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Measuring Criteria Weights by Means of Dimension Theory

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Abstract

Measuring criteria weights in multicriteria decision making is a key issue in order to amalgamate information when reality is being described from several different points of view. In this paper we propose a method for evaluating those weights taking advantage of Dimension Theory, which allows the representation of the set of alternatives within a real space, provided that decision maker preferences satisfy certain consistency conditions. Such a representation allows a first information about possible underlying criteria in decision maker's mind. In particular, we propose to measure the importance of those underlying criteria by means of all possible representations associated to the dimension of the binary preference relations between criteria, each one being understood as a linear order of the set of alternatives.

Keywords: Multicriteria Decision Analysis, Valued Preference Relations, Dimension Theory.

1 Introduction

Most multicriteria decision making models assume that reality is being explained by means of a finite set of criteria, each one being represented by a linear order. It is commonly assumed that these criteria are already known, so their importance can be measured in terms of weights, i.e., a real value. Alternatively, some models consider that it is enough to know the relative position of each criteria in the real line (see, e.g., [8, 16, 19]). An quite extended approach, perhaps the most popular one lately, was proposed by Saaty [18], based upon a preference matrix A where a_{ij} represents the degree to which the decision maker considers that criteria i is more important than criteria j (assigning, for example, the values $a_{ij} = 1$ when the criteria are equally important, $a_{ij} = 3$ when i is slightly more important than j, $a_{ij} = 5$ when i is strongly more importance than j, $a_{ij} = 7$ when i has demonstrate more importance than j, and $a_{ij} = 9$ when criteria i is absolutely more importance than j). Taking into account this information, Saaty [18] determine the importance (measures as weights) of each criteria. But Saaty´s method find serious difficulties in building up Saaty's matrix (because of discrimination between *moderate*, *strong*, *demonstrable* and *absolutely* categories, or because of implied consistency conditions). For example, Saaty [18] and some other decision making procedures do not accept incomparability between criteria, in such a way that criteria define a linear order.

In this paper we want to stress the role of those inconsistencies in preference modelling, if we really pretend to model decision maker's mind. Complex decision making problems do need methodologies for a better understanding rather than choice proposals (*Aid for Knowledge* is more appropriate to our objectives than *Aid for Decisions*). In this context, searching for a representative geometrical representation will play a key role, as a natural way for describing the complexity of the problem decision maker is faced to. Our approach here is based upon such a simple argument: a good representation of alternatives should give hints about the structure of the decision making problem, and should help decision makers to understand the problem, and therefore determine the importance of those criteria explaining their preferences.

The paper is organized as follows: in section 2 we present a short review of classical dimension theory, pointing out some limitations of the model. In section 3 we present a new dimension concept that avoids those limitations, extending the concept to valued preference relations. In section 4 we propose a new method to determine criteria weights by means of the information comparing those criteria. Finally, in section 5 we stress the relevance of our approach.

2 Classical dimension theory translated to valued preference relations

Dimension theory was initially developed by Dushnik-Miller [7] for crisp partial orders $R \subset X \times X$, i.e, mappings

$$\mu^R : X \times X \to \{0,1\}$$

where $X = \{x_1, x_2, \ldots, x_n\}$ represents a finite set of alternatives and $\mu^R(x_i, x_j) = 1$ whenever $x_i R x_j$ and $\mu^R(x_i, x_j) = 0$ otherwise. Being a partial order implies that the following conditions hold: non reflexivity ($\mu^R(x_i, x_i) = 0 \quad \forall x_i \in X$), antisymmetry ($\mu^R(x_i, x_j) = 1 \Rightarrow \mu^R(x_j, x_i) = 0$), and transitivity ($\mu^R(x_i, x_j) = \mu^R(x_j, x_k) = 1 \Rightarrow \mu^R(x_i, x_k) = 1$). In particular, based on a result due to Szpilrajn [20] proving that every partial order can be represented as an intersection of linear orders, the dimension of a partial order R was defined by Dushnik-Miller [7] in the following way.

Definition 2.1 (Dushnik-Miller 1941) *Given X a finite set of alternatives, the dimension of a partial order relation R, $dim(R)$, is the minimum number of linear orders L_t in X, such that*

$$R = \bigcap_t L_t.$$

Hence, each element $x_i \in X$ in a crisp partial order set (*poset*) R with dimension d can be represented in the real space $(x_i^1, \ldots, x_i^d) \in R^d$ in such a way that $x_i R x_j$ if and only if

$$x_i^k > x_j^k \quad \forall k \in \{1, \ldots d\} \quad \forall x_i, x_j \in X$$

Of course the associated representation is not unique (see Trotter [21]), but within decision making each one of these linear orders of the set of alternatives suggests a possible underlying criteria playing a role in decision maker's mind. By $[x_{[i_1]}, x_{[i_2]}, \cdots, x_{[i_d]}]$ we shall denote here the linear order such that $x_{[i_j]} > x_{[i_k]}$ for all $j < k$.

Then, given X a finite set of alternatives, a valued preference relation in X is a fuzzy subset of the cartesian product $X \times X$. A valued preference relation in X will be characterized by its membership function

$$\mu : X \times X \to [0, 1]$$

where $\mu(x_i, x_j)$ will represent the degree to which alternative x_i is preferred to alternative x_j. We shall assume here that such a preference intensity is referred to a *strict* preference, so by definition $\mu(x_i, x_i) = 0 \quad \forall x_i \in X$.

A natural way to bring into this valued framework classical dimension theory would be to associate a dimension to each α-cut of the valued preference relation: once a value $\alpha \in (0, 1]$ has been fixed, the α-cut of μ is defined as the crisp binary relation R^α in X such that

$$x_i R^\alpha x_j \quad \Longleftrightarrow \quad \mu(x_i, x_j) \geq \alpha$$

Meanwhile R^α is a partial order set, a dimension $d(\alpha)$ can be associated to such α-cut. A *dimension function* can be then defined as the mapping

$$d : [0, 1] \to N$$

where $d(\alpha) = dim(R^\alpha)$ whenever such a dimension is well defined, see [14] (notice that crisp dimension of all α-cuts were taken also into account in [6] in order to obtain operative bounds). However, this approach requires antisymmetry and transitivity for each α-cut.

For example, in case our valued preference relation is max-min transitive, i.e.,

$$\mu(x_i, x_j) \geq \min\{\mu(x_i, x_k), \mu(x_k, x_j)\}$$

for all $x_i, x_j, x_k \in X$, then R^α is a partial order set whenever antisymmetry holds, i.e., meanwhile those α-cuts do not show 2-order cycles. In particular (see [15]), R^α is antisymmetric for all $\alpha > \alpha_2$, being

$$\alpha_2 = \max_{x_i \neq x_j} \min\{\mu(x_i, x_j), \mu(x_j, x_i)\}$$

Therefore, since μ is max-min transitive if and only if every α-cut R^α is transitive (see [11] but also [6]), if μ is max-min transitive, then dimension is defined for $R^\alpha, \forall \alpha > \alpha_2$.

But imposing strong consistency restrictions seems unrealistic when dealing with valued preference relations, even if the number of alternatives is relatively small. As pointed out in [14, 15], more effort should be devoted to understand and explain decision maker inconsistencies (accepted inconsistencies are some times extremely informative). The following section presents an interesting less restrictive approach that generalizes classical dimension.

3 Dimension theory for arbitrary preference relations

Since representation of arbitrary binary preference relation must be an objective, the following result shows that any strict preference relation can be represented in terms of unions and intersections of linear orders (see [14, 15] but also [9]). In this way we offer a representation in a real space avoiding main restrictions of the classical approach.

Theorem 3.1 *Let $X = \{x_1, ..., x_n\}$ be a finite set of alternatives, and let us consider*

$$\mathcal{C} = \{L/L \text{ linear order on } X\}$$

Then for every non-reflexive crisp binary relation R on X there exists a family of linear orders $\{L_{st}\}_{s,t} \subset \mathcal{C}$ such that

$$R = \bigcup_s \bigcap_t L_{st}$$

Proof: see [14].

Hence, a generalized dimension function can be translated into the valued preference framework, which will allow a realistic approach to dimension function, since no strong condition is needed for every α-cut.

Definition 3.1 *Let us consider X a finite set of alternatives. The* generalized dimension *of a crisp binary relation R is the minimum number of different linear orders, L_{st}, such that*

$$R = \bigcup_s \bigcap_t L_{st}$$

Of course, practical implementation of generalized dimension is still subject to the resolution of its algorithmic complexity , also present in the classical definition (see [22] but also [23]). But the existence of the above general representation is always assured, no matter if it is a partial ordered set or not. This is an important issue when dealing with preferences in practice.

Notice also that under our approach intransitivity due to a missing preference will be not confused with intransitivity due to a cycle: these two situations will have different representation, as shown in the next two examples (equal dimension but different representation).

Example 3.1 *Let us consider* $X = \{x_1, x_2, x_3\}$ *and* R *a crisp binary relation showing a 3-cycle, in such a way that its membership function verifies*

$$\mu_R(x_i, x_j) = \begin{cases} 1 & \textit{if } (x_i, x_j) = (x_1, x_2), (x_2, x_3), (x_3, x_1) \\ 0 & \textit{otherwise} \end{cases}$$

Then R *can be represented as*

$$R = P_1 \cup P_2 \cup P_3$$

where P_1, P_2 *and* P_3 *are strict partial orders respectively defined by:*

$$\mu_{P_1}(x_i, x_j) = \begin{cases} 1 & \textit{if } (x_i, x_j) = (x_1, x_2) \\ 0 & \textit{otherwise} \end{cases}$$

$$\mu_{P_2}(x_i, x_j) = \begin{cases} 1 & \textit{if } (x_i, x_j) = (x_2, x_3) \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\mu_{P_3}(x_i, x_j) = \begin{cases} 1 & \textit{if } (x_i, x_j) = (x_3, x_1) \\ 0 & \textit{otherwise} \end{cases}$$

Hence,

$$R = \big([x_1, x_2, x_3] \cap [x_3, x_1, x_2]\big) \cup \big([x_1, x_2, x_3] \cap [x_2, x_3, x_1]\big) \cup \big([x_3, x_1, x_2] \cap [x_2, x_3, x_1]\big)$$

in such a way that $Dim(R) = 3$.

Example 3.2 *Let us consider* R' *with*

$$\mu_{R'}(x_i, x_j) = \begin{cases} 1 & \textit{if } (x_i, x_j) = (x_1, x_2), (x_2, x_3) \\ 0 & \textit{in other cases} \end{cases}$$

then R' *can be represented as the union*

$$R' = P_1 \cup P_2$$

where

$$\mu_{P_1}(x_i, x_j) = \begin{cases} 1 & \textit{if } (x_i, x_j) = (x_1, x_2) \\ 0 & \textit{otherwise} \end{cases}$$

$$\mu_{P_2}(x_i, x_j) = \begin{cases} 1 & \textit{if } (x_i, x_j) = (x_2, x_3) \\ 0 & \textit{otherwise} \end{cases}$$

Hence,

$$R'_3 = \big([x_1, x_2, x_3] \cap [x_3, x_1, x_2]\big) \cup \big([x_1, x_2, x_3] \cap [x_2, x_3, x_1]\big)$$

but still $Dim(R') = 3$.

Example 3.3 *Let us consider now R'' with*

$$\mu_{R''}(x_i, x_j) = \begin{cases} 1 & \text{if } (x_i, x_j) = (x_1, x_2), (x_1, x_3) \\ 0 & \text{in other cases} \end{cases}$$

then R'' is a partial order set with

$$R'' = [x_1, x_2, x_3] \cap [x_1, x_3, x_2]$$

being in this case $Dim(R'') = 2$.

It should be pointed out that, in general, generalized dimension of a partial order may not be equal to its classical dimension value. From a practical point of view, we expect that each representation will suggest different explanation of preferences (the search for a meaningful representation will be more relevant than the dimension value).

3.1 Dimension function of arbitrary preference relations

Based upon the above generalized representation of crisp preferences we can therefore assure the existence of a *generalized* dimension function (see [14, 15]).

Definition 3.2 *Given a valued strict preference relation*

$$\mu : X \times X \to [0, 1]$$

its generalized dimension function *is given by the mapping*

$$\begin{array}{llll} D: & (0,1] & \to & N \\ & \alpha & \to & D(\alpha) = Dim(R^\alpha) \end{array}$$

where $Dim(R^\alpha)$ is the generalized dimension of R^α.

This approach will then lead to a *generalized dimension function* showing the generalized dimension for every α-cut, no matter if our valued preference relation μ is max-min transitive or not. In general, X being a finite set of alternatives, the interval $(0, 1]$ is divided in two subsets, depending on the existence of union operators in the above generalized representation.

A standard dimension analysis under approach is suggested in the following example, where only three alternatives are considered.

Example 3.4 *Let us consider $X = \{x_1, x_2, x_3\}$ and let μ be the strict valued preference relation depicted in figure 1.*

$$\mu = \begin{pmatrix} 0 & 0.2 & 0.3 \\ 0.4 & 0 & 0.6 \\ 0.9 & 0.1 & 0 \end{pmatrix}$$

Seven different α-cuts intervals can be considered:

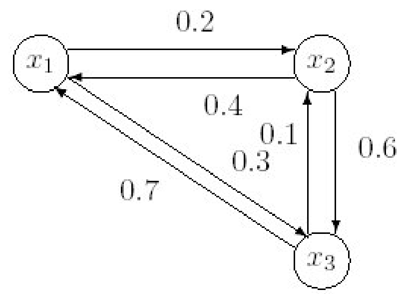

Figure 1: Binary valued relation in example 3.4

1. *When* $\alpha \leq 0.1$*, we have*

$$\mu^{R^\alpha} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

*This relation shows cycles (*e.g.*,* $x_1 > x_3, x_3 > x_1$*) but it is transitive. This relation can be obtained as*

$$R^\alpha = L_1 \cup L_2$$

where

$$\mu^{L_1} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mu^{L_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

That is,

$$R^\alpha = ([x_1, x_2, x_3]) \cup ([x_3, x_2, x_1])$$

and $dim(R^\alpha) = 2$.

2. *When* $0.1 < \alpha \leq 0.2$*,* R^α *also shows cycles and transitivity does not hold:*

$$\mu^{R^\alpha} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

In this case, R^α *can be obtained as the union of two posets*

$$R^\alpha = L_1 \cup R_1$$

where

$$\mu^{R_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

in such a way that

$$R^\alpha = ([x_1, x_2, x_3]) \cup ([x_3, x_2, x_1] \cap [x_2, x_3, x_1])$$

and $dim(R^\alpha) = 3$.

3. *When* $0.2 < \alpha \leq 0.3$, *relation* R^α *still shows cycles but it is transitive again:*

$$\mu^{R^\alpha} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Hence,

$$R^\alpha = L_3 \cup R_2$$

where

$$\mu^{L_3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mu^{R_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$R^\alpha = ([x_2, x_3, x_1]) \cup ([x_2, x_1, x_3] \cap [x_1, x_3, x_2])$$

in such a way that $dim(R^\alpha) = 3$.

4. *When* $0.3 < \alpha \leq 0.4$, *the* α*-cut shows no cycles and it is transitive:*

$$R^\alpha = L_3 = [x_2, x_3, x_1]$$

Therefore, $dim(R^\alpha) = 1$.

For higher values, $\alpha > 0.3$, *the relation* R^α *will not show cycles.*

5. *When* $0.4 < \alpha \leq 0.6$, *however, the relation* R^α *becomes non transitive:*

$$\mu^{R^\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and the union operator it is again needed:

$$R^\alpha = R_3 \cup R_4$$

where

$$\mu^{R_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mu^{R_4} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

in such a way that

$$R^\alpha = ([x_2, x_3, x_1] \cap [x_3, x_1, x_2]) \cup ([x_1, x_2, x_3] \cap [x_2, x_3, x_1])$$

and $dim(R^\alpha) = 3$.

6. *When* $0.6 < \alpha \leq 0.9$*, the relation* R^α *defines the previous* R_3 *poset:*

$$\mu^{R^\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$R^\alpha = [x_2, x_3, x_1] \cap [x_3, x_1, x_2]$$

in such a way that $dim(R^\alpha) = 2$.

7. *When* $\alpha > 0.9$*, the relation* R^α *is the empty relation.*

$$\mu^{R^\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $dim(R^\alpha) = 2$ *and*

$$R^\alpha = [x_1, x_2, x_3] \cap [x_3, x_2, x_1].$$

4 Determination of weights

If descriptive tools are useful in order to help decision maker capture complexity of a problem, the search for good representation models should be a main objective. Dimension theory seems a natural alternative under this framework, and will allow us to avoid the restrictive classical assumption of a family of criteria C being imbedded in the real line, i.e., $dim(C) = 1$ (it is important to stress that this is the only possibility being allowed in standard methods). In this section we will show how we can take advantage of the above sequence of representations, in order to determine the importance of each criteria. It is again extremely important to realize that we are avoiding restrictive constraints being assumed in standard approaches (see [8, 16, 18, 19]), both in the crisp and valued context (see also [9]).

Our objective is the next subsections is to get information about the importance of criteria taking into account the above sequence of representations for a given valued preference relation between criteria

$$\mu_C : C \times C :\longrightarrow: [0, 1]$$

being associated to each value $\alpha \in (0, 1]$. Lets address before the crisp case.

4.1 Partial order of criteria

Let

$$\mu_C : C \times C :\longrightarrow \{0, 1\}$$

be a crisp preference relation between criteria. Let us suppose first that μ_C defines a partial order, and let $R = \bigcap_{k=1,d} L_k$ be a representation of R, being understood and accepted by the decision maker.

Definition 4.1 *Let C be the set of criteria and let L be a lineal order on C. We will say that*

$$F_L : C \longrightarrow [0,1]$$

is a fair allocation rule for the pair (C, L) if and only if:

- *If $c_i L c_j$ then $F_L(c_i) < F_L(c_j)$.*
- $\sum_{c \in C} F_L(c) = 1$.

It is easy to see that most standard models in the literature (see, e.g., [18, 19, 8]) are *fair allocation rules* in the above sense (additional information may be required in each particular case). Two interesting examples of fair allocation rule are the following, both based on the position of the j-th criteria in the above ordering, r_j:

1. $F(c_j) = \dfrac{\frac{1}{r_j}}{\sum_{i=1}^{n} \frac{1}{r_i}}$.
2. $W_j = \dfrac{(n - r_j + 1)}{\sum_{i=1}^{n} (n - r_i + 1)}$.

Once we have a *fair allocation rule* F_L for each lineal order L, we only need to aggregate the weights

$$(F_L(c_1), F_L(c_2), \dots F_L(c_{|C|}))$$

in such a way that the sum of final weights is one (notice that there are alternative options, see [1, 3]).

So, given a representation of the crisp preference relation $R = \bigcap_{k=1}^{d} L_k$, a family of *fair allocation rules* for this representation $\{F_{L_k}\}$, and given $\phi : [0,1]^d \longrightarrow [0,1]$ an aggregation operator (see [4] but also [2, 5]), the aggregated weights of every criteria $(W_1, W_2, \dots W_{|C|})$ can be obtained as

$$W_i = \phi\left(F_{L_1}(c_i), \dots, F_{L_d}(c_i)\right) \;\; \forall i = 1, \dots, |C|$$

Obviously, aggregation operators can not be chosen arbitrarily, since the final sum of weights must be one.

Definition 4.2 *Let $\phi : [0,1]^d \longrightarrow [0,1]$ be an aggregation operator. We will say that this operator is doubly efficient with respect to the sum if*

$$\sum_{i=1}^{|C|} \phi(w_i^1, w_i^2, \dots, w_i^d) = 1$$

whenever $\sum_{j=1}^{|C|} w_i^j = 1 \;\; \forall i = 1, 2, \dots, d$

Proposition 4.1 *If ϕ is a additive aggregation operator i.e.*

$$\phi(x_1, \ldots, x_d) = \sum_{k=1}^{d} a_k x_k \tag{1}$$

then ϕ is doubly efficient with respect to the sum.

Proof: Since $\phi(1, \ldots, 1) = 1$ by definition of aggregation operator (see [4]), then we have that $\sum_{k=1}^{d} a_k = 1$. Then,

$$\sum_{i=1}^{|C|} \phi(w_i^1, w_i^2, \ldots, w_i^d) = \sum_{i=1}^{|C|}\sum_{r=1}^{d} a_r w_i^r = \sum_{r=1}^{d} a_r \sum_{i=1}^{|C|} w_i^r = \sum_{r=1}^{d} a_r = 1$$

So, additive aggregation operators define *fair allocation rules.*

Example 4.1 *Let C_1, C_2, C_3 be the criteria of a given decision making problem, and let*

$$R = \mu_C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Hence, $c_1 < c_3$ (the importance of c_1 is less than c_3), $c_2 < c_3$ and $c_1 || c_2$ (incomparable importance of these two criteria). In this case, $dim(R) = 2$ because $R = [c_2, c_1, c_3] \bigcap [c_1, c_2, c_3]$. If decision maker accepts and understands this representation, and for example

$$F_{L_k}(c_j) = w_j = \frac{\frac{1}{r_j}}{\sum_{i=1}^{n} \frac{1}{r_i}}$$

then the weights for L_1 are $(\frac{3}{11}, \frac{2}{11}, \frac{6}{11})$, and the weights for L_2 are $(\frac{2}{11}, \frac{3}{11}, \frac{6}{11})$. If ϕ is the median aggregate operator we get $W = (\frac{5}{22}, \frac{5}{22}, \frac{12}{22})$.

4.2 Valued preferences of criteria

Let $\mu_C : C \times C :\longrightarrow [0, 1]$ be a valued preference relation between criteria, in such a way that $\mu_C(c_i, c_j)$ represents the degree to which c_i is more important than c_j for the decision maker. As already pointed out, a standard generalization of the classical representation needs max-min transitive valued preference relations, plus the condition $\mu_C(c, c) = 0$, so a representation of criteria can be obtained for each $\alpha \in (0, 1]$ in order to determine the importance (weights) of criteria. It is important to notice that even in this case, the weighting function $W(\alpha)$ depends on the attitude of the decision maker: different people with the same valued preference relation, if forced to be crisp, can face different problems depending on their exigency level.

We can therefore aggregate all possible attitudes of the decision maker, for example as $\int_0^1 W(\alpha)d\alpha$. And in case we know the probability distribution of the attitude, defined by means of their density function $a(\alpha)$, we can aggregate $W(\alpha)$ as $\int_0^1 W(\alpha)\ a(\alpha)d\alpha$. Of course, other aggregations are possible.

4.3 Generalized dimension function

The above sections allow us to obtain the importance of each criteria when decision maker preferences between criteria are max-min transitive, but of course this is not always possible. In order to determine the weights in a more general case, we can introduce generalized dimension [14, 15], which allows a representation of arbitrary binary relations, as already shown in section 3.

Given R a binary preference relation represented as

$$R = \bigcup_{r=1}^{k} \bigcap_{s=1}^{d_r} L_{rs}$$

we can aggregate this information taking into account a *fair allocation rule* for each L_{rs}. Now we need to aggregate the information in two steps, in order to obtain the aggregated importance of each criteria.

Let us denote by $w_j^{rs} = F_{L_{r,s}}(c_j)$ the weight of the j-th criteria in the lineal order L_{rs}. First, we aggregate the information contained in $\cap_{s=1,d_r} L_{rs}$, obtaining

$$w_j^r = \phi_r(w_j^{r,1}, w_j^{r,2}, \ldots, w_j^{r,d_r})$$

and the aggregated weight W_j will be the aggregation of $w_j^r, \forall r = 1, \ldots, k$:

$$W_j = \varphi(w_j^1, w_j^2, \ldots, w_j^k)$$

Hence, given

$$R = \cup_{r=1,k} \cap_{s=1,d_r} L_{rs}$$

a representation of the binary preference relation,

$$\phi_r : [0,1]^{d_r} \longrightarrow [0,1]$$

a family of aggregation operator for $r = 1, \ldots k$ and

$$\varphi : [0,1]^k \longrightarrow [0,1]$$

another aggregation operator, in this case we need to impose that

$$\sum_{j=1}^{|C|} \varphi\left(\phi_1\left(w_j^{11}, \ldots w_j^{1,d_1}\right), \ldots, \phi_k\left(w_j^{k1}, \ldots w_j^{k,d_k}\right)\right)$$

takes value 1, for all w_j^{rs} such that

$$\sum_{j=1}^{|C|} w_j^{rs} = 1 \quad \forall s = 1, \dots, d_r \; ; \; \forall r = 1, \dots, k$$

Proposition 4.2 *Given $R = \cup_{r=1}^{k} \cap_{s=1}^{d_r} L_{rs}$ a representation of the binary preference relation, $\phi_r : [0,1]^{d_r} \longrightarrow [0,1]$ a family of aggregation operators ($r = 1, \dots k$), and $\varphi : [0,1]^k \longrightarrow [0,1]$ another aggregation operator, if φ and $\{\phi_r\}_{r=1}^{k}$ are aggregation operators being double efficient with respect to the sum, then*

$$\sum_{j=1}^{|c|} \varphi \left(\phi_1 \left(w_j^{11}, \dots w_j^{1,d_1} \right), \dots, \phi_k \left(w_j^{k1}, \dots w_j^{k,d_k} \right) \right)$$

is one, for all w_j^{rs} such that

$$\sum_{j=1}^{|C|} w_j^{rs} = 1 \quad \forall s = 1, \dots, d_r \; ; \; \forall r = 1, \dots, k$$

Proof: Let us denote

$$w_j^r = \phi_r(w_j^{r1}, \dots, w_j^{rd_r})$$

Since φ is a aggregation operator being double efficient with respect to the sum, we only need to prove that

$$\sum_{j=1}^{|C|} w_j^r = 1, \forall r = 1, \dots k$$

Fixed $r \in \{1, \dots k\}$,

$$\sum_{j=1}^{|C|} w_j^r = \sum_{j=1}^{|C|} \phi_r(w_j^{r1}, \dots, w_j^{rd_r})$$

where $\sum_{s=1,d_r} w_j^{rs} = 1, \forall j$. Then, since ϕ_r is doubly efficient,

$$\sum_{j=1}^{|C|} \phi_r(w_j^{r1}, \dots, w_j^{rd_r}) = 1$$

and the result holds.

Proposition 4.3 *Under the previous conditions, if φ and ϕ are additive aggregation rules then $\sum_{i=1}^{|C|} W_i = 1$.*

Proof: direct from 4.1 and 4.2.

It is important to point out that we have been able to represent the dimension function for any value of α and for any valued preference relation.

Example 4.2 *Let us consider $C = \{C_1, C_2, C_3\}$ a set of three criteria, and let μ be the strict valued preference relation analyzed in the example 3.4. If we consider the operators and allocation rule given in such example, the weights for each alpha cut are:*

1. *When $\alpha \leq 0.1$, $R^\alpha = ([c_1, c_2, c_3]) \cup ([c_3, c_2, c_1])$, so $W(\alpha) = \left(\frac{4}{11}, \frac{3}{11}, \frac{4}{11}\right)$*
2. *When $0.1 < \alpha \leq 0.2$, $R^\alpha = ([c_1, c_2, c_3]) \cup ([c_3, c_2, c_1] \cap [c_2, c_3, c_1])$, so $W(\alpha) = \left(\frac{4}{11}, \frac{2.75}{11}, \frac{4.25}{11}\right)$*
3. *When $0.2 < \alpha \leq 0.3$, $R^\alpha = ([c_2, c_3, c_1]) \cup ([c_2, c_1, c_3] \cap [c_1, c_3, c_2])$, so $W(\alpha) = \left(\frac{4.25}{11}, \frac{3}{11}, \frac{3.75}{11}\right)$*
4. *When $0.3 < \alpha \leq 0.4$, $R^\alpha = [c_2, c_3, c_1]$, so $W(\alpha) = \left(\frac{6}{11}, \frac{2}{11}, \frac{3}{11}\right)$.*
5. *When $0.4 < \alpha \leq 0.6$, $R^\alpha = ([c_2, c_3, c_1] \cap [c_3, c_1, c_2]) \cup ([c_1, c_2, c_3] \cap [c_2, c_3, c_1])$, so $W(\alpha) = \left(\frac{4.25}{11}, \frac{3.25}{11}, \frac{3.5}{11}\right)$.*
6. *When $0.6 < \alpha \leq 0.9$, $R^\alpha = [c_2, c_3, c_1] \cap [c_3, c_1, c_2]$, so $W(\alpha) = \left(\frac{4.5}{11}, \frac{4}{11}, \frac{2.5}{11}\right)$.*
7. *When $0.9 < \alpha$, $R^\alpha = [c_1, c_2, c_3] \cap [c_3, c_2, c_1]$, so $W(\alpha) = \left(\frac{4}{11}, \frac{3}{11}, \frac{4}{11}\right)$.*

If we consider that any attitude of the decision maker is possible, the final criteria weights can be aggregated as $\int_0^1 W(\alpha) d\alpha$. Therefore,

$$W = \left(\frac{4.425}{11}, \frac{3.225}{11}, \frac{3.35}{11}\right) = (0.4022, 0.2931, 0.3045)$$

5 Final remarks

In this paper we continue the research initiated in [10] in order to develop an alternative method to determine the importance of criteria in decision making, obtained from possible representations of preferences within real space. This method will be specially useful when incomparability among criteria appears.

It must be noticed that key degrees of freedom in our approach should be carefully fixed through experience: aggregation operator is not unique, neither the associated representation or the consistency behind (see [11]). The role of decision makers is extremely relevant, since representations, in order to be useful, should be understandable and manageable by them (see [13]). Our generalized dimension approach to preference relations opens interesting possibilities for decision making aid, so importance of criteria can be put in clear, perhaps acknowledging that such an importance valuation does not imply the assumption of a linear order in the set of criteria. In this sense, we again want to stress the key role representation tools play in order to get a good understanding of complex decision making problem. Representation techniques should be considered as part of a necessary data mining analysis, previous to any decision making procedure (see, e.g., [13, 12]). In this paper we have explored a particular representation technique based upon dimension theory, assuming that this approach should give natural hints about underlying criteria and their relative importance.

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