Mathware & Soft Computing 13 (2006) 139-156

Searching Degrees of Self-Contradiction in Atanassov's Fuzzy Sets*

E. Castiñeira, S. Cubillo and C. Torres
Dpto. Matemática Aplicada
Universidad Politécnica de Madrid
28660 Boadilla del Monte, Madrid (Spain)
{*ecastineira,scubillo,ctorres*}*@fi.upm.es*

Abstract

In [11] and [12] Trillas *et al.* introduced the study of contradiction in the framework of Fuzzy Logic because of the significance to avoid contradictory *outputs* in the processes of inference. Later, the study of contradiction in the framework of intuitionistic or Atanassov's fuzzy sets was initiated in [6] and [5]. The aim of this work is to go into the problem of measuring the self-contradiction in the case of intuitionistic fuzzy sets, since it is interesting to know not only if a set is contradictory, but also the extend to which this property holds. The study of self-contradiction is tackled from two different aspects: the self-contradiction with respect to a specific intuitionistic fuzzy negation, and the self-contradiction without depending on any negation. To the purpose of measuring the contradiction degrees, in both cases, some functions based on a previous geometrical study are defined. Also, some properties of these functions are shown as well as some particular relations among them.

Keywords: Intuitionistic (or Atanassov's) fuzzy sets, intuitionistic fuzzy generators and fuzzy negations, degrees of contradiction.

1 Introduction

1.1 Due to the significance to avoid contradictory *outputs* in the processes of inference, Trillas *et al.* (see [11] and [12]) addressed the study of contradiction in framework of Fuzzy Logic introducing the concept of contradictory set. These papers formalise the idea that a set is self-contradictory (or contradictory to be short) if it violates the principle of not contradiction in the following sense: the statement "If x is P, then x is not P" holds with some degree of truth. So, they established

* This paper is supported by CICYT (Spain) under Project TIN 2005-08943-C02-01, and by UPM-CAM (Spain) under Project R05/11240.

that the fuzzy set associated with the predicate P, and determined by μ_P, is contradictory if "$\mu_P(x) \to \mu_{\neg P}(x)$ for all x" representing the implication "$\to$" by means of the reticular order $\leq$ of $[0,1]$, that is, μ_P is self-contradictory regarding a strong negation N, or N-self-contradictory, if $\mu_P \leq N \circ \mu_P$. The condition $\mu \leq N \circ \mu$ is equivalent to $\operatorname{Sup}_x(\mu(x)) \leq \alpha_N$, where α_N is the fixed point of N. Nevertheless, the extent to which this condition holds, that is, how contradictory μ is, is a matter for consideration, since μ can behave quite differently regarding this characteristic. For example, if the fuzzy set determined by μ verifies that $\operatorname{Sup}_x(\mu(x)) = \alpha_N$, then a minimal variation in this supreme could produce a non-N-contradictory fuzzy set. But, if $\operatorname{Sup}_x(\mu(x))$ is much smaller than α_N, then small changes in this supreme do not modify the contradictoriness of the disturbed μ. The need to speak not only of contradiction but also of degrees of contradiction was later raised in [3] and [4], where some functions were considered for the purpose of determining (or measuring) these degrees.

1.2 The intuitionistic fuzzy sets, as it is well known, were introduced by K. T. Atanassov in 1983 and they include fuzzy sets as a particular case. Since then many papers have been published in the theoretical framework of these sets as well as in that of their applications. The formal definition of intuitionistic fuzzy set is the following:

Definition 1.1. ([1]) *An intuitionistic fuzzy set (IFS) A, in the universe $X \neq \emptyset$, is a set given as $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ where $\mu_A : X \to [0,1]$, $\nu_A : X \to [0,1]$ are called membership and non-membership functions, respectively, and such that $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.*

This set could be considered as a L-fuzzy set as defined by Goguen ([9]) being, in this case, $L = \{(\alpha_1, \alpha_2) \in [0,1]^2 : \alpha_1 + \alpha_2 \leq 1\}$, with the partial order $\leq_L$ defined as follows: given $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2) \in L$,

$$\boldsymbol{\alpha} \leq_L \boldsymbol{\beta} \iff \alpha_1 \leq \beta_1 \ \& \ \alpha_2 \geq \beta_2 .$$

$(L, \leq_L)$ is a complete lattice with smallest element, $\mathbf{0_L} = (0,1)$, and greatest element, $\mathbf{1_L} = (1,0)$.

So, an IFS A is a L-fuzzy set whose L-membership function $\chi^A \in L^X = \{\chi : X \to L\}$ is defined for each $x \in X$ as $\chi^A(x) = (\mu_A(x), \nu_A(x))$. Let us denote the set of all intuitionistic fuzzy sets on X as $\mathcal{IF}(X)$.

Furthermore, K. Atanassov also proposes an early negation operator in the IFS and, later in [2], a more general definition is established at the same time that the intuitionisc fuzzy generators are introduced to build intuitionistic fuzzy negations. Recall that a decreasing function $\mathcal{N} : L \to L$ is an intuitionistic fuzzy negation (IFN) if $\mathcal{N}(\mathbf{0_L}) = \mathbf{1_L}$ and $\mathcal{N}(\mathbf{1_L}) = \mathbf{0_L}$ hold. Moreover, $\mathcal{N}$ is a strong IFN if the equality $\mathcal{N}(\mathcal{N}(\boldsymbol{\alpha})) = \boldsymbol{\alpha}$ holds for all $\boldsymbol{\alpha} \in L$. Moreover Deschrijver *et al.* in [7] and [8] focus on the problem of generating and characterizing a strong intuitionistic fuzzy negation and they prove that any strong IFN $\mathcal{N}$ is characterized by a strong negation $N : [0,1] \to [0,1]$ throughout the formula

$\mathcal{N}(\alpha_1, \alpha_2) = (N(1-\alpha_2), 1 - N(\alpha_1))$, for all $(\alpha_1, \alpha_2) \in L$. Regarding strong fuzzy negations, they were characterized by Trillas in [10]. He showed that N is a strong negation if and only if there exists an order automorphism in the unit interval, $g : [0,1] \to [0,1]$, such that $N(\alpha) = g^{-1}(1 - g(\alpha))$, for all $\alpha \in [0,1]$. Thus a strong IFN $\mathcal{N}$ is also determined by an order automorphism g in $[0,1]$.

1.3 The study of contradiction in the framework of IFS was initiated in [6] where the concept of contradictory IFS was introduced, and some necessary conditions to be a contradictory set, as well as some sufficient conditions, were obtained. Since it is interesting to know not only if a set is contradictory, but also the extend to which this property holds, the problem of measuring the contradiction in the case of IFS was initiated in [5] introducing some functions to this purpose.

In this work, we go into this topic and the paper is organized as follows: Section 2 deals with self-contradiction regarding a specific negation; first we tackle contradiction from a geometrical point of view to find some relation suggesting the way to measure how contradictory an IFS is. Secondly, we propose some measures to determine the searched degrees of contradiction. After defining these functions some of their properties are proved as well as some relations among them. An analogous study with the contradiction without depending on any negation is shown in section 3. Finally, we raise some conclusions.

2 Measuring $\mathcal{N}$-self-contradiction in $\mathcal{IF}(X)$

Similarly to the fuzzy case, an IFS A is said to be a self-contradictory set with respect to some strong IFN, $\mathcal{N}$, if $\chi^A(x) \leq_L (\mathcal{N} \circ \chi^A)(x)$ for all $x \in X$, where χ^A is the L-membership function of A.

Before defining the functions that measure the contradiction degrees, we analyse the regions of L in which the contradictory sets for a given negation are located. This study motivates us to define the mentioned functions.

2.1 Regions of $\mathcal{N}$-contradiction

In [3] it is proved that, given $A \in \mathcal{IF}(X)$, with $\chi^A = (\mu_A, \nu_A) \in L^X$, and $\mathcal{N}$ a strong IFN, associated with the strong negation N, then

(i) A is $\mathcal{N}$-contradictory $\Leftrightarrow N(\mu_A(x)) + \nu_A(x) \geq 1$ for all $x \in X$.

(ii) A is $\mathcal{N}$-contradictory $\Leftrightarrow g(\mu_A(x)) + g(1-\nu_A(x)) \leq 1$ for all $x \in X$, provided g is the generator of N.

Preceding inequalities, that are equivalent, determine a region free of contradiction in L and another one where the contradictory sets must remain. Let us see those regions for some particular negations and afterwards in the general case.

2.1.1 $\mathcal{N}_s$-contradiction with standard negation $\mathcal{N}_s(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1)$

If we consider the standard negation, $\mathcal{N}_s$, that is given by $N = 1 - id$, where the generator g is the identity; then the above statements become: A is $\mathcal{N}$-contradictory if and only if $1 - \mu_A(x) + \nu_A(x) \geq 1$, or $\nu_A(x) \geq \mu_A(x)\ \ \forall x \in X$.

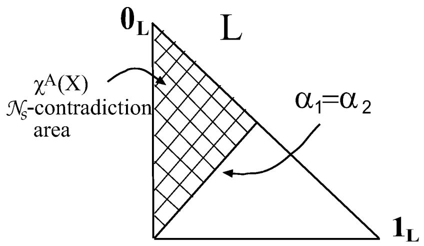

Figure 1: $\mathcal{N}_s$-contradiction area.

So, A is $\mathcal{N}_s$-contradictory if and only if $\chi^A(X) = \{\chi^A(x) : x \in X\} \subset \{(\alpha_1, \alpha_2) \in L : \alpha_1 \leq \alpha_2\}$; therefore, the image of X under χ^A, that we also call *range of* A, should be inside of the region showed in figure 1, and the line $\alpha_1 = \alpha_2$ is the boundary between the contradictory and non-contradictory regions.

2.1.2 $\mathcal{N}_g$-contradiction being $\mathcal{N}_g$ the strong IFN associated with a Sugeno's negation

The order automorphism $g(\alpha) = \frac{\ln(1+\alpha)}{\ln 2}$ determines the strong Sugeno's negation $N_g(\alpha) = \frac{1-\alpha}{1+\alpha}$. In this case, the set $A \in \mathcal{IF}(X)$ is $\mathcal{N}_g$-contradictory if and only if

$$\frac{1 - \mu_A(x)}{1 + \mu_A(x)} + \nu_A(x) \geq 1 \qquad \forall x \in X$$

Thus, A is $\mathcal{N}_g$-contradictory if and only if (fig. 2(a))

$$\begin{aligned} \chi^A(X) &\subset \{(\alpha_1, \alpha_2) \in L : \alpha_2 + \alpha_1\alpha_2 - 2\alpha_1 \geq 0\} \\ &= \left\{(\alpha_1, \alpha_2) \in L : \alpha_2 \geq \frac{2\alpha_1}{1 + \alpha_1}\right\} \end{aligned}$$

2.1.3 $\mathcal{N}_r$-contradiction with $\mathcal{N}_r$ determined by $g_r(\alpha) = \alpha^r$, $r > 0$

Let us consider the family of strong negations $\{N_r\}_{r>0}$, where for each $r > 0$ the automorphism determining N_r is $g_r(\alpha) = \alpha^r$. This family includes as a particular case the standard negation for $r = 1$, and for each $r > 0$ it is $N_r(\alpha) = (1 - \alpha^r)^{1/r}$ with fixed point $\alpha_{N_r} = \frac{1}{2^{1/r}}$.

$A \in \mathcal{IF}(X)$ is $\mathcal{N}_r$-contradictory, where $\mathcal{N}_r$ is the IFN associated with N_r (or with g_r), if and only if

$$\chi^A(X) \subset \{(\alpha_1, \alpha_2) \in L : \alpha_1^r + (1 - \alpha_2)^r \leq 1\}$$

For each $r > 0$ the curve $\alpha_1^r + (1 - \alpha_2)^r = 1$ is the boundary delimiting the contradiction region, and if an IFS takes some L-value under that curve, then it is not $\mathcal{N}_r$-contradictory.

In particular, A is $\mathcal{N}_2$-contradictory if and only if the image of X under χ^A is inside or on the circumference with centre $\mathbf{0_L}$ and radius 1 (fig. 2(b)). Let us point out that the boundary curve of the contradiction region, with equation $\alpha_1^2 + (1-\alpha_2)^2 = 1$, intersects the line with equation $\alpha_1 + \alpha_2 = 1$ at the point $(\alpha_{N_g}, 1-\alpha_{N_g}) = \big(1/\sqrt{2}, 1-1/\sqrt{2}\big)$, where α_{N_g} is the fixed point of N_g.

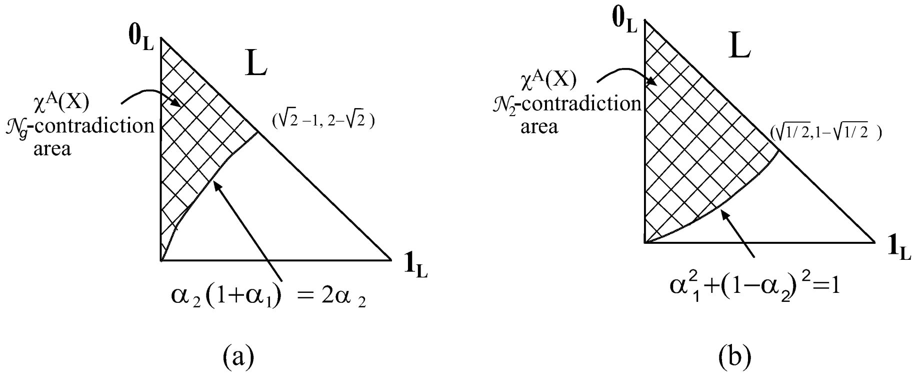

Figure 2: (a) $\mathcal{N}_g$-contradiction area with a Sugeno's negation, and (b) $\mathcal{N}_2$-contradiction area with $\mathcal{N}_2$ determined by $g_2(\alpha) = \alpha^2$.

Let us note that the more r increases, the more the curve $\alpha_1^r + (1-\alpha_2)^r = 1$ comes closer to axis α_1 (except for $\alpha_1 = 1$); to be precise, when $r \to \infty$, the family of functions $\{1-(1-\alpha_1^r)^{1/r}\}_{r>0}$ is pointwise convergent to the null function for all $\alpha_1 \in [0,1)$ and to 1 at $\alpha_1 = 1$; therefore, the non-$\mathcal{N}_r$-contradiction region decreases. Furthermore, when $r \to 0$, the family of functions $\{(1-(1-\alpha_2)^r)^{1/r}\}_{r>0}$ converges for all $\alpha_2 \in [0,1)$ to the null function, and for $\alpha_2 = 1$ converges to 1; that is, the more r decreases, the more the curve delimiting the contradiction region comes closer to axis α_2 (except for $\alpha_2 = 1$), and then the non-$\mathcal{N}_r$-contradiction region spreads when r decreases.

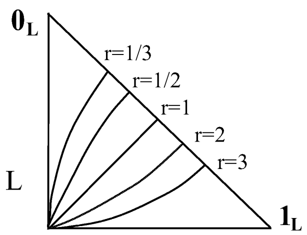

Figure 3: Curves $\alpha_1^r + (1-\alpha_2)^r = 1$.

On the other hand, if $0 < r < s$ the curve $\alpha_1^s + (1-\alpha_2)^s = 1$ is under the curve $\alpha_1^r + (1-\alpha_2)^r = 1$ (in figure 3 some of them are showed), and, if

$A \in \mathcal{IF}(X)$ is $\mathcal{N}_r$-contradictory, A is $\mathcal{N}_s$-contradictory for all $s > r$. Indeed, if $r < s$ it is $\alpha_1^r > \alpha_1^s$ for all $\alpha_1 \in (0,1)$, and, as $g_{\frac{1}{r}}$ is increasing and $1/s < 1/r$, it is $(1-\alpha_1^r)^{1/r} < (1-\alpha_1^s)^{1/r} < (1-\alpha_1^s)^{1/s}$, from which it follows that the coordinate α_2 of the curve related to s is smaller than the one related to r.

Finally, let us observe that the aforementioned family of curves almost "cover" the lattice L with the exception of the axis without the origin, that is:

$$\cup_{r>0} \{(\alpha_1, \alpha_2) \in L : \alpha_1^r + (1-\alpha_2)^r = 1\} = \\ L \setminus \{(\alpha_1, \alpha_2) \neq (0,0) : \alpha_1 = 0 \text{ or } \alpha_2 = 0\}.$$

2.1.4 General case of $\mathcal{N}$-contradiction

If $\mathcal{N}$ is a strong IFN associated with the strong negation N, a set $A \in \mathcal{IF}(X)$ is $\mathcal{N}$-contradictory if and only if

$$\chi^A(X) \subset \{(\alpha_1, \alpha_2) \in L : N(\alpha_1) + \alpha_2 \geq 1\},$$

and the boundary curve delimiting the contradiction region, $N(\alpha_1) + \alpha_2 = 1$, and that we name $\mathcal{N}$-*boundary curve* and we note $\mathcal{L}_{\mathcal{N}}$, verifies the following properties:
(1) $\alpha_2 = 1 - N(\alpha_1)$ is an increasing function of α_1.
(2) Its range contains the point (0,0).
(3) The intersection of $N(\alpha_1) + \alpha_2 = 1$ and $\alpha_1 + \alpha_2 = 1$ is the point $(\alpha_N, 1 - \alpha_N)$, being α_N the fixed point of N.

2.2 Degrees of $\mathcal{N}$-contradiction

As we noted in the introduction, it is important to measure how much contradictory a set is, and not only in the fuzzy case, but also in the intuitionistic one. In fact, the IFS with L-membership function $\chi^{\mathbf{0_L}}(x) = \mathbf{0_L}$ for all $x \in X$ is $\mathcal{N}$-contradictory for any IFN $\mathcal{N}$, and $A \in \mathcal{IF}(X)$ taking all its L-values on the $\mathcal{N}$-boundary curve, $N(\alpha_1) + \alpha_2 = 1$ (where N is the strong negation associated with $\mathcal{N}$), is also $\mathcal{N}$-contradictory. Nevertheless, small disturbances in the L-values of A could return a new set, very similar to A, but not $\mathcal{N}$-contradictory, whereas small disturbances on $\mathbf{0_L}$ will never change its nature of contradictoriness. So, it seems quite suitable to assign the value 0 as the degree of $\mathcal{N}$-contradiction of A, and also of any set taking L-values on the $\mathcal{N}$-boundary curve or underneath it. Analogously, it seems appropriate to assign positive degree to a set whose range is above the boundary curve and it will be as much higher as the range is farther away from the curve. Taking in account these comments, we will define different functions that could be used to determine the contradiction degrees.

Definition 2.1. *Let $A \in \mathcal{IF}(X)$ determined by $\chi^A = (\mu_A, \nu_A) \in L^X$; then*
(i) $\mathcal{C}_1^{\mathcal{N}}(A) = \text{Max}\left(0, \operatorname*{Inf}_{x \in X}(N(\mu_A(x)) + \nu_A(x) - 1)\right)$ *is the $\mathcal{N}$-contradiction degree of A according to the strong negation N associated with $\mathcal{N}$, or $\mathcal{C}_1^{\mathcal{N}}$-contradiction degree of A.*

(ii) $\mathcal{C}_2^{\mathcal{N}}(A) = \text{Max}\left(0, 1 - \sup_{x\in X}(g(\mu_A(x)) + g(1-\nu_A(x)))\right)$ *is the* $\mathcal{N}$*-contradiction degree of* A *according to the automorphism* g *determining* $\mathcal{N}$*, or* $\mathcal{C}_2^{\mathcal{N}}$*-contradiction degree of* A*.*
(iii) The contradiction degree according to the distance to the non-$\mathcal{N}$*-contradiction region is*

$$\mathcal{C}_3^{\mathcal{N}}(A) = \frac{d(\chi^A(X), L_{\mathcal{N}})}{d(\mathbf{0_L}, L_{\mathcal{N}})},$$

where d *is the euclidean distance, and* $L_{\mathcal{N}} = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in L : N(\alpha_1) + \alpha_2 \leq 1\}$ *is the* $\mathcal{N}$*-contradiction free region, that is, the region below the curve* $\mathcal{L}_{\mathcal{N}}$*; thus,*

$$d(\chi^A(X), L_{\mathcal{N}}) = \text{Inf}\left\{d(\chi^A(x), \boldsymbol{\alpha}) : x \in X, \boldsymbol{\alpha} \in L_{\mathcal{N}}\right\}$$

and $d(\mathbf{0_L}, L_{\mathcal{N}}) = \text{Inf}\{d(\mathbf{0_L}, \boldsymbol{\alpha}) : \boldsymbol{\alpha} \in L_{\mathcal{N}}\}$*. We will say that* $\mathcal{C}_3^{\mathcal{N}}(A)$ *is the* $\mathcal{C}_3^{\mathcal{N}}$*-contradiction degree of* A*.*

The above three functions take their values in $[0, 1]$ and, in general, they are different measures, as the following example shows.

Example 2.2. Let $A \in \mathcal{IF}([0, 1])$ with L-membership function $\chi^A(x) = (x/4, 1 - x/2)$ (fig.4), and let us consider the strong IFN $\mathcal{N}$ determined by the fuzzy negation $N(x) = \sqrt{1 - x^2}$, with $g(x) = x^2$. Then:

$$\mathcal{C}_1^{\mathcal{N}}(A) = \text{Max}\left(0, \inf_{x\in[0,1]}\left(\sqrt{1 - \left(\frac{x}{4}\right)^2} - \frac{x}{2}\right)\right) = \frac{\sqrt{15} - 2}{4},$$

$$\mathcal{C}_2^{\mathcal{N}}(A) = \text{Max}\left(0, 1 - \sup_{x\in[0,1]}\left(\left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^2\right)\right) = \frac{11}{16}.$$

And, finally

$$\mathcal{C}_3^{\mathcal{N}}(A) = \frac{d(\chi^A(X), L_{\mathcal{N}})}{d(\mathbf{0_L}, L_{\mathcal{N}})} = 1 - \frac{\sqrt{5}}{4}.$$

◁

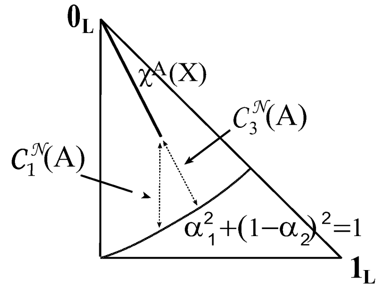

Figure 4: Geometrical interpretation of different contradiction degrees.

Remark 2.3. The function $\mathcal{C}_1^{\mathcal{N}}$ is motivated by the characterization of the contradiction (i) of 2.1, whereas $\mathcal{C}_2^{\mathcal{N}}$ is originated by (ii). Although both characterizations are equivalent, $\mathcal{C}_1^{\mathcal{N}}$ and $\mathcal{C}_2^{\mathcal{N}}$ do not match up, as it is showed in the above example 2.2. Besides, $\mathcal{C}_3^{\mathcal{N}}$ represents a relative distance: the euclidean distance between the range of an IFS and the non-$\mathcal{N}$-contradiction region of L, $L_{\mathcal{N}}$, relative to the distance between the "most contradictory" set and the same region; whereas $\mathcal{C}_1^{\mathcal{N}}$ represents the infimum of the distances between the ordinates of the L-values of the IFS and those of $\mathcal{L}_{\mathcal{N}}$ with the same abscises, that is, the infimum on the $x \in X$ of euclidean distances $d((\mu_A(x), \nu_A(x)), (\mu_A(x), \alpha_2))$ being $(\mu_A(x), \alpha_2) \in \mathcal{L}_{\mathcal{N}}$ (see the figure 4). Regarding $\mathcal{C}_2^{\mathcal{N}}$, it is possible to find some geometrical interpretations in some particular cases.

Theorem 2.4. *For $i = 1, 2, 3$, the function $\mathcal{C}_i^{\mathcal{N}} : \mathcal{IF}(X) \to [0, 1]$ given for each $A \in \mathcal{IF}(X)$ and for each IFN $\mathcal{N}$ as in the definition 2.1 satisfies:*
(i) If $\mathbf{0_L}$ denotes the IFS such that $\chi^{\mathbf{0_L}}(x) = \mathbf{0_L}$ for all $x \in X$, then $\mathcal{C}_i^{\mathcal{N}}(\mathbf{0_L}) = 1$.
(ii) If $A \in \mathcal{IF}(X)$ satisfies the condition $\operatorname*{Inf}_{x\in X} \nu_A(x) = 0$, then $\mathcal{C}_i^{\mathcal{N}}(A) = 0$.
(iii) $\mathcal{C}_i^{\mathcal{N}}$ is anti-monotonic with respect to the orders $\leq_L$ in L and the usual one of $\mathbb{R}$: If $A, B \in \mathcal{IF}(X)$ with $\chi^A \leq_L \chi^B$ (that is, $\chi^A(x) \leq_L \chi^B(x)$ for all $x \in X$), then $\mathcal{C}_i^{\mathcal{N}}(B) \leq \mathcal{C}_i^{\mathcal{N}}(A)$.

Proof: (i) $\mathcal{C}_i^{\mathcal{N}}(\mathbf{0_L}) = 1$ for $i = 1, 2, 3$ trivially.

(ii) Let $A \in \mathcal{IF}(X)$ be a set such that $\operatorname*{Inf}_{x\in X} \nu_A(x) = 0$. Since $N(\mu_A(x)) - 1 \leq 0$ for all $x \in X$, where N is the fuzzy negation associated with the IFN $\mathcal{N}$, then $\operatorname*{Inf}_{x\in X}(N(\mu_A(x)) + \nu_A(x) - 1) \leq \operatorname*{Inf}_{x\in X} \nu_A(x) = 0$, and therefore $\mathcal{C}_1^{\mathcal{N}}(A) = \operatorname{Max}\left(0, \operatorname*{Inf}_{x\in X}(N(\mu_A(x)) + \nu_A(x) - 1)\right) = 0$.

If g is the automorphism associated with $\mathcal{N}$, then

$$\operatorname*{Sup}_{x\in X} g(1 - \nu_A(x)) = g\left(\operatorname*{Sup}_{x\in X}(1 - \nu_A(x))\right) = g\left(1 - \operatorname*{Inf}_{x\in X} \nu_A(x)\right) = g(1) = 1,$$

and so $\operatorname*{Sup}_{x\in X}(g(\mu_A(x)) + g(1 - \nu_A(x)) \geq \operatorname*{Sup}_{x\in X} g(1 - \nu_A(x)) = 1$, consequently, it holds that

$$\mathcal{C}_2^{\mathcal{N}}(A) = \operatorname{Max}\left(0, 1 - \operatorname*{Sup}_{x\in X}(g(\mu_A(x)) + g(1 - \nu_A(x)))\right) = 0.$$

Since $\operatorname*{Inf}_{x\in X} \nu_A(x) = 0$ there exists some $(\alpha_0, 0)$ belonging to the closure of $\chi^A(X)$, $\overline{\chi^A(X)}$. If $\alpha_0 > 0$ there exists some $(\alpha_1, \alpha_2) \in \chi^A(X)$ such that $N(\alpha_1) + \alpha_2 < 1$ and then $d(\chi^A(X), L_{\mathcal{N}}) = 0$. If $\alpha_0 = 0$ it is $(0, 0) \in \overline{\chi^A(X)} \cap \mathcal{L}_{\mathcal{N}}$, then $d(\chi^A(X), L_{\mathcal{N}}) = 0$ and so $\mathcal{C}_3^{\mathcal{N}}(A) = 0$.

(iii) Let us see that the three functions are anti-monotonic. If $\chi^A \leq_L \chi^B$ then

$$\left.\begin{array}{lcl} \mu_A(x) \leq \mu_B(x) & \text{and} & N(\mu_A(x)) \geq N(\mu_B(x)) \\ \nu_A(x) \geq \nu_B(x) & \text{and} & 1 - \nu_A(x) \leq 1 - \nu_B(x) \end{array}\right\} \forall x \in X,$$

therefore, $N(\mu_A(x)) + \nu_A(x) \geq N(\mu_B(x)) + \nu_B(x)$ for all $x \in X$ and it follows that $\mathcal{C}_1^{\mathcal{N}}(B) \leq \mathcal{C}_1^{\mathcal{N}}(A)$. Moreover,

$$\operatorname*{Sup}_{x\in X} (g(\mu_B(x)) + g(1-\nu_B(x))) \geq \operatorname*{Sup}_{x\in X} (g(\mu_A(x)) + g(1-\nu_A(x)))$$

and so $\mathcal{C}_2^{\mathcal{N}}(B) \leq \mathcal{C}_2^{\mathcal{N}}(A)$.

Finally, let us see the anti-monotony of $\mathcal{C}_3^{\mathcal{N}}$. For this, we will prove that all $x \in X$ satisfies $d(\chi^A(x), L_{\mathcal{N}}) \geq d(\chi^B(x), L_{\mathcal{N}})$ provided $\chi^A \leq_L \chi^B$. Let $\chi^A(x) = (\alpha_1^A, \alpha_2^A) \in L$ and $\chi^B(x) = (\alpha_1^B, \alpha_2^B) \in L$. If $d(\chi^A(x), L_{\mathcal{N}}) = 0$ then $\chi^A(x) = (\alpha_1^A, \alpha_2^A) \in \overline{L_{\mathcal{N}}} = L_{\mathcal{N}}$, thus $N(\alpha_1^A) + \alpha_2^A \leq 1$. Since $\chi^A \leq_L \chi^B$ then

$$N(\alpha_1^B) + \alpha_2^B \leq N(\alpha_1^A) + \alpha_2^A \leq 1$$

and therefore $d(\chi^B(x), L_{\mathcal{N}}) = 0$ and the required inequality is satisfied. Now, let us suppose that $d(\chi^A(x), L_{\mathcal{N}}) > 0$ then this distance is reached on the boundary of $L_{\mathcal{N}}$, $\mathcal{L}_{\mathcal{N}}$, since $L_{\mathcal{N}}$ is a compact set. Let $\boldsymbol{\alpha}^* = (\alpha_1^*, \alpha_2^*) \in \mathcal{L}_{\mathcal{N}}$ be the point such that

$$d(\chi^A(x), L_{\mathcal{N}}) = d(\chi^A(x), \boldsymbol{\alpha}^*)$$

and we consider the following four regions contained in the set $[\chi^A(x), \mathbf{1_L}] = \{\boldsymbol{\alpha} \in L : \chi^A(x) \leq_L \boldsymbol{\alpha}\}$ (see figure 5): $L_{\mathcal{N}} \cap [\chi^A(x), \mathbf{1_L}]$, $[\chi^A(x), \boldsymbol{\alpha}^*] = \{\boldsymbol{\alpha} \in L : \chi^A(x) \leq_L \boldsymbol{\alpha} \leq_L \boldsymbol{\alpha}^*\}$,

$$R_l = \left\{ \boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in [\chi^A(x), \mathbf{1_L}] : \begin{array}{l} N(\alpha_1) + \alpha_2 \geq 1 \\ \alpha_2^* \geq \alpha_2 \end{array} \right\}$$

and

$$R_r = \left\{ \boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in [\chi^A(x), \mathbf{1_L}] : \begin{array}{l} N(\alpha_1) + \alpha_2 \geq 1 \\ \alpha_1^* \leq \alpha_1 \end{array} \right\}$$

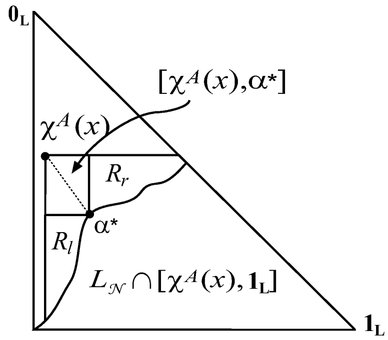

Figure 5: The four regions in $[\chi^A(x), \mathbf{1_L}]$.

Note that the union of these four sets is the whole L-interval $[\chi^A(x), \mathbf{1_L}]$ and we assume $\chi^B(x) \in [\chi^A(x), \mathbf{1_L}]$.

If $\chi^B(x) \in L_{\mathcal{N}} \cap [\chi^A(x), \mathbf{1_L}]$ then $d(\chi^B(x), L_{\mathcal{N}}) = 0 \leq d(\chi^A(x), L_{\mathcal{N}})$.

Since the diameter $D([\chi^A(x), \boldsymbol{\alpha}^*]) = \text{Sup}\{d(\boldsymbol{\alpha}, \boldsymbol{\beta}) : \boldsymbol{\alpha}, \boldsymbol{\beta} \in [\chi^A(x), \boldsymbol{\alpha}^*]\}$ is exactly the distance between $\chi^A(x)$ and $\boldsymbol{\alpha}^*$, if $\chi^B(x) \in [\chi^A(x), \boldsymbol{\alpha}^*]$ then the required inequality is satisfied because of

$$d(\chi^B(x), L_{\mathcal{N}}) \leq d(\chi^B(x), \boldsymbol{\alpha}^*) \leq D([\chi^A(x), \boldsymbol{\alpha}^*]) = d(\chi^A(x), \boldsymbol{\alpha}^*).$$

Now let us suppose that $\chi^B(x) = (\alpha_1^B, \alpha_2^B) \in R_l$ and we consider the point of $\mathcal{L}_{\mathcal{N}}$ with the second coordinate α_2^B, that is $(N^{-1}(1-\alpha_2^B), \alpha_2^B)$ (see figure 6(a)), so

$$d(\chi^B(x), L_{\mathcal{N}}) \leq d((\alpha_1^B, \alpha_2^B), (N^{-1}(1-\alpha_2^B), \alpha_2^B)) = N^{-1}(1-\alpha_2^B) - \alpha_1^B \qquad (1)$$

Moreover, $\alpha_2^B \leq \alpha_2^*$ implies

$$N^{-1}(1-\alpha_2^B) \leq N^{-1}(1-\alpha_2^*) = \alpha_1^* \qquad (2)$$

From (1) and (2)

$$d(\chi^B(x), L_{\mathcal{N}}) \leq \alpha_1^* - \alpha_1^B \leq \alpha_1^* - \alpha_1^A \leq d(\chi^A(x), \boldsymbol{\alpha}^*)$$

is obtained.

The last possibility is $\chi^B(x) \in R_r$, and let us see that the required inequality is also satisfied. We consider the point of $\mathcal{L}_{\mathcal{N}}$ with first coordinate α_1^B, that is $(\alpha_1^B, 1 - N(\alpha_1^B))$ (see figure 6(b)), then

$$d(\chi^B(x), L_{\mathcal{N}}) \leq d((\alpha_1^B, \alpha_2^B), (\alpha_1^B, 1 - N(\alpha_1^B)) = \alpha_2^B - (1 - N(\alpha_1^B)) \qquad (3)$$

Moreover, $\alpha_1^* \leq \alpha_1^B$ implies

$$\alpha_2^* = 1 - N(\alpha_1^*) \leq 1 - N(\alpha_1^B) \qquad (4)$$

From (3) and (4) the following is obtained:

$$d(\chi^B(x), L_{\mathcal{N}}) \leq \alpha_2^B - \alpha_2^* \leq d(\chi^A(x), \boldsymbol{\alpha}^*)$$

□

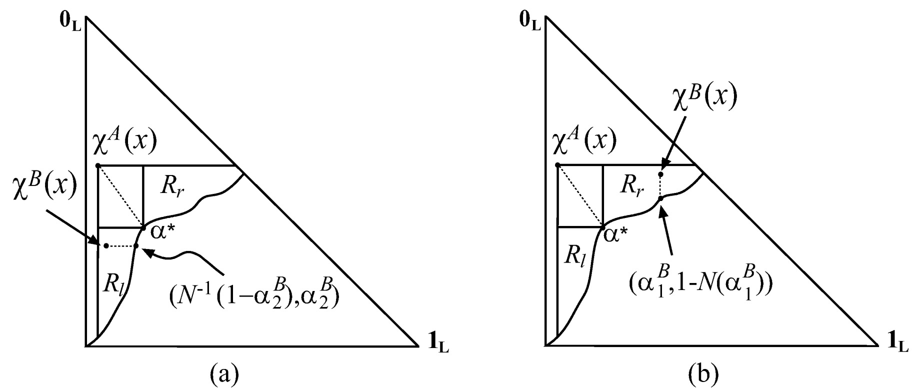

Figure 6: Illustration of theorem 2.4(iii) proof.

Remark 2.5. Let us see some considerations:
(a) The inequality $\operatorname*{Sup}_{x\in X}\mu_A(x) \le 1 - \operatorname*{Inf}_{x\in X}\nu_A(x)$ is satisfied since $\mu_A(x) \le 1-\nu_A(x)$ for all $x \in X$, so if $\operatorname*{Sup}_{x\in X}\mu_A(x) = 1$ then $\mathcal{C}_i^{\mathcal{N}}(A) = 0$ for all $i = 1,2,3$ and for all strong IFN $\mathcal{N}$.
(b) There exist many sets that are contradictory regarding any negation and there exist many sets that are non-contradictory with regard to any negation. Indeed, if $A \in \mathcal{IF}(X)$ satisfies $\operatorname*{Sup}_{x\in X}\chi^A(x) = (0,\alpha_2^*)$ then A is $\mathcal{N}$-contradictory for all $\mathcal{N}$ strong IFN. Furthemore if $\alpha_2^* > 0$ then $\mathcal{C}_1^{\mathcal{N}}(A) = \operatorname*{Inf}_{x\in X}\nu_A(x) = \alpha_2^* > 0$ for all $\mathcal{N}$, $\mathcal{C}_2^{\mathcal{N}}(A) > 0$ and $\mathcal{C}_3^{\mathcal{N}}(A) > 0$ being values $\mathcal{C}_2^{\mathcal{N}}(A)$ and $\mathcal{C}_3^{\mathcal{N}}(A)$ dependent on $\mathcal{N}$; but if $\alpha_2^* = 0$ then $\mathcal{C}_i^{\mathcal{N}}(A) = 0$ for all $\mathcal{N}$ and for $i = 1,2,3$. In short, if all L-values of A are on the axe $\alpha_1 = 0$, then A is $\mathcal{N}$-contradictory for all $\mathcal{N}$. Meanwhile if $\operatorname*{Sup}_{x\in X}\chi^A(x) = (\alpha_1^*,0)$, where $\alpha_1^* > 0$ and $(0,0) \notin \overline{\chi^A(X)}$ then A is non-$\mathcal{N}$-contradictory for all strong IFN $\mathcal{N}$ (see proposition 3.4). However, if A does not satisfy any of these possibilities, then A is a contradictory set regarding countless negations and, at the same time, it is a non-contradictory set with respect to other countless negations as it is shown in the following result.

Proposition 2.6. *Let $A \in \mathcal{IF}(A)$ with L-membership function $\chi^A = (\mu_A,\nu_A) \in L^X$. If $\operatorname*{Sup}_{x\in X}\chi^A(x) = \boldsymbol{\alpha}^* = (\alpha_1^*,\alpha_2^*)$ with $\alpha_1^* > 0$ and $\alpha_2^* > 0$, then the following is satisfied:*
(i) A is $\mathcal{N}$-contradictory for all $\mathcal{N}$ such that its associated fuzzy negation N satisfies $1 < N(\alpha_1^*) + \alpha_2^*$. Moreover, $\mathcal{C}_i^{\mathcal{N}}(A) > 0$ for $i = 1,2,3$, with $\mathcal{C}_i^{\mathcal{N}}$ according to the formula in definition 2.1.
(ii) A is non-$\mathcal{N}$-contradictory for all $\mathcal{N}$ such that the fixed point α_N of the fuzzy negation associated with $\mathcal{N}$ verifies $\alpha_N < \alpha_1^*$.

Proof: (i) For all $x \in X$

$$N(\mu_A(x)) + \nu_A(x) \ge N(\alpha_1^*) + \alpha_2^* > 1$$

holds, and so A is $\mathcal{N}$-contradictory. Moreover

$$\mathcal{C}_1^{\mathcal{N}}(A) = \operatorname*{Inf}_{x\in X}(N(\mu_A(x)) + \nu_A(x) - 1) \ge N(\alpha_1^*) + \alpha_2^* - 1 > 0.$$

Furthermore, if g is the automorphism associated with N, condition $1 < N(\alpha_1^*)+\alpha_2^*$ is equivalent to $1 > g(\alpha_1^*) + g(1-\alpha_2^*)$ and thus

$$g(\mu_A(x)) + g(1-\nu_A(x)) \le g(\alpha_1^*) + g(1-\alpha_2^*) < 1 \quad \forall x \in X$$

therefore $\mathcal{C}_2^{\mathcal{N}}(A) > 0$.

Now, let us suppose that $\mathcal{C}_3^{\mathcal{N}}(A) = 0$, then $d(\chi^A(X), L_{\mathcal{N}}) = 0$ and since $\chi^A(X) \subset [\mathbf{0_L},\boldsymbol{\alpha}^*]$, it is $d([\mathbf{0_L},\boldsymbol{\alpha}^*], L_{\mathcal{N}}) = 0$. As $[\mathbf{0_L},\boldsymbol{\alpha}^*]$ and $L_{\mathcal{N}}$ are compact sets of $\mathbb{R}^2$ with the usual topology, then there exists $\boldsymbol{\beta} = (\beta_1,\beta_2) \in [\mathbf{0_L},\boldsymbol{\alpha}^*] \cap L_{\mathcal{N}}$, and consequently

$$1 < N(\alpha_1^*) + \alpha_2^* \le N(\beta_1) + \beta_2 \le 1$$

which is absurd, therefore $\mathcal{C}_3^{\mathcal{N}}(A) > 0$.

(ii) If the fixed point of N, α_N, satisfies $\alpha_N < \alpha_1^* = \operatorname{Sup}_{x\in X} \mu_A(x)$ then there exists $x_0 \in X$ such that $\alpha_N < \mu_A(x_0) \leq \alpha_1^*$, thus

$$N(\mu_A(x_0)) + \nu_A(x_0) < N(\alpha_N) + \nu_A(x_0) \leq \alpha_N + 1 - \mu_A(x_0) < 1$$

holds, and therefore A is a non-$\mathcal{N}$-contradictory set. □

In the following, we reach some relations between the measures $\mathcal{C}_1^{\mathcal{N}}$, $\mathcal{C}_2^{\mathcal{N}}$ and $\mathcal{C}_3^{\mathcal{N}}$ defined in 2.1 in some particular cases.

Proposition 2.7. *Let $\mathcal{N}_s$ be the standard IFN, then for all $A \in \mathcal{IF}(X)$ it holds:*

$$\mathcal{C}_1^{\mathcal{N}_s}(A) = \mathcal{C}_2^{\mathcal{N}_s}(A) = \mathcal{C}_3^{\mathcal{N}_s}(A).$$

Proof: Since the fuzzy negation associated with $\mathcal{N}_s$ is the standard negation $N = 1 - id$, then

$$\mathcal{C}_1^{\mathcal{N}_s}(A) = \operatorname{Max}\left(0, \operatorname{Inf}_{x\in X}(\nu_A(x) - \mu_A(x))\right)$$

Moreover, as the automorphism associated with $\mathcal{N}_s$ is $g = id$ then:

$$\begin{aligned}\mathcal{C}_2^{\mathcal{N}_s}(A) &= \operatorname{Max}\left(0, 1 - \operatorname{Sup}_{x\in X}(\mu_A(x) + 1 - \nu_A(x))\right)\\ &= \operatorname{Max}\left(0, \operatorname{Inf}_{x\in X}(\nu_A(x) - \mu_A(x))\right)\end{aligned}$$

Thus $\mathcal{C}_1^{\mathcal{N}_s}(A) = \mathcal{C}_2^{\mathcal{N}_s}(A)$.

If $\chi^A(X) \cap \{(\alpha_1, \alpha_2) \in L : \alpha_2 \leq \alpha_1\} \neq \emptyset$ then $\mathcal{C}_1^{\mathcal{N}_s}(A) = \mathcal{C}_2^{\mathcal{N}_s}(A) = \mathcal{C}_3^{\mathcal{N}_s}(A) = 0$ holds. Let us suppose that $\nu_A(x) > \mu_A(x)$ for all $x \in X$, then for each $x \in X$ (see figure 7)

$$d(\chi^A(x), L_{\mathcal{N}_s}) = d(\chi^A(x), \mathcal{L}_{\mathcal{N}_s}) = (\nu_A(x) - \mu_A(x)) \sin\frac{\pi}{4} = (\nu_A(x) - \mu_A(x))\frac{\sqrt{2}}{2}$$

holds; therefore

$$\begin{aligned}\mathcal{C}_3^{\mathcal{N}_s}(A) &= \frac{d(\chi^A(X), L_{\mathcal{N}_s})}{d(\mathbf{0}_{\boldsymbol{L}}, L_{\mathcal{N}_s})} = \frac{\operatorname{Inf}_{x\in X}(\nu_A(x) - \mu_A(x))\sqrt{2}/2}{d(\mathbf{0}_{\boldsymbol{L}}, \mathcal{L}_{\mathcal{N}_s})}\\ &= \operatorname{Inf}_{x\in X}(\nu_A(x) - \mu_A(x)) = \mathcal{C}_1^{\mathcal{N}_s}(A)\end{aligned}$$

since $d(\mathbf{0}_{\boldsymbol{L}}, L_{\mathcal{N}_s}) = \sqrt{2}/2$. □

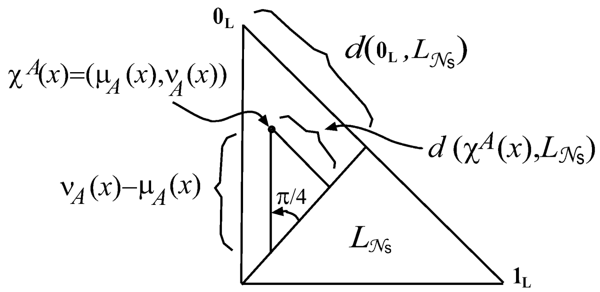

Figure 7: Proof of proposition 2.7.

Proposition 2.8. *If $\mathcal{N}_g$ is the IFN generated by the automorphism $g(\alpha) = \alpha^2$, then for all $A \in \mathcal{IF}(X)$ the following is satisfied:*

$$\mathcal{C}_2^{\mathcal{N}_g}(A) = 1 - (1 - \mathcal{C}_3^{\mathcal{N}_g}(A))^2.$$

Proof: If A is $\mathcal{N}_g$-contradictory then $\mathcal{C}_2^{\mathcal{N}_g}(A) = \mathcal{C}_3^{\mathcal{N}_g}(A) = 0$ and the required inequality holds. If A is non-$\mathcal{N}_g$-contradictory, since $\mathcal{L}_{\mathcal{N}_g}$ is the circumference with centre $\mathbf{0_L}$ and radius 1, $d(\chi^A(x), \mathcal{L}_{\mathcal{N}_g}) = 1 - d(\chi^A(x), \mathbf{0_L})$ for all $x \in X$, and then

$$\begin{aligned}
\mathcal{C}_2^{\mathcal{N}_g}(A) &= 1 - \operatorname*{Sup}_{x\in X}(\mu_A(x)^2 + (1-\nu_A(x))^2) = 1 - \operatorname*{Sup}_{x\in X} d(\chi^A(x), \mathbf{0_L})^2 \\
&= 1 - \operatorname*{Sup}_{x\in X}(1 - d(\chi^A(x), \mathcal{L}_{\mathcal{N}_g}))^2 = 1 - \left(1 - \left(\operatorname*{Inf}_{x\in X} d(\chi^A(x), \mathcal{L}_{\mathcal{N}_g})\right)^2\right) \\
&= 1 - (1 - \mathcal{C}_3^{\mathcal{N}_g}(A)^2).
\end{aligned}$$

□

Remark 2.9. From previous preposition the next comment is followed: if $\chi^A(X)$ is a compact set, the supremum in definition $\mathcal{C}_2^{\mathcal{N}_g}(A)$ and the infimum in definition $\mathcal{C}_3^{\mathcal{N}_g}(A)$ are obtained from the same point $\chi^A(x_0)$, provided $g(\alpha) = \alpha^2$, despite $\mathcal{C}_2^{\mathcal{N}_g}(A) \neq \mathcal{C}_3^{\mathcal{N}_g}(A)$. Nevertheless this fact does not happen when we deal with functions $\mathcal{C}_1^{\mathcal{N}_g}$ and $\mathcal{C}_3^{\mathcal{N}_g}$, that is, we can find an IFS A such that $\mathcal{C}_1^{\mathcal{N}_g}(A)$ and $\mathcal{C}_3^{\mathcal{N}_g}(A)$ are obtained from different points of L. Indeed, let $X \neq \emptyset$ be any universe of discourse whose cardinal is greater than 1, and we consider $A \in \mathcal{IF}(X)$ such that $\chi^A(X) = \{(1/2, 1/2), (0, \alpha)\}$ where $1 - 1/\sqrt{2} < \alpha < (\sqrt{3} - 1)/2$. There exist $x_1, x_2 \in X$, such that $x_1 \neq x_2$, $\chi^A(x_1) = (1/2, 1/2)$ and $\chi^A(x_2) = (0, \alpha)$. Since the point in $\mathcal{L}_{\mathcal{N}_g}$ with $\alpha_1 = 1/2$ is $(1/2, 1 - \sqrt{3}/2)$ the following is satisfied (see figure 8):

$$\begin{aligned}
\mathcal{C}_1^{\mathcal{N}_g}(A) &= \operatorname{Min}\left\{d\left((1/2, 1/2), (1/2, 1 - \sqrt{3}/2)\right), d((0, \alpha), (0, 0))\right\} \\
&= \operatorname{Min}\{\sqrt{3}/2 - 1/2, \alpha\} = \alpha = d(\chi^A(x_2), (0, 0))
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_3^{\mathcal{N}_g}(A) &= \text{Min}\left\{d\left((1/2,1/2),\mathcal{L}_{\mathcal{N}_g}\right), d((0,\alpha),\mathcal{L}_{\mathcal{N}_g})\right\} \\
&= \text{Min}\{d\left((1/2,1/2),\mathcal{L}_{\mathcal{N}_g}\right), d((0,\alpha),(0,0))\} \\
&= \text{Min}\{1-1/\sqrt{2},\alpha\} = 1-1/\sqrt{2} = d(\chi^A(x_1),\mathcal{L}_{\mathcal{N}_g})
\end{aligned}$$

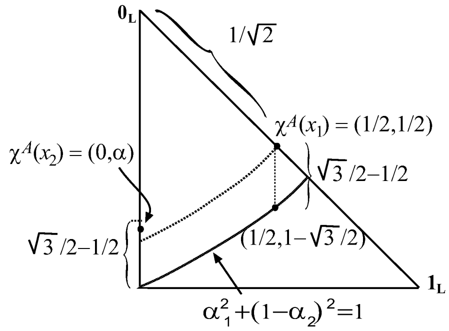

Figure 8: Illustration of remark 2.9.

3 Measuring Self-contradiction in $\mathcal{IF}(X)$

The previous section establishes the contradiction of an IFS related to a chosen negation. We now address contradiction more generally, without depending on a specific IFN. In [6] an IFS $A \in \mathcal{IF}(X)$ was defined self-contradictory (or contradictory to be short) if A is $\mathcal{N}$-self-contradictory regarding some strong IFN $\mathcal{N}$. The aim of this section is to define some functions to measure how self-contradictory an IFS is in the same line as the functions to measure $\mathcal{N}$-self-contradiction. To that purpose, it is very useful to show some results.

Proposition 3.1. **([6])** *Given $A \in \mathcal{IF}(X)$ with L-membership function $\chi^A = (\mu_A, \nu_A) \in L^X$, the following holds:*
(i) If A is self-contradictory, then $\text{Sup}_{x\in X}\, \mu_A(x) < 1$.
(ii) If $\text{Inf}_{x\in X}\, \nu_A(x) > 0$, then A is self-contradictory.

Corollary 3.2. *If $A \in \mathcal{IF}(X)$, with membership function $\mu_A \in [0,1]^X$, is contradictory, then $\text{Sup}_{x\in X}(\mu_A(x) - \nu_A(x)) < 1$.*

Proposition 3.3. *Let $A \in \mathcal{IF}(X)$ with L-membership function $\chi^A = (\mu_A, \nu_A) \in L^X$. If A is self-contradictory then for all $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to\infty} \nu_A(x_n) = 0$, $\lim_{n\to\infty} \mu_A(x_n) = 0$ is satisfied.*

Proof: Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to\infty} \nu_A(x_n) = 0$, then $\lim_{n\to\infty}(1-\nu_A(x_n)) = 1$ and, if g is an order automorphism, $\lim_{n\to\infty} g(1-\nu_A(x_n)) = 1$ provided g is a continuous function. Besides, since A is $\mathcal{N}$-contradictory for all $\mathcal{N}$, in particular for $\mathcal{N} = \mathcal{N}_g$, then

$$0 \leq g(\mu_A(x_n)) \leq 1 - g(1 - \nu_A(x_n)), \quad \forall n \in \mathbb{N}$$

Therefore, $\lim_{n\to\infty} g(\mu_A(x_n)) = 0$, and so $\lim_{n\to\infty} \mu_A(x_n) = 0$. □

The reciprocal of condition (ii) in proposition 3.1 is not true since if $X = [0,1]$ and $A \in L^{[0,1]}$ such that $\chi^A(x) = (x/2, x/2)$ for all $x \in [0,1]$, then A is a self-contradictory set and, however, $\operatorname{Inf}_{x\in[0,1]} \nu_A(x) = 0$. Nevertheless, we have the following result:

Proposition 3.4. *Let $A \in \mathcal{IF}(X)$ with L-membership function $\chi^A = (\mu_A, \nu_A) \in L^X$ such that $\operatorname{Inf}_{x\in X} \nu_A(x) = 0$ and $(0,0) \notin \overline{\chi^A(X)}$, then A is non-self-contradictory.*

Proof: If there exists $(\alpha, 0) \in \chi^A(X)$ with $\alpha > 0$, as it is previously shown, A is non-$\mathcal{N}$-contradictory for all strong IFN $\mathcal{N}$. If for all $\alpha \in [0,1]$, $(\alpha, 0) \notin \chi^A(X)$, then there exists $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to\infty} \nu_A(x_n) = 0$. Thus, the sequence $\{(\mu_A(x_n), \nu_A(x_n))\}_{n\in\mathbb{N}} \subset L^X$ has a convergent subsequence $\{(\mu_A(x_{n_k}), \nu_A(x_{n_k}))\}_{k\in\mathbb{N}}$ since L is a compact set. Moreover, $\lim_{k\to\infty} \mu_A(x_{n_k}) = \alpha > 0$ provided $(0,0) \notin \overline{\chi^A(X)}$, and this implies A is non-self-contradictory because if A were self-contradictory then, according to the previous proposition, $\lim_{k\to\infty} \mu_A(x_{n_k}) = 0$ which is absurd. □

Corollary 3.5. *Let $A \in \mathcal{IF}(X)$ with L-membership function $\chi^A = (\mu_A, \nu_A) \in L^X$ and such that $(0,0) \notin \overline{\chi^A(X)}$, then the following is verified:*

$$\operatorname{Inf}_{x\in X} \nu_A(x) = 0 \Longleftrightarrow A \text{ is non-self-contradictory}$$

Definition 3.6. *Let $A \in \mathcal{IF}(X)$ with L-membership function $\chi^A = (\mu^A, \nu^A) \in L^X$; then the $\mathcal{C}_i$-contradiction degree of A, for i=1,2, is defined as follows:*

(i) $\mathcal{C}_1(A) = \operatorname{Inf}_{x\in X} \nu_A(x)$.

(ii) $$\mathcal{C}_2(A) = \begin{cases} 0 & \text{if } \operatorname{Inf}_{x\in X} \nu_A(x) = 0 \\ \operatorname{Inf}_{x\in X} \dfrac{1 - \mu_A(x) + \nu_A(x)}{2} & \text{in other case} \end{cases}$$

Essentially, the definition of $\mathcal{C}_1$ is based on point (ii) of the proposition 3.1, whereas $\mathcal{C}_2$ is motivated by the corollary 3.2 taking into account the proposition 3.4.

Example 3.7. Let $A, B \in \mathcal{IF}([0,1])$ determined by $\chi^A(x) = (1/4, 1/4)$ and $\chi^B(x) = (3/4, 1/4)$ for all $x \in [0,1]$, respectively. Then $\mathcal{C}_1(A) = \mathcal{C}_1(B) = 1/4$; however, $\mathcal{C}_2(A) = 1/2$ and $\mathcal{C}_2(B) = 1/4$. How could we explain these results? As both sets are contradictory, it is obvious that the two measures should be positive for them. The first one measures how much each set needs to stop being self-contradictory, as that is just what is missing to "touch" the axis α_1. But, how to interpret that the degree for $\mathcal{C}_2$ is greater for A? A possible answer is that the set A is $\mathcal{N}$-contradictory for the same negations $\mathcal{N}$ that B and, furthermore, for a lot more of them, that is, there are more negations $\mathcal{N}$ that make A contradictory than that make B contradictory. In this sense, the measure $\mathcal{C}_2$ provides more information than $\mathcal{C}_1$ about contradictoriness. $\triangleleft$

Remark 3.8. On the one hand, it is evident that the function $\mathcal{C}_1$ measures the euclidean distance from the range of a contradictory set $A \in \mathcal{IF}(X)$ to the axis α_1 (that we denote $\mathcal{L}_1$):

$$\mathcal{C}_1(A) = d(\chi^A(X), \mathcal{L}_1) = \frac{d(\chi^A(X), \mathcal{L}_1)}{d(\mathbf{0_L}, \mathcal{L}_1)}.$$

On the other hand,

$$\mathcal{C}_2(A) = \begin{cases} 0, & \text{if } \operatorname*{Inf}_{x\in X} \nu_A(x) = 0 \\ \dfrac{d_1(\chi^A(X), \mathbf{1_L})}{d_1(\mathbf{0_L}, \mathbf{1_L})}, & \text{in other case} \end{cases}$$

that is, the function $\mathcal{C}_2$ measures the reticular or Hamming distance between the range of A and $\mathbf{1_L}$, relative to the reticular distance from $\mathbf{0_L}$ to $\mathbf{1_L}$. These geometrical interpretations of the measures $\mathcal{C}_1$ y $\mathcal{C}_2$ suggest another way to measure the contradiction degree.

Definition 3.9. *The function $\mathcal{C}_3 : \mathcal{IF}(X) \to [0,1]$ is defined for each $A \in \mathcal{IF}(X)$, with L-membership function $\chi^A = (\mu_A, \nu_A)$, as follows:*

$$\mathcal{C}_3(A) = \begin{cases} 0, & \textit{if } \operatorname*{Inf}_{x\in X} \nu_A(x_n) = 0 \\ \dfrac{d(\chi^A(X), \mathbf{1_L})}{d(\mathbf{0_L}, \mathbf{1_L})} & \textit{in other case.} \end{cases}$$

Proposition 3.10. *For all $A \in \mathcal{IF}(X)$ the following is satisfied:*

$$\mathcal{C}_1(A) \le \mathcal{C}_2(A) \le \mathcal{C}_3(A) \quad \forall A \in \mathcal{IF}(X)$$

Proof: As $\nu_A(x) \le 1 - \mu_A(x)$ for all $x \in X$, the inequality $\nu_A(x) \le (\nu_A(x) + 1 - \mu_A(x))/2$ follows trivially for all $x \in X$, and therefore $\mathcal{C}_1(A) \le \mathcal{C}_2(A)$.

From $\nu_A(x)^2 - 2\nu_A(x)(1-\mu_A(x)) + (1-\mu_A(x))^2 \ge 0$ for all $x \in X$, the inequality $\nu_A(x)^2 + (1-\mu_A(x))^2 + 2\nu_A(x)(1-\mu_A(x)) \le 2(\nu_A(x)^2 + (1-\mu_A(x))^2)$ follows for all $x \in X$, and then $\nu_A(x) + 1 - \mu_A(x) \le \sqrt{2}\sqrt{\nu_A(x)^2 + (1-\mu_A(x))^2}$ for all $x \in X$. Therefore

$$\mathcal{C}_2(A) = \operatorname*{Inf}_{x\in X} \frac{\nu_A(x) + 1 - \mu_A(x)}{2} \le \operatorname*{Inf}_{x\in X} \frac{\sqrt{\nu_A(x)^2 + (1-\mu_A(x))^2}}{\sqrt{2}} = \mathcal{C}_3(A)$$

$\square$

Finally, similar properties to those ones of measures that depend on some strong IFN are now obtained trivially.

Theorem 3.11. *For each $i = 1, 2, 3$, the function $\mathcal{C}_i$ satisfies:*
(i) $\mathcal{C}_i(\mathbf{0_L}) = 1$.
(ii) If $A \in \mathcal{IF}(X)$ *verifies* $\operatorname*{Inf}_{x\in X} \nu_A(x) = 0$, *then* $\mathcal{C}_i(A) = 0$.
(iii) $\mathcal{C}_i$ *is anti-monotonic respect to the orders* $\leq_L$ *into* L *and the usual of* $\mathbb{R}$*: If* $A, B \in \mathcal{IF}(X)$ *such that* $\chi^A \leq_L \chi^B$, *then* $\mathcal{C}_i(B) \leq \mathcal{C}_i(A)$.

4 Conclusions

The study of self-contradiction in the framework of IFS has been tackled from two aspects. The first one was the contradiction depending on a particular strong negation. In order to find out some models to measure how self-contradictory an IFS is, we have proposed some functions based on a previous geometrical study. The fundamental properties of these functions have been proved. Furthermore, some results about the relations among these functions have been attained. Finally, as the second aspect, the contradiction without depending on any negation was studied throughout similar steps.

References

[1] Atanassov, K.T. (1999) "Intuitionistic fuzzy sets", *(Physica-Verlag, Heidelberg, New York).*

[2] Bustince, H., Kacprzyk, J. and Mohedano, V. (2000) "Intuitionistic fuzzy generators - Application to intuitionistic fuzzy complementation", *Fuzzy Sets and Systems* **114**, 485-504.

[3] Castiñeira, E., Cubillo, S. and Bellido, S. (2002) "Degrees of Contradiction in Fuzzy Sets Theory", *Proceedings IPMU'02*, Annecy (France), 171-176.

[4] Castiñeira, E., Cubillo, S. and Bellido, S. (2002) "Contradicción entre dos conjuntos", *Actas ESTYLF'02*, León (Spain), 379-383 (in Spanish).

[5] Castiñeira, E., Cubillo, S., Torres, C. and Zarzosa, V. (2005) "Selft-Contradiction Degrees in Intuitionistic Fuzzy Sets", *Proceedings of Joint EUSFLAT-LFA 2005*, Barcelona (Spain), 455-460.

[6] Cubillo, S., Castiñeira, E. (2004) "Contradiciton in Intuitionistic Fuzzy Sets", *Proceedings IPMU'04*, Perugia (Italy), 2180-2186.

[7] Deschrijver, G., Cornelis, C. and Kerre, E. (2002), "Intuitionistic fuzzy connectives revisited", *Proceedings IPMU'02*, 1839-1844.

[8] Deschrijver, G., Cornelis, C. and Kerre, E. (2004), "On the representation of intuitionistic fuzzy t-norms and t-conorms", *IEEE Transactions on Fuzzy Systems*, **12** (1), 45-61.

[9] Goguen, J.A. (1967) "L-Fuzzy Sets", *Journal of Mathematical Analysis and Applications* **18**(1), 623-668.

[10] Trillas, E. (1979) "Sobre funciones de negación en la teoría de conjuntos difusos", *Stochastica* **III/1**, 47-60 (in Spanish). Reprinted (English version) (1998) in *Advances of Fuzzy Logic* (eds. S. Barro et altri), 31-43 (Ed. Universidad de Santiago de Compostela).

[11] Trillas, E., Alsina, C. and Jacas, J. (1999), "On Contradiction in Fuzzy Logic", *Soft Computing*, **3**(4), 197-199.

[12] Trillas, E. and Cubillo, S. (1999), "On Non-Contradictory Input/Output Couples in Zadeh's CRI", *Proceedings NAFIPS*, 28-32, New York.