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Lexicographic Combinations of Preference Relations in the Context of Possibilistic Decision Theory

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Abstract

In Possibilistic Decision Theory (PDT), decisions are ranked by a pessimistic or by an optimistic qualitative criteria. The preference relations induced by these criteria have been axiomatized by corresponding sets of rationality postulates, both à la von Neumann and Morgenstern and à la Savage. In this paper we first address a particular issue regarding the axiomatic systems of PDT à la von Neumann and Morgenstern. Namely, we show how to adapt the axiomatic systems for the pessimistic and optimistic criteria when some finiteness assumptions in the original model are dropped. Second, we show that a recent axiomatic approach by Giang and Shenoy using binary utilities can be captured by preference relations defined as lexicographic refinements of the above two criteria. We also provide an axiomatic characterization of these lexicographic refinements.

1 Introduction

In [3] Dubois and Prade proposed an axiomatic qualitative counterpart of von Neumann and Morgenstern's Expected Utility Theory [8] where uncertainty is modeled by possibility distributions on the set of states or situations instead of probability distributions. Since then, a number of contributions (e.g. [1, 4, 5, 6]) have been made in order to develop an axiomatic basis for possibilistic decision theory, both in the style of von Neumann and Morgenstern (where a possibility distribution on the states is assumed to be given) and in the style of Savage (where the distribution is induced from a preference relation on the set of decisions).

In Possibilistic Decision Theory two main axiomatic systems have been studied, that can be respectively represented by two types of qualitative[1] utility functionals, called *pessimistic* and *optimistic*, which are generalizations of the well-known Wald's maximin and maximax criteria. For example, assume the set of possible

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[1] In the sense that they only involve the minimum, the maximum and an order-reversing operators

outcomes in a decision problem is $X = \{x_1, x_2, x_3, x_4\}$ and that the preferences of the decision maker is $x_1 \prec x_2 \prec x_3 \prec x_4$, where $\prec$ reads "less preferred than". Given a decision d that may result on the outcomes x_1 or x_4 and a decision d' that results on x_2 or x_3, a decision maker following the pessimistic criterion would choose d' since the worst outcome of d', x_2, is preferred to the worst outcome of d, x_1. Conversely, a decision maker following the optimistic criterion would choose d since the best outcome of d, x_4, is preferred to the best outcome of d', x_3. The possibilistic pessimistic and optimistic criteria modulate the maximin and maximax criteria when the possible outcomes of a decision are not equally plausible.

As already pointed out, for example in [6], these criteria may fail to provide intuitive results in some situations. For instance, following the above example, consider two further decisions: d_1 which always results on x_1, and d_4 which always results on x_4. Then according to the pessimistic criterion d and d_1 are equally preferred since both have the same worst outcome. However, it seems reasonable to prefer d in such a case since there is a chance to obtain the best outcome x_4 while with d_1 we would obtain x_1 for sure. Analogously, d and d_4 are indifferent with respect to the optimistic criteria, while it seems reasonable to prefer d_4 since the best outcome is guaranteed.

To remedy this lack of discriminatory power, which is inherited by the optimistic and pessimistic possibilistic utilities, several solutions have been proposed. In particular several ways of refining the resulting preference relation among decisions have been considered. Dubois et al. [2] consider the refinement of one of the possibilistic utilities by lexicographically combining it with additional criteria to break ties. For instance, in the above example, if the decision maker uses the optimistic criterion to refine indifferences yielded by the pessimistic criterion, then he would choose d in front of d_1. Fargier and Sabaddin [5] consider refinements[2] of the possibilistic utilities that agree with the Expected Utility(EU) model. Finally, Giang and Shenoy [6] propose a modified possibilistic utility based on a two-dimensionally valued utility function, axiomatized in the von Neumann-Morgenstern style[3], they argue it provides more intuitive results. They claim moreover that this new model should not be viewed as a (lexicographic) combination of pessimistic and optimistic utilities like the previously mentioned one in [2]. Weng also considers in [9] a general axiomatic system still in the possibilistic framework capable of coping with the above utility models and some refinements of them.

One of the motivations of this paper is to show that Giang and Shenoy's model can indeed be captured by a particular lexicographic refinement of the above mentioned pessimistic and optimistic criteria. This is shown in Section 3, after briefly recalling the basic framework of possibilistic decision framework in Section 2. This result has lead us to study a new axiomatic characterization of lexicographic refinements of possibilistic utilities, which is alternative to the one described in [2]. To do so, and for the sake of having a technically simpler framework to work on, in Section 4 we first adapt the original axiomatic systems for the pessimistic and

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[2] By means of lexicographic orderings (leximax, leximin) but not on the criteria themselves but on the representative vectors $((\pi_d(x_1), u(x_1)), ..., (\pi_d(x_n), u(x_n)))$ induced by each decision d.

[3] Giang and Shenoy's binary possibilistic utility has been recently axiomatized in the Savagean framework by Weng [10].

optimistic criteria to the case where the uncertainty and utility scales V and U are taken both to be the real unit interval [0, 1], as opposed to the original (and usual so far) framework where U and V were finite linear and commensurate scales. Finally, in Section 5, using this new framework we provide an axiomatic approach to characterize general lexicographic refinements of possibilistic utilities, improving [2], and paying attention to some cases of particular interest. We end up with some concluding remarks.

2 The Basic Framework of Possibilistic Decision Theory

In a possibilistic decision problem one typically considers a (finite) set S of possible states of the world and the uncertainty about what is the actual state of the world is represented by a normalized possibility distribution

$$\pi_0 : S \to V$$

with values on a *finite* totally ordered uncertainty scale $(V, \leq, 0_V, 1_V)$. Decisions are modeled as mappings

$$d : S \to X$$

from the set of situations to a *finite* set of possible consequences (or prizes) X, where $d(s)$ denotes the consequence obtained by decision d when the state s occurs. Each decision d induces a (normalized) possibility distribution on consequences $\pi_d : X \to V$ defined as

$$\pi_d(x) = \max\{\pi_0(s) \mid s \in S, d(s) = x\} \ .$$

A normalized possibility distribution on X is also called a *possibilistic lottery*. We shall also use the expression $[\pi(x_1)/x_1, \pi(x_2)/x_2, ..., \pi(x_n)/x_n]$ to denote a lottery π where the outcome x_i has associated plausibility level $\pi(x_i)$, with the convention that impossible consequences (consequences x with $\pi(x) = 0$) are omitted from the list. The set of lotteries will be denoted by $\Pi(X)$. Notice that $\Pi(X)$ is closed under the operation of *standard possibilistic mixture* defined as follows. Given n possibility distributions $\pi_1, ..., \pi_n$ from $\Pi(X)$ and n values $\lambda_1, ..., \lambda_n$ from V such that $\max(\lambda_1, ..., \lambda_n) = 1_V$, then $[\lambda_1/\pi_1, ..., \lambda_n/\pi_n]$ is the (normalized) possibility distribution defined as

$$[\lambda_1/\pi_1, ..., \lambda_n/\pi_n](x) = \max_{i=1,...,n} \min(\lambda_i, \pi_i(x)) \ . \tag{1}$$

This possibilistic mixture construction allows us to express not only simple lotteries but also compound lotteries. Notice that each consequence $x \in X$ can be viewed also as a lottery π_x where $\pi_x(x) = 1_V$ and $\pi_x(y) = 0_V$ for $y \neq x$. When no confusion exists, we will simply use x to also denote π_x. Similarly, we shall also denote by A both a subset $A \subseteq X$ and the possibility distribution on X such that $\pi(z) = 1_V$ if $z \in A$ and $\pi(z) = 0_V$ otherwise.With this convention, we can consider X as included in $\Pi(X)$.

In the framework of possibilistic decision theory à la von Neumann and Morgenstern, from the decision maker point of view, a decision d is equivalently described by its induced lottery π_d, hence, ranking decisions amounts to rank lotteries. Therefore, the main concern will be on the definition of preference relations in the set of possibility distributions on consequences (i.e. possibilistic lotteries) and the axiomatic systems of rationality postulates which characterize them.

Given a utility function $u : X \to U$ representing the decision maker's preferences on consequences, where $(U, \leq_U, 0_U, 1_U)$ is a *finite* totally ordered utility scale (a consequence x is at least as preferred as x' whenever $u(x') \leq_U u(x)$), the basic possibilistic model introduced by Dubois and Prade [3] propose to define two preference relations among lotteries according to an optimistic or a pessimistic criterion represented by Sugeno-like integrals which generalize the well-known Wald's maximin and maximax criteria. Namely,

$$d \preceq^- d' \text{ iff } QU^-(\pi_d \mid u) \leq QU^-(\pi'_d \mid u) \ ,$$
$$d \preceq^+ d' \text{ iff } QU^+(\pi_d \mid u) \leq QU^+(\pi'_d \mid u) \ ,$$

where

$$\begin{array}{c} QU^-(\pi_d \mid u) = \min_{x \in X} \max(n(\pi_d(x)), u(x)) \ , \\ QU^+(\pi_d \mid u) = \max_{x \in X} \min(h(\pi_d(x)), u(x)) \ , \end{array}$$

with $n = n_U \circ h$, n_U being the reversing involution on U and with $h : V \to U$ being an *onto*[4] order-preserving mapping linking the uncertainty and utility scales.

$QU^-(\pi_d \mid u)$ evaluates to what extent all possible consequences of d are good, hence $QU^-(\cdot \mid u)$ models a pessimistic criterion, while $QU^+(\cdot \mid u)$ represents an optimistic behavior since $QU^+(\pi_d \mid u)$ evaluates to what extent at least one possible consequence of d is good. Notice that both criteria are qualitative in the sense that they only involve the minimum, the maximum and the order reversing operators, plus an order-preserving mapping.

In [3, 1], the authors study two axiomatic systems in the style of von Neumann and Morgenstern (VNM). Namely, the first set of rationality postulates $\mathcal{S}_P$ for a preference ordering $\sqsubseteq$ on lotteries is the following one:

A1 (*Structure*): $\sqsubseteq$ is a total pre-order (i.e. $\sqsubseteq$ is reflexive, transitive and total)

A2$^-$ (*Uncertainty aversion*): $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.

A3 (*Independence*): $\pi_1 \sim \pi_2 \Rightarrow [\alpha/\pi_1, \beta/\pi] \sim [\alpha/\pi_2, \beta/\pi]$.

A4$^-$ (*Continuity$^-$*): $\forall \pi \in \Pi(X)\ \exists \lambda \in V$ s.t. $\pi \sim [1/\overline{x}, \lambda/\underline{x}]$,

where $\pi_1 \sim \pi_2$ means $\pi_1 \sqsubseteq \pi_2$ and $\pi_2 \sqsubseteq \pi_1$, and $\overline{x}$ and $\underline{x}$ denote respectively a best and a worst consequence according to $\sqsubseteq$. The second system of postulates $\mathcal{S}_O$ consists of **A1**, **A3** together with

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[4] The onto condition is not necessary for defining the utilities but it is a technical condition required for representation purposes, see e.g. [1].

A2$^+$(*Uncertainty attraction*): $\pi \leq \pi' \Rightarrow \pi \sqsubseteq \pi'$.

A4$^+$(*Continuity$^+$*): $\forall \pi \in \Pi(X)\ \exists \lambda \in V$ s.t. $\pi \sim [\lambda/\overline{x}, 1/\underline{x}]$.

It has been shown that a preference relation satisfying the first axiom system $\mathcal{S}_P$ is always representable by a pessimistic utility function QU^-, and a preference relation obeying the second system $\mathcal{S}_O$ is always representable by an optimistic utility function QU^+. In [4], these two utility functionals are also justified by axiomatic systems in the style of Savage. The difference is that in the VNM approach, a possibility function on states is assumed to be given, whereas in the latter approach such a function is deduced from a preference relation on the set of actions.

The previous systems of axioms were extended in [1] to cope with *generalized possibilistic mixtures* operations $[\lambda_1/\pi_1, ..., \lambda_n/\pi_n]_\star$ induced by a t-norm-like operation $\star$ on V and defined for any $x \in X$ as:

$$[\lambda_1/\pi_1, ..., \lambda_n/\pi_n]_\star(x) = \max_{i=1,...,n} \lambda_i \star \pi_i(x)\ . \tag{2}$$

Then, if one replaces the original mixtures (1) by these ones in the above axiomatic systems, call them $\mathcal{S}_P^\star$ and $\mathcal{S}_O^\star$, it was shown [1] that the preference relations obeying $\mathcal{S}_P^\star$ and $\mathcal{S}_O^\star$ can still be represented respectively by the following pessimistic and optimistic utilities:

$$QU_\star^-(\pi_d \mid u) = \min_{x \in X} n(\pi_d(x) \star \lambda_x), \tag{3}$$

$$QU_\star^+(\pi_d \mid u) = \max_{x \in X} h(\pi_d(x) \star \mu_x), \tag{4}$$

with $n(\lambda_x) = u(x) = h(\mu_x)$, and n is as above.

3 Coping with Giang-Shenoy's Utility Systems

Remaining in the same possibilistic framework, Giang and Shenoy propose [6] the next system $\mathcal{S}_B$ of four axioms for a preference relation $\preceq$ on lotteries with a min-based mixture operation (1).

B1(*Total pre-order*): $\preceq$ is reflexive, transitive and complete.

B2(*Qualitative monotonicity*): for any $\lambda, \mu \in V$ with $\max(\lambda, \mu) = 1$,

$[\lambda/\overline{x}, \mu/\underline{x}] \preceq [\lambda'/\overline{x}, \mu'/\underline{x}]$ if[5] $(\lambda \leq \lambda'$ and $\mu' \leq \mu)$

B3(*Substitutability*): $\pi_1 \sim \pi_2 \Rightarrow [\alpha/\pi_1, \mu/\pi] \sim [\alpha/\pi_2, \mu/\pi]$.

B4(*Continuity*): $\forall x \in X, \exists \lambda, \mu \in V$ with $\max(\lambda, \mu) = 1$ such that $x \sim [\lambda/\overline{x}, \mu/\underline{x}]$.

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[5] The original condition in [6] was ($\lambda \leq \lambda'$ and $\mu = \mu' = 1$) or ($\lambda < 1$ and $\lambda' = 1$) or ($\lambda = \lambda' = 1$ and $\mu \geq \mu'$), but this condition is indeed equivalent to the one used above, see [7, Lemma 4]

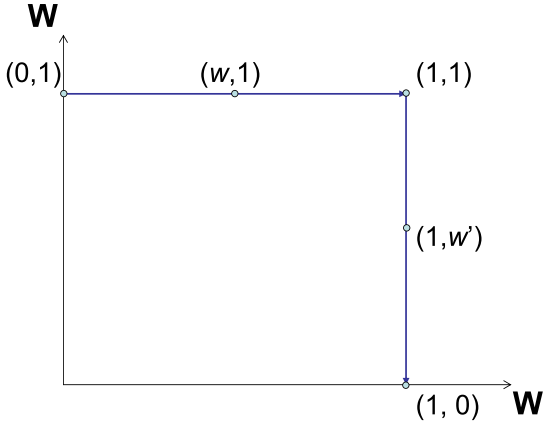

Figure 1: Ordering in U_W.

The authors show that this axiomatic system $\mathcal{S}_B$ is weaker than $\mathcal{S}_P$ and $\mathcal{S}_O$ and that preference relations satisfying axioms B1 through B4 are representable by utility functions PU similar to QU^+ but taking values on a two-dimensional scale, but still linearly ordered. Indeed, given a finite linearly ordered utility scale $(W, \leq)$, one can define a corresponding linearly ordered *binary*[6] utility scale $(U_W, \underline{\ll})$, where $U_W = \{(a,b) \mid a, b \in W, \max(a,b) = 1\}$ and $\underline{\ll}$ is defined as

$$(a,b) \underline{\ll} (a',b') \text{ iff } (a \leq a') \text{ and } (b \geq b'),$$

and whose strict part of $\underline{\ll}$ is defined as:

$$(a,b) \ll (a',b') \text{ iff } (a < a') \text{ or } (b > b').$$

This gives the following linear ordering

$$(0,1) \ll ... \ll (w,1) \ll ... \ll (1,1) \ll ... \ll (1,w') \ll ... \ll (1,0),$$

where $0 < w, w' < 1$ of W, and it is depicted in Figure 1.

Then, given a pair of order-preserving mappings $k_1, k_2 : V \to W$ such that $k_i(0) = 0, k_i(1) = 1$, and a binary utility assessment of consequences $u : X \to U_W$, the utility function $PU : \Pi(X) \to U_W$ is defined as

$$PU(\pi \mid k, u) = \overline{\max}_{x \in X} \overline{\min}(k(\pi(x)), u(x))$$

where $k(v) = (k_1(v), k_2(v))$ for any $v \in V$ and $\overline{\min}$ and $\overline{\max}$ denote the point-wise extension of min and max to $W \times W$. This kind of utility function PU induces a total pre-ordering among possibility distributions

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[6] This is the term used by the authors.

$$\pi \preceq \pi' \text{ if } PU(\pi \mid k,u) \lll PU(\pi' \mid k,u)$$

that satisfies the axioms of system $\mathcal{S}_B$, and conversely.

It can be shown that such pre-ordering can be seen in fact as a lexicographic ordering in terms of suitable evaluations of the pessimistic and optimistic utilities QU^- and QU^+. This is based on the following trivial observation.

Lemma 1 *Let* $a, b, a', b' \in W$ *such that* $\max(a, b) = \max(a', b') = 1$*, and let* $n : W \to W$ *the order-reversing involution on* W*. Then:*

$$(a, b) \lll (a', b') \textit{ iff } (a, n(b)) \leq_{lex} (a', n(b')) \textit{ iff } (n(b), a) \leq_{lex} (n(b'), a')$$

where $\leq_{lex}$ *is the lexicographic ordering on* $W \times W$ *induced by the ordering* $\leq$ *on* W*.*

Proof: It is enough to check the property for the strict part of the orderings since (a, b) is indifferent to (a', b') in any of the orderings if, and only, if $(a, b) = (a', b')$. We first prove below the equivalence $(a, b) \ll (a', b')$ iff $(a, n(b)) <_{lex} (a', n(b'))$.

Recall that, by definition, $(a, b) \ll (a', b')$ iff $a < a'$ or $b > b'$. As for one direction, assume $(a, b) \ll (a', b')$. If $a < a'$ it is obvious that $(a, n(b)) <_{lex} (a', n(b'))$. If $b > b'$ then $n(b) < n(b')$ and $a' = 1$ since $b' < 1$, which obviously implies $(a, n(b)) <_{lex} (a', n(b'))$ as well. As for the other direction, assume $(a, n(b)) <_{lex} (a', n(b'))$. If $a < a'$ we are done. Otherwise, if $a = a'$, then it must be $n(b) < n(b')$, hence $b > b'$ and we are done again.

As for the other equivalence, $(a, n(b)) \leq_{lex} (a', n(b'))$ iff $(n(b), a) \leq_{lex} (n(b'), a')$, due to the constraint $\max(a, b) = \max(a', b') = 1$, it is a matter of routine to check that the equivalence actually holds in each of the four possible situations ($a = a' = 1$, $a = b' = 1$, $a' = b = 1$ and $a' = b' = 1$). □

Now, given a binary utility assignment $u : X \to U_W$, if we consider its projections $u_1, u_2 : X \to U$, i.e. $u(x) = (u_1(x), u_2(x))$, of course with the constraint $\max(u_1(x), u_2(x)) = 1$ for any $x \in X$, then we can express $PU(\pi \mid u, k) = (PU^1(\pi \mid k_1, u_1), PU^2(\pi \mid k_2, u_2))$, where

$$PU^i(\pi \mid k_i, u_i) = \max_{x \in X} \min(k_i(\pi(x)), u_i(x))$$

for $i = 1, 2$. Noticing that $n(PU^2(\pi \mid k_2, u_2)) = QU^-(\pi \mid k_2, u_2^*)$, where $u_2^*(x) = n(u_2(x))$, the next representation it is just a matter of routine checking.

Theorem 1 *The preference ordering induced by* $PU(\cdot \mid k, u)$ *is the lexicographic refinement of the ordering induced by* $QU^+(\cdot \mid k_1, u_1)$ *by the ordering induced by* $QU^-(\cdot \mid k_2, u_2^*)$ *(or viceversa). That is, for any lottery* π*:*

$$PU(\pi \mid k, u) \lll PU(\pi' \mid k, u)$$

$$\textit{iff}$$

$$(QU^+(\pi \mid k_1, u_1), QU^-(\pi \mid k_2, u_2^*)) \leq_{lex} (QU^+(\pi' \mid k_1, u_1), QU^-(\pi' \mid k_2, u_2^*))$$

$$\textit{iff}$$

$$(QU^-(\pi \mid k_2, u_2^*), QU^+(\pi \mid k_1, u_1)) \leq_{lex} (QU^-(\pi' \mid k_2, u_2^*), QU^+(\pi' \mid k_1, u_1)).$$

However, notice that in this representation u_1 and u_2^* *are not independent utility assignments.* Indeed, since $\max(u_1(x), u_2(x)) = 1$ for all x, then $\min(n(u_1(x)), u_2^*(x)) = 0$, i.e. $u_1(x) < 1$ implies $u_2^*(x) = 0$.

4 Pessimistic, Optimistic and Binary Utilities on [0, 1]

While keeping the set X finite, it is not difficult to adapt the pessimistic and optimistic utility axiomatic systems $\mathcal{S}_P^\star$ and $\mathcal{S}_O^\star$ to preference relations defined over $\Pi_{[0,1]}(X)$, the set of possibilistic lotteries with $V = [0,1]$, and mixture operations (2) defined by an arbitrary t-norm operation $\otimes$ on $[0,1]$. In fact, as we will see, it is enough to introduce a uniqueness condition for the paramaters λ and μ in axioms A4$^-$ and $A4^+$. Then the whole framework becomes notationally much simpler. Indeed, let us consider the axiomatic systems $\mathcal{S}_{P!}^\otimes = \{\mathbf{A1}, \mathbf{A2}^-, \mathbf{A3}, \mathbf{A4!}^-\}$ and $\mathcal{S}_{O!}^\otimes = \{\mathbf{A1}, \mathbf{A2}^+, \mathbf{A3}, \mathbf{A4!}^+\}$, where

A4!$^-$: for all $\pi \in \Pi_{[0,1]}(X)$ there exists *a unique* λ such that $\pi \sim [1/\overline{x}, \lambda/\underline{x}]_\otimes$

A4!$^+$: for all $\pi \in \Pi_{[0,1]}(X)$ there exists *a unique* μ such that $\pi \sim [\mu/\overline{x}, 1/\underline{x}]_\otimes$

In the following $\oplus$ will denote the corresponding dual t-conorm of $\otimes$ (i.e. $\oplus(t,s) = 1 - \otimes(1-t, 1-s)$ for any $t, s \in [0,1]$).

Theorem 2 *A binary relation* $\preceq$ *on* $\Pi_{[0,1]}(X)$ *satisfies the axioms* $\mathcal{S}_{P!}^\otimes$ *iff there exists* $u : X \to [0,1]$ *such that, for any* $\pi_1, \pi_2 \in \Pi_{[0,1]}(X)$, $\pi_1 \preceq \pi_2$ *iff* $QU_\otimes^-(\pi_1 \mid u) \leq QU_\otimes^-(\pi_2 \mid u)$*, where* $QU_\otimes^-(\pi \mid u) = \min_{x\in X} \oplus(1 - \pi(x), u(x))$.

Proof: One direction is not difficult, let us verify the other. For each $x \in X$ there exists a unique λ_x such that $x \sim [1/\overline{x}, \lambda_x/\underline{x}]_\otimes$. Then define $u : X \to [0,1]$ by

$$u(x) = 1 - \lambda_x.$$

It is clear that $u(\overline{x}) = 1$. By axiom A4!$^-$, $\underline{x} \sim [1/\overline{x}, \mu/\underline{x}]_\otimes$ for some(unique) μ, then by axiom A3, $\underline{x} = [1/\underline{x}, 1/\underline{x}]_\otimes \sim [1/[1/\overline{x}, \mu/\underline{x}]_\otimes, 1/\underline{x}]_\otimes = [1/\overline{x}, 1/\underline{x}]_\otimes$, hence $u(\underline{x}) = 0$.

Now, for any $\pi \in \Pi_{[0,1]}(X)$, define $QU(\pi) = 1 - \lambda_\pi$, where $\pi \sim [1/\overline{x}, \lambda_\pi/\underline{x}]_\otimes$. Due to axiom A4!$^-$, QU is well defined (indeed, by axiom A4!$^-$, one can check that $[1/\overline{x}, \mu/\underline{x}]_\otimes \sim [1/\overline{x}, \lambda/\underline{x}]_\otimes$ iff $\lambda = \mu$) and represents $\preceq$. We want to prove that $QU = QU_\otimes^-(\cdot \mid u)$. This is done in the following steps:

- It is easy to check that QU and $QU_\otimes^-(\cdot \mid u)$ coincide over the lotteries $[1/\overline{x}, \lambda/\underline{x}]_\otimes$. Moreover, by definition of u, $QU(x) = u(x)$ for all $x \in X$.

- $QU([1/x, \lambda/y]_\otimes) = \min(u(x), \oplus(1 - \lambda, u(y)))$. Indeed, A4!$^-$ guarantees there exist α and β such that $x \sim [1/\overline{x}, \alpha/\underline{x}]_\otimes$ and $y \sim [1/\overline{x}, \beta/\underline{x}]_\otimes$. Using A3, we have

$$[1/x,\lambda/y] \sim [1/[1/\overline{x},\alpha/\underline{x}],\lambda/[1/\overline{x},\beta/\underline{x}]] \\ = [1/\overline{x},\max(\alpha,\otimes(\lambda,\beta))/\underline{x}].$$

Hence $QU([1/x,\lambda/y]) = 1-\max(\alpha,\otimes(\lambda,\beta)) = \min(1-\alpha,\oplus(1-\lambda,1-\beta)) = \min(u(x),\oplus(1-\lambda,u(y))$.

- $QU([1/\pi_1,1/\pi_2]_\otimes) = \min(QU(\pi_1),Q(\pi_2))$. Indeed, there exist α and β such that $[1/\pi_1,1/\pi_2]_\otimes \sim [1/[1/\overline{x},\alpha/\underline{x}]_\otimes,1/[1/\overline{x},\beta/\underline{x}]_\otimes]_\otimes = [1/\overline{x},\max(\alpha,\beta)/\underline{x}]_\otimes$, therefore $QU([1/\pi_1,1/\pi_2]_\otimes) = 1-\max(\alpha,\beta) = \min(1-\alpha,1-\beta) = \min(QU(\pi_1),Q(\pi_2))$.
- $QU(\pi) = \min_{x\in X}\oplus(1-\pi(x),u(x))$. Since π is normalized, let x_j be such that $\pi(x_j) = 1$. Without loss of generality assume $j = 1$. Defining $\pi_i = [1/x_1,\pi(x_i)/x_i]_\otimes$ for $i > 1$, then $\pi = \max_{i>1}\pi_i$, hence $QU(\pi) = \min_i QU(\pi_i) = \min_i \min(u(x_1),\oplus(1-\pi(x_i),u(x_i))) = \min_{x\in X}\oplus(1-\pi(x),u(x))$.

This ends the proof. □

In an analogous form, one can also prove a similar characterization of preference relations induced by optimistic possibilistic utilities.

Theorem 3 *A binary relation* $\preceq$ *on* $\Pi_{[0,1]}(X)$ *satisfies the axioms* $\mathcal{S}^{\otimes}_{O!}$ *iff there exists* $u : X \to [0,1]$ *such that the utility* $QU^{+}_{\otimes}(\cdot \mid u)$*, defined as* $QU^{+}_{\otimes}(\pi \mid u) = \max_{x\in X}\otimes(\pi(x),u(x))$*, represents* $\preceq$*.*

Proof: Similar to the previous case. In this case, for each $x \in X$ there exists μ_x such that $x \sim [1/\underline{x},\mu_x/\overline{x}]_\otimes$. Then define $u : X \to [0,1]$ by $u(x) = \mu_x$ and then define $QU([1/\underline{x},\mu/\overline{x}]_\otimes) = \mu$ and check that $QU(\cdot) = QU^{+}_{\otimes}(\cdot \mid u)$. □

Finally, one can show that a similar adaptation of Giang-Shenoy axiom system $\mathcal{S}_B$ for standard possibilistic mixtures (i.e. with $\otimes = \min$) can be done in order to have a representation theorem in terms of binary utilities on $[0,1]$, and again the issue is to guarantee that two *distinguished* lotteries $[\lambda/\overline{x},\mu/\underline{x}]$ and $[\lambda'/\overline{x},\mu'/\underline{x}]$ are indifferent only in the case $\lambda = \lambda'$ and $\mu = \mu'$. Indeed, let us consider the following enforcement of postulate **B2**:

B2!(*Qualitative monotonicity*): for any $\lambda,\mu \in [0,1]$ with $\max(\lambda,\mu) = 1$,

$$[\lambda/\overline{x},\mu/\underline{x}] \preceq [\lambda'/\overline{x},\mu'/\underline{x}] \text{ iff } (\lambda \leq \lambda' \text{ and } \mu' \leq \mu)$$

Notice that we have only formally changed and "if" by and "iff". Then one can prove the following modified representation theorem.

Theorem 4 *A binary relation* $\preceq$ *on* $\Pi_{[0,1]}(X)$*, equipped with the* min*-possibilistic mixture operation, satisfies the axioms* $\mathcal{S}_{B!} = \{\mathbf{B1},\mathbf{B2!},\mathbf{B3},\mathbf{B4}\}$ *iff there exists* $u : X \to U_{[0,1]}$ *such that the utility*

$$PU(\pi \mid u) = \overline{\max}_{x\in X}\overline{\min}(\pi(x),u(x)),$$

represents $\preceq$ *in the following sense: it holds that* $\pi \preceq \pi'$ *iff* $PU(\pi \mid u)\underline{\ll}PU(\pi' \mid u)$*, for any* $\pi,\pi' \in \Pi_{[0,1]}(X)$*.*

The proof is an easy adaptation of [6, Theorem 2] and will be omitted.

Finally, let us remark that for generalized $\otimes$-possibilistic mixtures on the set of lotteries $\Pi_{[0,1]}(X)$ with $\otimes$ being the algebraic product, Giang and Shenoy [7] have recently proposed another axiomatic system for decision making where uncertainty is modeled by likelihood functions. Their system of postulates, call it $\mathcal{S}_{B!}^{\odot}$, is actually the same than $\mathcal{S}_{B!}$ but applied to product-possibilistic mixtures, and it is characterized again by binary utilities, formally similar to the PU above, hence using $[0,1]$ as uncertainty and utility scales as well, but exchanging min by product.

Theorem 5 ([7]) *A binary relation $\preceq$ on $\Pi_{[0,1]}(X)$, equipped with the product-possibilistic mixture operation, satisfies the axioms $\mathcal{S}_{B!}^{\odot} = \{\mathbf{B1}, \mathbf{B2!}, \mathbf{B3}, \mathbf{B4}\}$ iff there exists $u : X \to U_{[0,1]}$, with $(0,1),(1,0) \in u(X)$, such that the utility*

$$PU^{\odot}(\pi \mid u) = \overline{\max}_{x\in X}\pi(x) \odot u(x),$$

represents $\preceq$, where $\pi(x) \odot (u_1(x), u_2(x)) = (\pi(x)\cdot u_1(x), \pi(x)\cdot u_2(x))$, i.e. it holds $\pi \preceq \pi'$ iff $PU^{\odot}(\pi \mid u) \underline{\ll} PU^{\odot}(\pi' \mid u)$, for any $\pi, \pi' \in \Pi_{[0,1]}(X)$.

5 Lexicographic Refinements: New Postulates

In this section we propose a general axiomatic approach for preference relations on $\Pi_{[0,1]}(X)$ which are representable by a lexicographic combination of one possibilistic utility with another one. Although the combination of the utilities could in principle be arbitrary (pessimistic-optimistic, optimistic-pessimistic, two pessimistics, two optimistics), for the sake of simplicity we will restrict ourselves to the case of lexicographic combination of an arbitrary pessimistic utility $QU_{\otimes}^{-}(\cdot \mid u_1)$ and an arbitrary optimistic utility $QU_{\otimes}^{+}(\cdot \mid u_2)$. The only *consistency condition* we will require to u_1 and u_2 is that they share at least one best outcome and one worst outcome. That is, we will assume there exist $\overline{x}$ and $\underline{x}$ in X such that $u_1(\overline{x}) = u_2(\overline{x}) = 1$ and $u_1(\underline{x}) = u_2(\underline{x}) = 0$.

For a given t-norm $\otimes$ and a given utility assignment $u : X \to [0,1]$, consider the pessimistic and optimistic preference orderings on $\Pi_{[0,1]}(X)$:

$$\pi \preceq_u^{-} \pi' \ \text{ if}_{\text{def}} \ QU_{\otimes}^{-}(\pi \mid u) \leq QU_{\otimes}^{-}(\pi' \mid u),$$

$$\pi \preceq_u^{+} \pi' \ \text{ if}_{\text{def}} \ QU_{\otimes}^{+}(\pi \mid u) \leq QU_{\otimes}^{+}(\pi' \mid u).$$

Using them, we can consider the binary utility functional $F_{\otimes}^{u_1,u_2} : \Pi_{[0,1]}(X) \to U \times U$ defined by

$$F_{\otimes}^{u_1,u_2}(\pi) = (QU_{\otimes}^{-}(\pi \mid u_1), QU_{\otimes}^{+}(\pi \mid u_2)).$$

We can define then on $\Pi_{[0,1]}(X)$ the total pre-ordering $\preceq_{u_1,u_2}^{lex}$ induced by $F_{\otimes}^{u_1,u_2}$ and by the lexicographic ordering $\leq_{lex}$ on $U \times U$. Namely,

$$\pi \preceq_{u_1,u_2}^{lex} \pi' \ \text{ if}_{\text{def}} \ F_{\otimes}^{u_1,u_2}(\pi) \leq_{lex} F_{\otimes}^{u_1,u_2}(\pi').$$

In other words, $\pi \preceq^{lex}_{u_1,u_2} \pi'$ if either $\pi \prec^{-}_{u_1} \pi'$ or ($\pi \sim^{-}_{u_1} \pi'$ and $\pi \preceq^{+}_{u_2} \pi'$).

Notation: in the rest of this paper, and for the sake of a simpler notation, we will denote the $\otimes$-mixture operation on $\Pi_{[0,1]}(X)$ simply by [...] instead of $[...]_{\otimes}$.

The following properties of $\preceq^{lex}_{u_1,u_2}$ are interesting and easy to check.

Proposition 1 *The following properties hold:*

(i) $\pi \preceq^{-}_{u_1} \pi'$ *iff* $[1/\pi, 1/\overline{x}] \preceq^{lex}_{u_1,u_2} [1/\pi', 1/\overline{x}]$.

(i) $\pi \preceq^{+}_{u_2} \pi'$ *iff* $[1/\pi, 1/\underline{x}] \preceq^{lex}_{u_1,u_2} [1/\pi', 1/\underline{x}]$.

(i) For all $x, x' \in X$, $x \preceq^{lex}_{u_1,u_2} x'$ *iff* $(u_1(x), u_2(x)) \leq_{lex} (u_1(x'), u_2(x'))$.

In view of these properties, let us consider the following system $\mathcal{S}^{\otimes}_{PO} = \{\mathbf{A1}, \mathbf{A3}, \mathbf{L2}, \mathbf{L4!}, \mathbf{L5}_{PO}\}$ of postulates for a preference relation $\preceq$ on $\Pi(X)_{[0,1]}$ where

L2: if $\pi \leq \pi'$ then $[1/\pi', 1/\overline{x}] \preceq [1/\pi, 1/\overline{x}]$ and $[1/\pi, 1/\underline{x}] \preceq [1/\pi', 1/\underline{x}]$.

L4!: for all $\pi \in \Pi(X)_{[0,1]}$, there exist *unique* $\lambda, \mu \in [0, 1]$ such that $[1/\pi, 1/\overline{x}] \sim [1/\overline{x}, \lambda/\underline{x}]$ and $[1/\pi, 1/\underline{x}] \sim [\mu/\overline{x}, 1/\underline{x}]$,

L5$_{PO}$: $\pi \preceq \pi'$ iff either $[1/\pi, 1/\overline{x}] \prec [1/\pi', 1/\overline{x}]$ or ($[1/\pi, 1/\overline{x}] \sim [1/\pi', 1/\overline{x}]$ and $[1/\pi, 1/\underline{x}] \preceq [1/\pi', 1/\underline{x}]$).

As usual, in the above postulates $\underline{x}$ and $\overline{x}$ denote respectively a minimal and maximal element of X with respect to $\preceq$.

Lemma 2 *If* $\preceq$ *satisfies* $\mathbf{A1}, \mathbf{L2}, \mathbf{L4!}$ *and* $\mathbf{L5}_{PO}$*, then* $\underline{x} \preceq \pi \preceq \overline{x}$ *for all* $\pi \in \Pi(X)_{[0,1]}$.

Proof: Let us prove that $\underline{x} \preceq \pi$, the other relation $\pi \preceq \overline{x}$ is proved analogously. By L5$_{PO}$, $\underline{x} \preceq \pi$ iff either $[1/\overline{x}, 1/\underline{x}] \prec [1/\pi, 1/\overline{x}]$ or ($[1/\overline{x}, 1/\underline{x}] \sim [1/\pi, 1/\overline{x}]$ and $\underline{x} \preceq [1/\pi, 1/\underline{x}]$). Notice that $\underline{x} \preceq [1/\pi, 1/\underline{x}]$ holds by L2. Therefore to prove $\underline{x} \preceq \pi$ it is enough to prove $[1/\overline{x}, 1/\underline{x}] \preceq [1/\pi, 1/\overline{x}]$. And to prove this, by L4!, there exists λ such that $[1/\pi, 1/\overline{x}] \sim [1/\overline{x}, \lambda/\underline{x}]$. Now, since $[1/\overline{x}, \lambda/\underline{x}] \leq [1/\overline{x}, 1/\underline{x}]$, by L2, we have $[1/\overline{x}, 1/\underline{x}] \preceq [1/\overline{x}, \lambda/\underline{x}] \sim [1/\pi, 1/\overline{x}]$. □

If $\overline{x}$ and $\underline{x}$ continue respectively denoting a maximal and minimal element of X w.r.t. a preference relation $\preceq$, then we can define two new relations $\preceq^{-}$ and $\preceq^{+}$ on $\Pi(X)_{[0,1]}$ as follows:

$$\begin{array}{lll} \pi \preceq^{-} \pi' & \text{if}_{\text{def}} & [1/\pi, 1/\overline{x}] \preceq [1/\pi', 1/\overline{x}], \\ \pi \preceq^{+} \pi' & \text{if}_{\text{def}} & [1/\pi, 1/\underline{x}] \preceq [1/\pi', 1/\underline{x}]. \end{array}$$

If $\preceq$ represents a lexicographic combination, then these new relations allows us to recover in a sense the properties of the original relations, as the following lemma shows.

Lemma 3 *Let $\preceq$ satisfy* $\mathbf{A1}, \mathbf{A3}, \mathbf{L2}$ *and* $\mathbf{L4!}$. *Let $\overline{x}$ and $\underline{x}$ be a maximal and minimal element of X w.r.t. $\preceq$. Then:*

(i) $\underline{x} \preceq^- \pi \preceq^- \overline{x}$, *and* $\underline{x} \preceq^+ \pi \preceq^+ \overline{x}$, *for all* π

(ii) $\preceq^-$ *satisfies the axioms* $\mathcal{S}^{\otimes}_{P!}$

(iii) $\preceq^+$ *satisfies the axioms* $\mathcal{S}^{\otimes}_{O!}$

Proof: We prove only the properties for $\preceq^-$, the ones for $\preceq^+$ can be proved in a similar way.

(i) By definition, $\underline{x} \preceq^- \pi$ iff $[1/\underline{x}, 1/\overline{x}] \preceq [1/\pi, 1/\overline{x}]$. But, by Axiom L4!, there exists λ such that $[1/\pi, 1/\overline{x}] \sim [\lambda/\underline{x}, 1/\overline{x}]$, and it is clear that $[1/\underline{x}, 1/\overline{x}] \preceq [\lambda/\underline{x}, 1/\overline{x}]$ by Axiom L2. On the other hand, by definition, $\pi \preceq^- \overline{x}$ iff $[1/\pi, 1/\overline{x}] \preceq [1/\overline{x}, 1/\overline{x}] = \overline{x}$, and again this is clear by the above Lemma 2.

(ii) We actually only need to prove that $\preceq^-$ satisfies Axiom A3. Namely, assume $[1/\pi_1, 1/\overline{x}] \sim [1/\pi_2, 1/\overline{x}]$. By A3 for $\preceq$, $[\alpha/[1/\pi_1, 1/\overline{x}], \beta/\pi] \sim [\alpha/[1/\pi_2, 1/\overline{x}], \beta/\pi]$, that is, $[\alpha/\pi_1, \alpha/\overline{x}, \beta/\pi] \sim [\alpha/\pi_2, \alpha/\overline{x}, \beta/\pi]$. Again, by A3, $[1/[\alpha/\pi_1, \alpha/\overline{x}, \beta/\pi], 1/\overline{x}] \sim [1/[\alpha/\pi_2, \alpha/\overline{x}, \beta/\pi], 1/\overline{x}]$, that is, $[\alpha/\pi_1, \beta/\pi,] \sim^- [\alpha/\pi_2, \beta/\pi]$. $\square$

Theorem 6 *A preference ordering $\preceq$ on $\Pi(X)_{[0,1]}$ satisfies the system of postulates $\mathcal{S}^{\otimes}_{PO}$ if, and only if, there exist two mapping $u_1, u_2 : X \to [0,1]$ with $u_1^{-1}(1) \cap u_2^{-1}(1) \neq \emptyset \neq u_1^{-1}(0) \cap u_2^{-1}(0)$ such that, for all $\pi, \pi' \in \Pi(X)_{[0,1]}$,*

$$\pi \preceq \pi' \text{ iff } F_{\otimes}^{u_1,u_2}(\pi) \leq_{lex} F_{\otimes}^{u_1,u_2}(\pi').$$

Proof: One direction is easy. As for the other direction, assume $\preceq$ satisfies (A1) through (L5$_{PO}$). By Lemma 3, its associated relations $\preceq^-$ and $\preceq^+$ satisfy the axioms $\mathcal{S}^{\otimes}_{P!}$ and $\mathcal{S}^{\otimes}_{O!}$ respectively. Therefore, by Theorems 2 and 3, we have:

- there exists $u_1 : X \to [0,1]$ such that $\preceq^- = \preceq^-_{u_1}$
- there exists $u_2 : X \to [0,1]$ such that $\preceq^+ = \preceq^+_{u_2}$

By Lemma 3, $u_1(\overline{x}) = u_2(\overline{x}) = 1$ and $u_1(\underline{x}) = u_2(\underline{x}) = 0$. For every $x \in X$, by axiom A4!$^-$, there exists λ_x such that $x \sim^- [1/\overline{x}, \lambda_x/\underline{x}]$, hence $QU^-_{\otimes}(x \mid u_1) = QU^-_{\otimes}([1/\overline{x}, \lambda_x/\underline{x}] \mid u_1)$, hence $u_1(x) = 1 - \lambda_x$. On the other hand, by axiom A4!$^+$, there exists μ_x such that $x \sim^+ [1/\underline{x}, \mu_x/\overline{x}]$, hence $u_2(x) = \mu_x$. Finally, by Axiom L5$_{PO}$, $\preceq$ is the lexicographic ordering defined by $\preceq^-_u$ and $\preceq^+_u$, in other words, we have for all π and π', $\pi \preceq \pi'$ iff $F_{\otimes}^{u_1,u_2}(\pi) \leq_{lex} F_{\otimes}^{u_1,u_2}(\pi')$. $\square$

It is easy to check that if we would replace axiom $\mathbf{L5}_{PO}$ by axiom $\mathbf{L5}_{OP}$, where

$\mathbf{L5}_{OP}$**:** $\pi \preceq \pi'$ iff either $[1/\pi, 1/\underline{x}] \prec [1/\pi', 1/\underline{x}]$ or $([1/\pi, 1/\underline{x}] \sim [1/\pi', 1/\underline{x}]$ and $[1/\pi, 1/\overline{x}] \preceq [1/\pi', 1/\overline{x}])$.

then the axiom system $\mathcal{S}^{\otimes}_{OP} = \{\mathbf{A1}, \mathbf{A3}, \mathbf{L2}, \mathbf{L4!}, \mathbf{L5}_{OP}\}$ would capture the preference relations over lotteries defined as the lexicographic refinement of an ordering induced by an optimistic criterion by an ordering induced by a pessimistic criterion.

In these representations, the utility assignments u_1 and u_2 are unrelated, except by the condition $u_1^{-1}(1) \cap u_2^{-1}(1) \neq \emptyset \neq u_1^{-1}(0) \cap u_2^{-1}(0)$, which says, as already mentioned, that they share a maximal and a minimal element of X. But it is not difficult to find suitable postulates to be added to the system $\mathcal{S}^{\otimes}_{PO}$ (or to $\mathcal{S}^{\otimes}_{OP}$) in order to guarantee some interesting further conditions on the utility assignments u_1 and u_2 in the representation theorems. For instance if one is interested in getting a representation where u_1 and u_2 are the same, the postulated to be added to $\mathcal{S}^{\otimes}_{PO}$ is:

L6: there exists λ such that $[1/x, 1/\overline{x}] \sim [1/\overline{x}, (1-\lambda)/\underline{x}]$ and $[1/x, 1/\underline{x}] \sim [\lambda/\overline{x}, 1/\underline{x}]$

Indeed, one can prove the following representation theorem.

Theorem 7 *A preference ordering $\preceq$ on $\Pi(X)_{[0,1]}$ satisfies the system of postulates $\mathcal{S}^{\otimes}_{PO}$ plus* **L6** *if, and only if, there exists a* single *mapping $u : X \to [0,1]$ with $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$ such that, for all $\pi, \pi' \in \Pi(X)_{[0,1]}$,*

$$\pi \preceq \pi' \text{ iff } F^{u,u}_{\otimes}(\pi) \leq_{lex} F^{u,u}_{\otimes}(\pi').$$

Proof: Inspecting the proof of Theorem 6, it turns out that, by using Axiom **L6**, $x \sim^{-} [1/\overline{x}, (1-\lambda_x)/\underline{x}]$ and $x \sim^{+} [1/\underline{x}, \lambda_x/\overline{x}]$, hence $u_1(x) = 1 - (1 - \lambda_x) = \lambda_x = u_2(x)$. Since this is for all $x \in X$, $u_1 = u_2$. □

Similarly, if one is interested in getting the utilities u_1 and u_2 related in the same way as in the Giang-Shenoy model, i.e. fulfilling $\min(u_1(x), 1 - u_2(x)) = 0$ for every $x \in X$ (see Theorem 1), the following additional postulate

L7: if $[1/x, 1/\overline{x}] \sim [1/\overline{x}, \lambda/\underline{x}]$ with $\lambda < 1$, then $[1/x, 1/\underline{x}] \sim [1/\overline{x}, 1/\underline{x}]$.

guarantees the required constraint, as the following theorem shows.

Theorem 8 *A preference ordering $\preceq$ on $\Pi(X)_{[0,1]}$ satisfies the system of postulates $\mathcal{S}^{\otimes}_{PO}$ plus* **L7** *if, and only if, there exist two mapping $u_1, u_2 : X \to [0,1]$ with $u_1^{-1}(1) \cap u_2^{-1}(1) \neq \emptyset \neq u_1^{-1}(0) \cap u_2^{-1}(0)$ and with $u_2(x) = 1$ if $u_1(x) > 0$, such that, for all $\pi, \pi' \in \Pi(X)_{[0,1]}$,*

$$\pi \preceq \pi' \text{ iff } F^{u_1,u_2}_{\otimes}(\pi) \leq_{lex} F^{u_1,u_2}_{\otimes}(\pi').$$

Proof: Inspecting the proof of Theorem 6 again, it turns out that, by using Axiom **L7**, if $x \sim^{-} [1/\overline{x}, \lambda_x/\underline{x}]$ with $\lambda_x < 1$ then $x \sim^{+} [1/\overline{x}, 1/\underline{x}]$, that is, if $u_1(x) = 1 - \lambda_x > 0$, then $u_2(x) = 1$. □

As an interesting corollary of this last theorem, we have that the systems $\mathcal{S}^{\otimes}_{PO}$ plus **L7** of lexicographic refinements actually capture both Giang-Shenoy systems $\mathcal{S}_{B!}$ and $\mathcal{S}^{\odot}_{B!}$ for particular choices of the t-norm $\otimes$.

Corollary 1 *For lotteries in* $\Pi_{[0,1]}(X)$ *equipped with the* $\otimes$*-possibilistic mixture operation, we have:*

1. *when* $\otimes = \min$*, the system* $\mathcal{S}_{PO}^{\otimes}$ *plus* **L7** *is equivalent to the system* $\mathcal{S}_{B!}$*;*
2. *when* $\otimes =$ *product, the system* $\mathcal{S}_{PO}^{\otimes}$ *plus* **L7** *is equivalent to the system* $\mathcal{S}_{B!}^{\odot}$*.*

6 Concluding Remarks

In this paper we have proposed a possibilistic decision framework à la von Neumann and Morgenstern style in which uncertainty and the decision maker's preferences can be measured in the real unit interval [0,1] instead of finite qualitative scales as in previous works, and the model is suitably adapted for capturing both pessimistic or optimistic behaviours.

Preference relations defined as a lexicographic combination of the pessimistic and optimistic criteria are characterized in an alternative way as done in [2], more directly in terms of the proper (refined) relation instead of in more explicit terms of the pessimistic and optimistic preferences involved. In particular, it has been shown that preference relations obeying the axiomatic system $\mathcal{S}_B$ proposed by Giang and Shenoy [6], and representable by binary possibilistic utilities, can indeed be also represented by preference relations defined as a lexicographic refinement of one pessimistic criterion by an optimistic criteria. Interestingly enough, the same relations are obtained independently if the optimistic criterion is refined by the pessimistic one, which is not usually the case in the general lexicographic refinements. This reinforces the idea (already claimed in [6]) that the system $\mathcal{S}_B$ indeed captures the common fragment of the pessimistic and optimistic systems $\mathcal{S}_P$ and $\mathcal{S}_O$.

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