will denote x_N, verifying $N(x_N) = x_N$.

3 On the satisfaction of the Non-Contradiction Law

Once equipped with aggregation operators and strong negations, the satisfaction of the Non-Contradiction law for aggregation operators allows for the following definition:

Definition 3.1 Let A be a binary aggregation operator and let N be a strong negation.

1. It is said that A *satisfies the Non-Contradiction (NC) principle with respect to (w.r.t.)* N when $A(x, N(x)) = 0$ holds for any $x \in [0,1]$.

2. It is said that A *satisfies NC* when there exists a strong negation N such that A satisfies NC w.r.t. N.

3. It is said that A *strictly satisfies NC* when it satisfies NC w.r.t. any strong negation.

Regarding the satisfaction of the NC law, let us first of all note that if an aggregation operator satisfies NC w.r.t. a given strong negation N, then it satisfies it for an infinite set of strong negations, and any weaker aggregation operator does also satisfy the principle:

Proposition 3.1 *Let A be a binary aggregation operator satisfying NC w.r.t. some strong negation N. Then:*

1. *A satisfies NC w.r.t. any strong negation N_1 such that $N_1 \leq N$.*
2. *Any binary aggregation operator B verifying $B \leq A$ satisfies NC w.r.t. N.*

Proof. Immediate thanks to the monotonicity of aggregation operators. ∎

Note also that it is easy to prove that given a strong negation N, the class made of all the operators satisfying NC w.r.t. N is closed under composition by means of any outer aggregation operator, i.e.:

Proposition 3.2 *Let N be a strong negation and let A_1, A_2 be two binary aggregation operators satisfying NC w.r.t. N. Then given any binary aggregation operator A, the binary aggregation operator $A(A_1, A_2)$, defined (see e.g. [2]) as $A(A_1, A_2)(x, y) = A(A_1(x, y), A_2(x, y))$ for any $x, y \in [0, 1]$, does also satisfy NC w.r.t. N.*

Proof. Obvious since any aggregation operator verifies the boundary condition $A(0, 0) = 0$. ∎

We will now provide two very simple but useful conditions that any aggregation operator must verify in order to satisfy NC:

Proposition 3.3 *Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. If A satisfies NC w.r.t. N, then A must verify the two following conditions:*

1. *$A(x_N, x_N) = 0$*
2. *Zero is an annihilator element for A, i.e., $A(x, 0) = A(0, x) = 0$ holds for any $x \in [0, 1]$.*

Proof. Both conditions are easily obtained from the definition of NC, the fact that x_N is the fixed point of N and the monotonicity of A. ∎

The above proposition clearly shows that not every category of aggregation operators is able to satisfy the NC law:

Corollary 3.1 *Let A be a binary aggregation operator. If A is either an averaging operator or a disjunctive operator, then it does not satisfy NC.*

Proof. Indeed, neither averaging operators (due to their idempotency) nor disjunctive operators (since they verify $A(x, x) \geq x$ for any $x \in [0, 1]$) satisfy the first condition of the last proposition. ∎

Therefore, aggregation operators satisfying NC may only be found among conjunctive and hybrid operators verifying the two conditions given in Proposition 3.3 (note that the second one is always true in the case of conjunctive operators). In fact, the following characterization may be obtained:

Proposition 3.4 *Let A be a binary aggregation operator and let N be a strong negation. A satisfies NC w.r.t. N if and only if for any $x, y \in [0,1]$ it is:*

$$A(x,y) = \begin{cases} 0, & \text{if } y \leq N(x) \\ B(x,y), & \text{otherwise} \end{cases}$$

where B is a binary non-decreasing operator verifying $B(1,1) = 1$.

Proof. If A is an aggregation operator satisfying NC w.r.t. N, the equality $A(x, N(x)) = 0$ for any $x \in [0,1]$ implies, by monotonicity, $A(x,y) = 0$ for any $y \leq N(x)$. The converse is obvious. ∎

Clearly, the above characterization provides either conjunctive operators (when B is chosen such that $B \leq Min$) or hybrid operators (otherwise). In addition:

Remark 3.1 Given a strong negation N, proposition 3.4 shows that:

1. The weakest aggregation operator satisfying NC w.r.t. N is the weakest aggregation operator, A_w (see e.g. [2]), defined, for any $x, y \in [0,1]$, as:

$$A_w(x,y) = \begin{cases} 0, & \text{if } (x,y) \neq (1,1) \\ 1, & \text{otherwise} \end{cases}$$

2. The strongest aggregation operator satisfying NC w.r.t. N is $A^{nc}_{(N)}$, defined, for any $x, y \in [0,1]$, as:

$$A^{nc}_{(N)}(x,y) = \begin{cases} 0, & \text{if } y \leq N(x) \\ 1, & \text{otherwise} \end{cases}$$

   Note that $A^{nc}_{(N)}$ is nothing else than the characteristic function of the set $\{(x,y) \in [0,1]^2 : y > N(x)\}$.

Regarding the strict satisfaction of the NC law (i.e., its satisfaction w.r.t. *any* strong negation), the following necessary condition may be stated:

Proposition 3.5 *Let A be a binary aggregation operator. If A strictly satisfies NC, then $A(x,y) = 0$ for all $(x,y) \in]0,1[^2 \cup \{(0,1),(1,0)\}$.*

Proof. If $x, y \neq 0, 1$, it is always possible to find a strong negation N such that $y = N(x)$, and therefore $A(x,y) = A(x, N(x)) = 0$. If $x = 0$ or $y = 0$, Proposition 3.3 has already shown that 0 is necessarily an annihilator element for A. ∎

Note that the above condition implies, in particular, that operators strictly satisfying NC have necessarily a diagonal section which is null except for the point $x = 1$, i.e., they verify $A(x,x) = 0$ for any $x \in [0,1[$. Moreover, such operators may be characterized as follows:

Proposition 3.6 *Let A be a binary aggregation operator. A strictly satisfies NC if and only if for any $x, y \in [0,1]$ it is:*

$$A(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2 \cup \{(0,1),(1,0)\} \\ B(x,y), & \text{otherwise} \end{cases}$$

where B is a binary non-decreasing operator verifying $B(1,1) = 1$.

Proof. If A strictly satisfies NC, Proposition 3.5 shows that it must be $A(x,y) = 0$ whenever $(x,y) \in [0,1[^2 \cup \{(0,1),(1,0)\}$. The converse is obvious. ∎

The above characterization shows that aggregation operators strictly satisfying NC may be either conjunctive operators (when B is chosen such that $B \leq Min$) or hybrid operators (otherwise), but that, in the latter case, they necessarily verify $A \leq Max$. In addition:

Remark 3.2

1. The weakest aggregation operator, A_w (see remark 3.1), strictly satisfies NC.
2. The strongest aggregation operator strictly satisfying NC is A^{nc}, defined, for any $x, y \in [0,1]$, as:

$$A^{nc}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2 \cup \{(0,1),(1,0)\} \\ 1, & \text{otherwise} \end{cases}$$

Note finally that, as it was the case for the standard satisfaction of the NC law, it is also easy to check that the class made of all the aggregation operators strictly satisfying NC is closed under composition by means of any outer aggregation operator. Moreover, it is also closed under transformation by means of any automorphism of the unit interval, that is, if A strictly satisfies NC, then for any automorphism $\varphi : [0,1] \to [0,1]$ the aggregation operator $A_\varphi = \varphi^{-1} \circ A \circ (\varphi \times \varphi)$ does also strictly satisfy NC.

4 On the satisfaction of the Excluded-Middle Law

When dealing with the EM principle, the following definitions, parallel to the ones given for the NC principle, can be established:

Definition 4.1 Let A be a binary aggregation operator and let N a strong negation.

1. It is said that A *satisfies the Excluded-Middle (EM) principle with respect to* N when $A(x, N(x)) = 1$ holds for any $x \in [0,1]$.
2. It is said that A *satisfies EM* when there exists a strong negation N such that A satisfies EM w.r.t. N.

3. It is said that A *strictly satisfies EM* when it satisfies EM w.r.t. any strong negation.

Given a strong negation N, it is obvious, from definitions 3.1 and 4.1, that if A satisfies NC w.r.t. N then it does not satisfy EM w.r.t. N, and vice-versa. Moreover, the same happens even when considering non-coincidental negations, i.e., if A (strictly) satisfies NC then it can not satisfy EM (and vice-versa). Indeed, proposition 3.3 shows that, in order to satisfy NC, A should verify, in particular, $A(0,1) = A(1,0) = 0$, and this obviously prevents A from satisfying EM.

More specifically, the Non-Contradiction and Excluded-Middle concepts have, as is well-known, a clear relationship by means of duality (see e.g. [15]). In the case of aggregation operators, let us first of all recall (see e.g. [2]) that if A is a binary aggregation operator and N is a strong negation, the operator $A_N : [0,1]^2 \to [0,1]$, defined as $A_N = N \circ A \circ (N \times N)$, is in turn a binary aggregation operator, called the N-*dual* operator of A. Then the duality between NC and EM in the framework of aggregation operators can be stated as follows:

Proposition 4.1 *Let A be a binary aggregation operator and let N be a strong negation. Then A satisfies NC w.r.t. N if and only if its N-dual operator, A_N, satisfies EM w.r.t. N.*

This result, whose proof is immediate, allows to easily translate the different conditions and characterizations found in Section 3 for the NC law to the case of the EM law. The main results are presented below, without proofs.

Proposition 4.2 *Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. If A satisfies EM w.r.t. N, then A must verify the two following conditions:*

1. *$A(x_N, x_N) = 1$.*
2. *One is an annihilator element for A, i.e., $A(x,1) = A(1,x) = 1$ holds for any $x \in [0,1]$.*

Corollary 4.1 *Let A be a binary aggregation operator. If A is either an averaging operator or a conjunctive operator, then it does not satisfy EM.*

Aggregation operators satisfying EM may thus only be found among the classes of disjunctive and hybrid operators, and can be characterized as follows:

Proposition 4.3 *Let A be a binary aggregation operator and let N be a strong negation. A satisfies EM w.r.t. N if and only if for any $x, y \in [0,1]$ it is:*

$$A(x,y) = \begin{cases} 1, & \text{if } y \geq N(x) \\ B(x,y), & \text{otherwise} \end{cases}$$

where $B : [0,1]^2 \to [0,1]$ is a non-decreasing operator verifying $B(0,0) = 0$.

When dealing with the strict satisfaction of the EM principle, the following characterization is available:

Proposition 4.4 *Let A be a binary aggregation operator. A strictly satisfies EM if and only if for any $x, y \in [0,1]$ it is:*

$$A(x,y) = \begin{cases} 1, & \text{if } (x,y) \in]0,1]^2 \cup \{(0,1),(1,0)\} \\ B(x,y), & \text{otherwise} \end{cases}$$

where $B : [0,1]^2 \to [0,1]$ is a non-decreasing binary operator verifying $B(0,0) = 0$.

Therefore, aggregation operators strictly satisfying the EM law are either disjunctive operators (when $B \geq Max$) or hybrid operators (otherwise), and, in the latter case, they verify $A \geq Min$.

5 Some Examples

According to the results found in previous sections, the situation regarding the satisfaction of the NC and EM laws in the aggregation operators framework is the following:

1. The class of *conjunctive* operators does not contain any operator satisfying EM, but contains operators satisfying NC, either w.r.t. some specific strong negation or w.r.t. any of them.
2. The class of *disjunctive* operators is dual to the previous one, that is, it does not contain operators satisfying NC but contains operators (strictly) satisfying EM.
3. *Averaging* operators do never satisfy NC, nor EM.
4. The class of *hybrid* operators contains both operators (strictly) satisfying NC as well as operators (strictly) satisfying EM.

In the following we provide examples of both conjunctive and hybrid operators satisfying NC and strictly satisfying NC. Examples of disjunctive/hybrid operators (strictly) satisfying EM may be easily build, by duality (see Proposition 4.1), from the previous ones.

5.1 Conjunctive Aggregation Operators

Given a strong negation N, any operator A constructed as in Proposition 3.4, but with the restriction $B \leq Min$ (in particular, B could be any conjunctive aggregation operator), is clearly a conjunctive aggregation operator satisfying NC w.r.t. N. Of course, the strongest aggregation operator in this class is obtained when taking $B = Min$, and in this case the resulting operator, given by

$$A(x,y) = \begin{cases} 0, & \text{if } y \leq N(x) \\ Min(x,y), & \text{otherwise} \end{cases}$$

is a *triangular norm* (or *t-norm*), i.e., a commutative, associative and non-decreasing operator with neutral element 1. This t-norm is known, when $N = 1 - Id$, as the nilpotent minimum ([8]). Note that in order to get t-norms verifying NC, it is necessary for B to behave as a t-norm (otherwise the overall operator A will loose some of the t-norms' properties), but this is not sufficient, since the associativity property may be lost (this happens, for example, if B is taken as the product t-norm). This means that Proposition 3.4 may be particularized to the case of t-norms in the following way:

Proposition 5.1 *Let $T : [0,1]^2 \to [0,1]$ be a t-norm and let N be a strong negation. T satisfies NC w.r.t. N if and only if there exists a t-norm T_0 such that for any $x, y \in [0,1]$ it is*

$$T(x,y) = \begin{cases} 0, & \text{if } y \leq N(x) \\ T_0(x,y), & \text{otherwise} \end{cases}$$

Aggregation operators as the ones given in Proposition 3.4 but taking B as a t-norm have been studied in the context of residuated lattices and theories based on left-continuous t-norms (see [6] and [7] for an overview and detailed references on this matter), where they are called N-annihilations. Therefore, given a strong negation N, the set of t-norms satisfying NC w.r.t. N coincides with the set of N-annihilations which end up in a t-norm. To that respect, the following results provide wide families of left-continuous t-norms satisfying NC:

- [5] characterizes those N-annihilations that end up in a t-norm when build upon a continuous t-norm.
- [3] generalizes the previous result to the case of left-continuous t-norm.

Regarding continuity, recall also that the only continuous t-norms satisfying NC are those which are isomorphic to the Łukasiewicz t-norm, that is, the following result is available (see e.g. [14]):

Proposition 5.2 *Let T be a continuous t-norm and let N be a strong negation. T satisfies NC w.r.t. N if and only if there exist an automorphism $\varphi : [0,1] \to [0,1]$ such that:*

$$T = W_\varphi = \varphi^{-1} \circ W \circ (\varphi \times \varphi) \quad \text{and} \quad N \leq N_\varphi = \varphi^{-1} \circ (1 - Id_{[0,1]}) \circ \varphi$$

where $W(x,y) = max(0, x+y-1)$ is the Łukasiewicz t-norm.

Note in addition that there are also non-left-continuous t-norms verifying this law, as, for instance, the one known as the drastic product, defined as:

$$Z(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2 \\ Min(x,y), & \text{otherwise} \end{cases}$$

When looking for conjunctive operators strictly satisfying NC, the characterization given in Proposition 3.6 provides a wide family of operator with this property, since it suffices to choose $B \leq Min$ in order to obtain one of them. The strongest operator in this class is the one obtained when choosing $B = Min$, i.e., the drastic product t-norm Z. Since this t-norm is the weakest one, it is, as a consequence, the only t-norm strictly satisfying NC. Of course, any weaker operator (such as for example the weakest aggregation operator A_w mentioned in remark 3.1) is a conjunctive aggregation operator strictly satisfying NC.

5.2 Hybrid Aggregation Operators

Given a strong negation N, any operator A constructed as in Proposition 3.4, with the restriction $B \not\leq Min$, is clearly a hybrid aggregation operator satisfying NC w.r.t. N. Therefore, wide families of these operators may be easily constructed by choosing B among either averaging, hybrid or disjunctive aggregation operators. Obviously, the strongest aggregation operator in this class is the one defined, for any $x, y \in [0,1]$, as:

$$A(x,y) = \begin{cases} 0, & \text{if } y \leq N(x) \\ 1, & \text{otherwise} \end{cases}$$

Regarding the most important known classes of hybrid aggregation operators (see [2] for details and appropriate references), note that neither uninorms, nor null-norms or N-self-dual aggregation operators will satisfy the NC principle, since none of them have the structure given in Proposition 3.4. Nevertheless, examples of hybrid aggregation operators satisfying the NC principle may be found, for example, in the class of *quasi-linear T-S operators* ([10]), that is, operators which, in the binary case, are of the form $QL_{T,S,\lambda,f}(x,y) = f^{-1}\left((1-\lambda)f(T(x,y)) + \lambda f(S(x,y))\right)$, where T is a t-norm, S is a t-conorm, $\lambda \in]0,1[$ and $f : [0,1] \to [-\infty, \infty]$ is a continuous and strictly monotone function such that $\{f(0), f(1)\} \neq \{-\infty, +\infty\}$. Indeed, it is possible to characterize the operators in this class satisfying NC in the following way:

Proposition 5.3 *Let $QL_{T,S,\lambda,f}$ be a binary quasi-linear T-S operator and let N be a strong negation. Then $QL_{T,S,\lambda,f}$ satisfies NC w.r.t. N if and only if it is $f(0) = \pm\infty$ and T satisfies NC w.r.t. N.*

Proof. If $QL_{T,S,\lambda,f}$ satisfies NC, then, according to Proposition 3.3, it must have zero as annihilator, and this happens (see [10]) if and only if it is $f(0) = \pm\infty$. In such circumstances, $QL_{T,S,\lambda,f}(x, N(x)) = 0$ implies $(1-\lambda)f(T(x,N(x))) + \lambda f(S(x,N(x))) = \pm\infty$, which, since $S(x,N(x)) \geq Max(x,N(x)) \neq 0$, may only happen if $T(x,N(x)) = 0$, i.e., T satisfies NC w.r.t. N. The converse is obvious. ∎

This result, along with the results on t-norms which have been mentioned before, provides a wide class of hybrid aggregation operators satisfying NC. This class includes, choosing $f = log$, the so-called *exponential convex T-S operators* ([2]), given by $E_{T,S,\lambda}(x,y) = T(x,y)^{1-\lambda} \cdot S(x,y)^{\lambda}$, whenever the t-norm T satisfies NC.

On the other hand, aggregation operators built as in Proposition 3.6 by means of any B such that $B \not\leq Min$ are hybrid operators strictly satisfying NC. In particular, the class of quasi-linear T-S aggregation operators includes some instances strictly satisfying NC. Indeed, it is not difficult to prove (similarly to Proposition 5.3) the following result:

Proposition 5.4 *A binary quasi-linear T-S operator $QL_{T,S,\lambda,f}$ strictly satisfies NC if and only if $f(0) = \pm\infty$ and T strictly satisfies NC.*

Note that, according to the results found before, the only t-norm strictly satisfying NC is the drastic product t-norm Z. This means, in particular, that the exponential convex combination of Z with any t-conorm S, i.e., the operator $E_{Z,S,\lambda}(x, y) = Z(x, y)^{1-\lambda} \cdot S(x, y)^{\lambda}$, is a hybrid operator strictly satisfying NC.

6 Conclusions and related works

This paper has characterized the classes of binary aggregation operators satisfying the Non-Contradiction and the Excluded-Middle laws, either w.r.t. a given strong negation or w.r.t. any strong negation (strict satisfaction). The results obtained are summarized in Figure 1, showing that:

- Averaging operators (mean operators) do never satisfy neither NC nor EM.

- On the other hand, the NC law may be used as a tool allowing to make distinctions inside the classes of conjunctive and hybrid operators. In the case of conjunctive operators, it appears that t-norms satisfying NC are exactly those t-norms built as the N-annihilation of other t-norms, and wide families of the latter may be found in the literature ([7]). Regarding the strict satisfaction of the NC law, there is just one t-norm with this property, namely the drastic product. When dealing with hybrid operators, even if there are wide families of them satisfying the NC law, it appears that the majority of known operators in this category (such as nullnorms, uninorms or symmetric sums) do not satisfy it. Nevertheless, examples of such operators may be found, for instance, among quasi-linear T-S operators.

- Similarly, the EM law may be used to distinguish among both disjunctive and hybrid operators. In both classes it is possible to find operators satisfying or strictly satisfying EM, as well as others that do not satisfy it.

Note finally that the way in which the NC/EM laws have been interpreted in this paper is not the only possible one. Indeed (see [15]), focussing on the NC law -it is similar for the EM law-, it appears that the fact "$p \wedge \neg p$ is impossible" has at least two interpretations: the one used in this paper, $p \wedge \neg p = 0$, based on the concept of falsity, and the one used in ancient Aristotelian logic, thought in terms of self-contradiction, that leads to the inequality $p \wedge \neg p \leq \neg(p \wedge \neg p)$. Even if both interpretations coincide in some particular structures (such as orthocomplemented lattices), they differ in many others, where the latter is clearly weaker than the former, thus leading to different results. For example, in the case of binary aggregation operators this new interpretation of the NC law translates into the inequality $A(x, N(x)) \leq N(A(x, N(x)))$ for any $x \in [0, 1]$, whose solutions include wider families of operators, among which one may find, for example, instances of averaging operators (see [11, 12]).

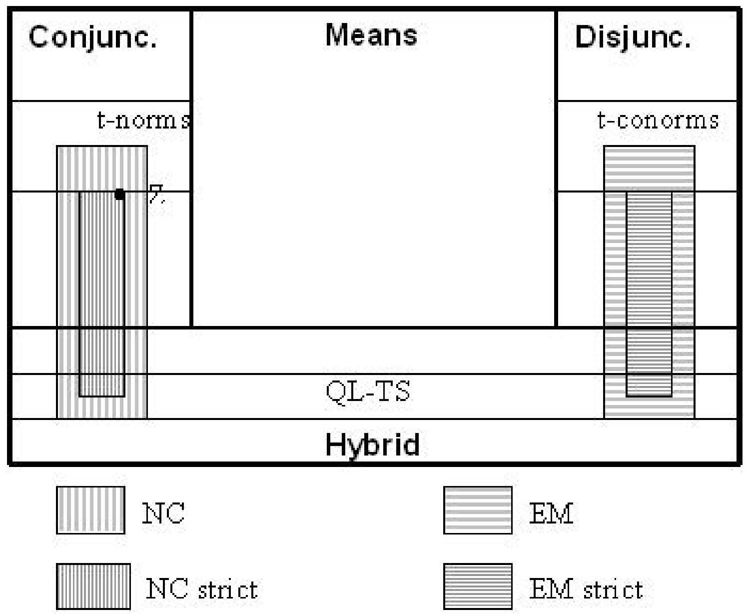

Figure 1: Aggregation operators and the NC/EM principles

References

[1] B. Bouchon-Meunier, editor. *Aggregation and Fusion of Imperfect Information*, volume 12 of *Studies in Fuzziness and Soft Computing.* Physica Verlag, Springer, 1998.

[2] T. Calvo, G. Mayor, R. Mesiar, editors. *Aggregation Operators: New Trends and Applications*, volume 97 of *Studies in Fuzziness and Soft Computing.* Physica Verlag, Springer, 2002.

[3] R. Cignoli, F. Esteva, L. Godo, F. Montagna. On a class of left-continuous t-norms. *Fuzzy Sets and Systems* **131**,(3), 283-296, 2002.

[4] D. Dubois, H. Prade. A review of fuzzy set aggregation connectives. *Information Sciences*, **36**, 85–121, 1985.

[5] S. Jenei. New family of triangular norms via contrapositive symmetrization of residuated implications. *Fuzzy Sets and Systems* **110**, 157-174, 1999.

[6] S. Jenei. Structure of Girard Monoids on [0,1]. In: *Topological and Algebraic Structures in Fuzzy Sets, A Handbook of Recent Developments in the Mathematics of Fuzzy Sets*, (E.P. Klement, S.E. Rodabaugh, eds.), Trends in Logic, volume 20. Kluwer Academic Publishers, Dordrecht, 277-308, 2003.

[7] S. Jenei. How to construct left-continuous triangular norms-state of the art. *Fuzzy Sets and Systems* **143**, 27-45, 2004.

[8] E. Klement, R. Mesiar, E. Pap. *Triangular Norms.* Trends in Logic, volume 8, Kluwer Academic Publishers, Dordrecht, 2000.

[9] R.B. Nelsen. *An Introduction to Copulas.* Lecture Notes in Statistics, volume 139, Springer, New York, 1999.

[10] A. Pradera, E. Trillas, T. Calvo. A general class of triangular norm-based aggregation operators: quasi-linear T-S operators. *International Journal of Approximate Reasoning* **30**, 57-72, 2002.

[11] A. Pradera, E. Trillas. Aggregation operators and the Non-Contradiction principle. In *Proceedings International Summer School on Aggregation Operators and their applications*, Lugano, Switzerland, 97-102, 2005.

[12] A. Pradera, E. Trillas. Aggregation Operators from the Ancient NC and EM point of view. *Kybernetika*, **42**, (3), 243–260, 2006.

[13] E. Trillas. Sobre funciones de negación en la teoría de los subconjuntos difusos. *Stochastica, III-1* (1979) 47-59 (in Spanish). Reprinted (English version) in *Advances of Fuzzy Logic* S. Barro et al. (eds), Universidad de Santiago de Compostela 31–43, 1998.

[14] E. Trillas, C. Alsina. Elkan's theoretical argument, reconsidered. *International Journal of Approximate Reasoning*, **26**, 145–152, 2001.

[15] E. Trillas, C. Alsina, A. Pradera. Searching for the roots of non-contradiction and excluded-middle. *International Journal of General Systems*, **31**, (5), 499–513, 2002.