Mathware & Soft Computing 10 (2003), 85-100

Fuzzy Morphological Operators in Image Processing.

P. Burillo, N. Frago, R. Fuentes
Depto. de Automática y Computación.
Universidad Pública de Navarra.
Campus de Arrosadía 31066, Pamplona, Navarra, Spain.
noe.frago@unavarra.es.

Abstract

First of all, in this paper we propose a family of fuzzy implication operators, which the generalised Luckasiewicz´s one, and to analyse the impacts of Smets and Magrez properties on these operators. The result of this approach will be a characterisation of a proposed family of inclusion grade operators (in Bandler and Kohout´s manner) that satisfies the axioms of Divyendu and Dogherty. Second, we propose a method to define fuzzy morphological operators (erosions and dilations). A family of fuzzy implication operators and the inclusion grade are the basis for this method.
Keywords: Implication operators, Inclusion grade, erosion and dilation.

1 Introduction

The expression Mathematical Morphology refers to the study of form and the structure. Morphological operations can be employed for many purposes, automatic image processing such as in artificial vision, industrial robotics and medicine. Essentially, mathematical morphology is a theory on morphological transformations. Serra [20] characterise the Binary morphological transformations with four principles: Compatibility under translation, compatibility under change of scale, Local Knowledge and semicontinuity.

These morphological transformations are based upon the intuitive notion of "fitting" a structuring element. It involves the study of the different ways in which a structuring element interacts with the image under study, modifies its shape, measures and reducing it to one other image, which is more expressive than the initial image. Mathematical Morphology was initially developed for analysis of binary images. The binary images are maps A: $\mathbf{U} \longrightarrow \{0, 1\}$, where U represents the Euclidean plane $\mathbf{R}^2$ or the Cartesian grid $\mathbf{Z}^2$ indistinctly, and in each point $x \in \mathbf{U}$ the value of image only can be 0 or 1, with which represents the black colour and the white.

The extension of binary morphological operators to greyscale morphology was formulated by Sternberg. Fuzzy Mathematical Morphology aims to extend the binary morphological operators to grey-level images. In the literature, several point of view are taken to define a Fuzzy Mathematical Morphology (erosion and dilation [3-8,13,18]. For instance, Sinha and Dougherty define the basic operations of Fuzzy Mathematical Morphology by notion of the inclusion grade for fuzzy subsets.

Several approach of the operator inclusion grade of fuzzy subsets A, B of a universe U, an example is of this approach is Zadeh´s, $A \subseteq B$ iff $A(x) \leq B(x)$ $\forall x \in U$. This definition of inclusion grade is to much sensitive to the values of individual elements of the universe U, which carries to several authors to propose weaker concepts. Bandler and Kohout [2] postulate the inclusion grade to which A is a subset of B (for A and B fuzzy subsets of U) is the membership $P(B): F(U) \longrightarrow [0,1]$, where F(U) is the set of all fuzzy set over universe U, such that, $P(B)(A)= \inf_{x \in \mathbf{U}} \{I(A, B)\}$ this gives of degree to with which A is a subset of B. Here "I" stand for a implication operator.

We define erosion and dilation with the inclusion grade operators as postulated by Bandler and Kohout [2]. For these authors, given A and B fuzzy subset of U and I a fuzzy implication operator, the degree in which A is a subset of B is given by $R(A, B) = \inf_{x \in \mathbf{U}} \{I(A(x), B(x))\}$.

Given A, $B \in F(U)$, $z \in U$ and denoting $-B$ the fuzzy subset such that $(-B)(x) = B(-x)$ $\forall x \in U$ and Bz the fuzzy subset such that $(Bz)(x) = B(x-z)$ $\forall x \in U$, then they define the erosion of a fuzzy subset A by another one B, called structuring element, as the fuzzy subset $\xi(A, B)$ whose membership function is $\xi(A, B)(z) = R(BZ, A)$. Dilatation is defined by duality $\mathcal{D}(A, B)(z)=(\xi(AC, -B)(z))C$, opening and closing are defined similarly to the binary case in terms of erosion and dilation.

Note, other fuzzy versions of the erosion and dilation operators can be found in available literature [3, 4, 7-8, 26].

2 Fuzzy implication operators and inclusion grade

In the Fuzzy Logic literature, that extends classical Boolean implications, the classical ones are considered $p \longrightarrow q$ for propositions p and q, area defined.

Table 1

¾ ®	**0**	**1**
0	1	1
1	0	1

Thus, are considered as maps I: [0,1]x[0,1] ⟶ [0,1], in such way that the true values of (p ⟶ q) are given by I(v(p), v(q)). Dubois and Prade [11-12] proposes an interesting classification of them. Kerre [14] gives a list of implications operators frequently used in the Bibliography. From an axiomatic point of view, Dubois and Prade [9,11,12] provides a list of properties which these operators must satisfy.

Concerning to the implication operators, different systems of axioms are used (Trillas and other authors [16,21,22,25]) depending on the purpose of its use. Note that, Lukasiewicz is the unique operator that verifies the whole Dubois and Prade´s properties [11-12], and in particular the Smets and Magrez´s axioms [21].

Numerous types of different operators are used to extend Boolean classic implication (Dubois, Prade, Trillas and others [11-12,22,23,1,16,21,14]), in fuzzy logic, as it is well known, the choice of operator depending on its practical efficiency.

In this paper we use Lukasiewicz generalised to define fuzzy operators: inclusion grade, erosion and dilation.

Definition 1: Lukasiewicz generalised operators are the maps L: [0, 1]x[0, 1]⟶[0, 1], such that L(a, b) = min(1, λ(a)+λ(1-b)) ∀a, b∈[0, 1] where function λ: [0, 1]⟶[0, 1] verifying λ(0) = 1 and λ(1) = 0.

We proceed now to analyse if the Lukasiewicz generalised operator verifies the Smets-Magrez axioms. In order to this, we will state the relations between the various axioms and properties of function λ. We can prove that the limit conditions required for the function λ is sufficient to assure that the operator L are an generalising of classical implication.

Let's remember that the Smets-Magrez´s axioms for implication operators, as being the essential properties for Lukasiewicz implication:
SM.1.- The value L(a, b) depends on values a and b.
SM.2.- Opposition law: L(a, b) = L(1-b, 1-a) ∀a, b∈[0, 1],
SM.3.- Exchange principle: L(a, I(b, c)) = L(b, I(a, c)) ∀a, b∈[0, 1],
SM.4.- L(., b) is non increasing ∀b ∈ [0, 1] and L(a, .) is non decreasing ∀a∈[0, 1]
SM.5.- ∀a, b ∈ [0, 1], a ≤ b ⇔ L(a, b) = 1 (implication defines an ordering)
SM.6.- L(1, b) = b ∀b∈[0, 1]
SM.7.- L(a, b) is continuous.

Theorem 2: Under the conditions stated in definition 1, the following properties are verified:
1) L(0, b) = 1 ∀b∈[0, 1], 2) L(a, 0) = λ(a) ∀a∈[0, 1], 3) L(a, 1) = 1 ∀a∈[0, 1]

4) $L(1, b) = \lambda(1-b) \quad \forall b \in [0, 1]$, 5) $L(a, b) = 1 \Leftrightarrow \lambda(a)+\lambda(1-b) \geq 1 \quad \forall a, b \in [0, 1]$
6) $L(a, b) = 0 \Leftrightarrow \lambda(a) = 0$ and $\lambda(1-b)=0$.

Theorem 3: Let L be in Definition 1 conditions:
a) L(., b) is non increasing in the first variable iff λ is non increasing
b) L(a, .) is non decreasing in the second variable iff λ is non increasing
c) L(a, b) is continuous iff λ is continuous.

Theorem 4: In the conditions of Definition 1, $L(a, b) = \min(1, \lambda(a)+\lambda(1-b))$ verifies the axioms, SM.2, SM.4, SM.5 and SM.7 iff λ and μ: $[0, 1] \longrightarrow [0, 1]$ verifies the conditions of the next table:

Table 2

1.- $\lambda(0) = 1$ and $\lambda(1) = 0$	**2.-** λ non increasing and μ non decreasing	
3.- $\lambda(p) = \mu(1-p) \quad \forall p \in [0, 1]$	**4.-** $p \leq q \Leftrightarrow \lambda(p)+\lambda(1-q) \geq 1$	**5.-** λ continuous

Theorem 5: Let L in the conditions of Definition 1 and λ verifying Table 2 then, the following properties are equivalent:
a) $L(a, L(b, c)) = L(b, L(a, c)) \quad \forall a, b, c \in [0, 1]$
b) $L(1, b) = b \quad \forall b \in [0, 1]$
c) $L(a, L(b, a)) = 1 \quad \forall a, b \in [0, 1]$
d) $\lambda(a) = 1-a \quad \forall a \in [0, 1]$

Therefore the only Lukasiewicz generalised operator that verifies all the Smets-Magrez axioms is for the function $\lambda(p)=1-p \quad \forall p \in [0, 1]$.

3 The inclusion grade

In the next paragraph, we will study the fuzzified concept of the set inclusion for fuzzy subsets in some universe U Finite, starting with the Sinha and Dogherty [7-8] proposition. The fuzzy subsets of U shall be denoted by F(U), and FR(UxV) is the class of the fuzzy relations in UxV.

Definition 6: We say that $R \in FR(F(U)xF(U))$ is an inclusion grade between fuzzy subsets, when the following axioms are satisfied:
IG.1.- $R(A, B) = 1 \Leftrightarrow A \subseteq B$
IG.2.- $R(A, B) = 0 \Leftrightarrow \exists\, x \in U$ such that $A(x) = 1$ and $B(x) = 0$
IG. 3.- R is non decreasing in its second argument $(B \subseteq C \Rightarrow R(A, B) \leq R(A, C))$
IG. 4.- R is non increasing in its first argument $(B \subseteq C \Rightarrow R(C, A) \leq R(B, A))$
IG. 5.- $R(A, B) = R(B^C, A^C)$
IG. 6.- $R(A \cup B, C) \geq \min(R(A, C), R(B, C))$
IG. 7.- $R(A, B \cap C) \geq \min(R(A, B), R(A, C))$

The properties of this list are concordant with those of the ordinary set inclusions. As an example let see that IG.1 is consistent with the crisp set inclusion: $A \subseteq B$ is necessary and sufficient in order that the grade which A is a subset of B is

1; If $x \in U$ such that A(x) = 1 and B(x) = 0 is necessary and sufficient so that the grade with which A is a subset of B is 0; IG.3 property maintains, for ordinary sets, the property of transitivity of inclusion; etc..

In the next theorem, we probe some properties of the inclusion grade.

Theorem 7: Under Definition 2 conditions the following is verified:
i) $R(A \cup B, C) = \min(R(A, C), R(B, C))$ ii) $R(A, B \cap C) = \min(R(A, B), R(A, C))$
iii) Property IG.3 is equivalent to $R(A, B \cup C) \geq \max(R(A, B), R(A, C))$
IV) Property IG.4 is equivalent to $R(A \cap B, C) \geq \max(R(A, C), R(B, C))$

We remark that the quoted authors include in their version of Definition 6, the property $R(A,B \cup C) \geq \max(R(A, B), R(A, C))$, to determine inclusions grades, producing therefore redundant family properties.

With respect to the existence of inclusion grade, Divyendu and Dougherty propose in [7-8] the relations

$$R(A, B)= \inf_{x \in \mathbf{U}} \{\min(1, \lambda(A(x))+\lambda(1-B(x)))\} \qquad \textbf{(1)}$$

As an inclusion grade for $A, B \in [0, 1]^U$ where $\lambda: [0, 1] \longrightarrow [0, 1]$ is a map that verifies the following table conditions:

Table 3

1.- λ is non increasing	**2.-** $\lambda(1) = 0$ and $\lambda(0) = 1$
3.- If there exist p and $q \in [0, 1]$ such that $\lambda(p) = \lambda(q) \geq 0.5$, then p = q	
4.- The equation $\lambda(p) = 0$ has a single solution	**5.-** $\lambda(p)+\lambda(1-p) \geq 1 \quad \forall p \in [0, 1]$

Theorem 8: Let $R \in FR(F(U)xF(U))$ be given by:

$$R(A, B) = \inf_{x \in \mathbf{U}} \{\min(1, \lambda(A(x))+\lambda(1-B(x)))\} \quad \forall A, B \in F(U).$$

Then R satisfies the properties from Definition 2 iff λ verifies the conditions of the following table:

Table 4

1.- $\lambda(1) = 0$ and $\lambda(0) = 1$,	**2**.- $p \leq q \Leftrightarrow \lambda(p)+\lambda(1-q) \geq 1$,	**3**.- λ is non increasing

Theorem 9: If $\lambda(0)=1$ and $\lambda(1)=0$, λ is non increasing and $\lambda(p)+\lambda(1-p)\geq 1 \quad \forall p \in [0, 1]$, then: $p \leq q \Rightarrow \lambda(p)+\lambda(1-q)\geq 1$. The converse implication is not true.

Theorem 10: A function $\lambda: [0, 1] \longrightarrow [0, 1]$ satisfies conditions of Table 4 iff there is a function $g: [0, 0.5] \longrightarrow [0.5, 1]$ for which
i) g(0) = 1, **ii)** g is strictly non increasing, **iii)** if $p \in [0, 0.5]$, then $\lambda(p) = g(p)$,
if $p \in (0.5, 1]$ and 1-p is a continuity point of g then $\lambda(p) = 1-g(1-p)$
if $p \in (0.5,1]$ and 1-p is a discontinuity point of g, then $1-g(1-p) \leq \lambda(p) \leq 1-g((1-p)+)$

Theorem 10 assure the existence of λ functions which verify the conditions of these Tables 4, for this it is enough to select functions $g: [0, 0.5] \longrightarrow [0.5, 1]$

strictly non increasing with g(0) = 1, and completely continuous, or piecewise continuous or continuous in zero or upper semi-continuous and continuous in zero. The functions that we propose, are just with a finite number of discontinuity points in [0, 1], that is piecewise continuous functions. The reason for this proposal is that they constitute a large number of functions, and these are easier to work with on the computer, as it is usual in Mathematical Morphology. In the same line it is useful the following Theorem.

Theorem 11: Let g: [0, ½]⟶[½, 1] a function verifying **i)** g(0)=1, **ii)** g is strictly non increasing and piecewise continuous. Then the function λ: [0, 1]⟶[0, 1] given by $\lambda(p)=\begin{cases} g(p) & \text{if } p \in \left[0, \frac{1}{2}\right] \\ 1-g(1-p) & \text{if } p \in \left(\frac{1}{2}, 1\right] \end{cases}$ satisfies Table 4 conditions and therefore the expression $R(A, B)= \inf_{x\in \mathbf{U}} \{\min(1, \lambda(A(x))+\lambda(1-B(x)))\}$ ∀ A and B∈ $[0, 1]^{\mathbf{U}}$ is an inclusion grade, which satisfies properties IG.1,...,IG.7 of Definition 6.

In this paper we have used the axioms of Smets and Magrez [21] to characterise a family of fuzzy implication operators, verifying axioms 1, 2, 4, 5 and 7 from these authors (Theorem 4-5), proving that the accomplishment of axioms 7 (continuity of L(a, b)) required the continuity of λ function for this operator. There are other weaker axiom families to characterise the fuzzy implication operators (Morsi [16]) in which instead of the continuity, the upper semi-continuity of these operators is required , this condition is verified with λ functions upper semi-continuous (See [5, 13]).

Now we have to note that the inclusion grade developed coincides with the formulation given by Bandler and Kohout [2], the generalised Lukasiewicz implications are used as fuzzy implications. L(a, b) = min(1, λ(a)+λ(1-b)) ∀a and b belonging to [0, 1], with λ function verifying the following properties: **1.-** λ(0)=1 and λ(1)=0, **2.-** λ increasing, **3.-** $p\le q \Leftrightarrow \lambda(p)+\lambda(1-q)\ge 1$, **4.-** λ continuous (upper semi-continuous).

Note: See proof of theorem 2-5 and 7-11 in [5,13].

4 Fuzzy mathematical morphology induced by generalised lukasiewicz operator

As an example of potential applications, these new inclusion grade operators have been used in the processing of the images, where they help to define the fundamental operations of erosion, dilatation, opening and closing thus having the way to the task of determining a Fuzzy Mathematical Morphology.

In this paper we have defined erosion in fuzzy mathematical morphology with R(A, B) operator and λ functions characterised by Table 2, and the basic morphological operations, erosion, dilation and closing-opening as the appropriate composition the fuzzy dilation and erosion. We have investigated the basic properties that verify these operators, and the paper closes with applications via image processing. The paper includes some examples as illustration of the effects of these operators.

In this section we formally define the various basic operations used in literature (see [3, 4, 7, 8, 26]), that are needed in the subsequent discussions.

The ***complement*** of a fuzzy subset A of **U,** denoted A^C, is the fuzzy subset of **U** defined as $A^C(x) = 1-A(x)\ \ \forall x\in \mathbf{U}$.

The ***translation*** of a fuzzy subset A of **U** by $v\in \mathbf{U}$, denoted A_v, is the fuzzy subset of **U** defined as $A_v(x)=A(x-v)\ \ \forall x\in \mathbf{U}$.

The ***reflection*** of a fuzzy subset A of **U,** is the fuzzy subset of U -A of **U** defined as $-A(x)=A(-x)\ \ \forall x\in \mathbf{U}$.

The ***scalar addition*** of a fuzzy subset A de **U,** and constant $\alpha\in[-1, 1]$, denoted as $A\alpha$, is defined as $(A\alpha)(x) = \min(1, \max(0, A(x)+\alpha))\ \ \forall x\in \mathbf{U}$.

Given $\alpha\in[0, 1]$ and A fuzzy subset of U, we denote $\alpha(A)$ as the fuzzy subset of U such that $\alpha(A)(x)=\alpha(A(x))$.

Let be a fuzzy subset A of $[0, 1]^{\mathbf{U}}$ and $v\in\mathbf{U}$, then, it is easily verified that:

1) $-(A_v) = (-A)_{(-v)}$, **2**) $(A_v)^C = (A^C)_v$, **3**) $(-A)^C = -(A^C)$.

The operator R(A, B) (**1**), with λ function verifying the conditions of Table 2, the characterisation given by Sinha and Dougherty [16-19] is:

$$\forall A, B\in[0, 1],\ R(A, B) = \sup_{\lambda(A^c)\lozenge(r-1)\subseteq\lambda(B)^c}\{r\},\ r\in[0, 1]$$

5 Erosion and dilation: definition and properties.

Let us consider the notion of erosion within the original formulation of mathematical morphology in Euclidean space U. Given A, B subsets of U, $x\in U$ and denoting $B_x=\{b+x \mid b\in B\}$ and $-B=\{x\in U \mid -x\in B\}$. Then the erosion of A by B is defined by $\xi(A,B)=A\lfloor(-B)=\bigcap_{b\in(-B)} A_b=\{x\in \mathbf{U} \mid B_x\subseteq A\}$ [14]. Resulting that $x\in \xi(A,B)$ if and only if $B_x\subseteq A$, this idea can be extended to the fuzzy situation, considering the erosion in a point $x\in \mathbf{U}$ as the grade with which the translated of B by x is included in A.

Now we are ready to explain the erosion and dilation as we can see in the Sinha and Dougherty´s work [7-8].

Definition 12: The erosion (dilation) of an image A by another B, being A and B fuzzy subset of **U**, is the fuzzy subset denoted by $\xi(A, B)$ ($\mathcal{D}(A, B)$), and it is defined by: $\xi(A, B)(z)=R(B_z, A)$ $\forall z\in \mathbf{U}$ ($\mathcal{D}(A, B)(z)=1-R((-B)_z, A^c)$ $\forall z\in \mathbf{U}$).

The B image is called "*structuring element*".

Note, other fuzzy versions of the operator erosion and dilation can be found in available literature [3, 4, 7, 8, 26].

Considering any A, B and C fuzzy subsets of U, if we study erosion and dilation operation from definition 1 and λ function satisfying Table 2 conditions, and erosion and dilation with the same definition, but λ verifying conditions from Table 1, we obtain that some properties are common in both cases.

Note that Table 1 is Sinha and Dougherty´s conditions and Table 2 are our conditions.

- Dilation is commutative $\mathcal{D}(A, B) = \mathcal{D}(B, A)$, erosion is not.
- Erosion and dilation are dual of each other, $\mathcal{D}(A^C, -B) = \xi(A, B)^C$.
- Erosion and dilation are invariant for translations. Give A, B fuzzy subset of **U** and $v\in \mathbf{U}$ then:

1) $\xi(A_v, B)=\xi(A, B)_v$, **2)** $\xi(A, B_v)=\xi(A, B)(-v)$, **3)** $\mathcal{D}(A_v, B) = \mathcal{D}(A, B)_v$
4) $\mathcal{D}(A, B_v) = \mathcal{D}(A, B)$

- Erosion and dilation are increasing in the first variable and decreasing in the second variable
- $\xi(A, B\cup C) = \xi(A, B)\cap\xi(A, C)$, $\xi(A\cap B, C) = \xi(A, C)\cap\xi(B, C)$,
  $\mathcal{D}(A, B\cup C) = \mathcal{D}(A, B)\cup\mathcal{D}(A, C)$, $\mathcal{D}(A\cup B, C) = \mathcal{D}(A, C)\cup\mathcal{D}(B, C)$,
- Let A, B be fuzzy subsets of U and K' crisps subset of U, then exist a bounded crisp subsets of U, such that:

$$\xi(A\cap K, B)=\xi(A, B)\cap K', \quad \mathcal{D}(A\cap K, B)=\mathcal{D}(A, B)\cap K'.$$

We also state that erosion and dilation verify the next principles of quantification: compatibility under translation, the local knowledge, and semi-continuity if the λ function is semicontinua. They are the first, third and fourth Serra's principles [20] to characterise the crisp morphological transformations. The second of these principles, compatibility under change of scale, is not verified [13].

Theorem 13: Let A and B be fuzzy subsets of $\mathbf{U}$ and $z \in \mathbf{U}$, then

1) $\xi(A, B)(z) = \sup\{1+\min(0, \alpha) \mid (\lambda(B)^C \alpha)_z \subseteq \lambda(A^C)\}$ and $0 \geq \alpha \geq -\sup_{x \in \mathbf{U}} \{\lambda(B(x))\}$

2) $\xi(A, B)(z) = \sup_{\lambda(B^C)_z \lozenge (\alpha - 1) \subseteq \lambda(A)^C} \{\alpha\}$ being α a real number belonging to $[0, 1]$

3) $\mathcal{D}(A, B)(z) = \sup_{x \in \mathbf{U}} \{\max(0, 1-\lambda((-B)_z(x))-\lambda(A(x)))\}$.

4) $\mathcal{D}(A, B)(z) = \inf_{(-\lambda(B^C))_{(-z)} \lozenge (-\alpha) \subseteq \lambda(A^C)^C} \{\alpha\}$ and $\alpha \in [0, 1]$

5) $\mathcal{D}(A, B)(z) = \inf_{(-\lambda(B^C))_z \lozenge \alpha \subseteq \lambda(A)} \{\alpha\}$ and $0 \geq \alpha \geq -\sup_{x \in \mathbf{U}} \{\lambda(B)^C(x)\}$

Proposition 14: Dilation and erosion are not associative.

It follows immediately from the previous definition that $\xi(A, \emptyset)(z) = 1 \ \forall z \in \mathbf{U}$ and $\xi(\emptyset, B)(z) = \inf_{x \in \mathbf{U}} \{\lambda(B(x))\} \ \forall z \in \mathbf{U}$ for each A and B fuzzy subsets of $\mathbf{U}$. Dilations verify that: $\mathcal{D}(\emptyset, B)(z) = 0 \ \forall z \in \mathbf{U}$ and $\xi(A, \emptyset)(z) = 0 \ \forall z \in \mathbf{U}$ for all A and $B \in F(\mathbf{U})$.

In theoretical notion, an image A is a map of $\mathbf{R}^2$ ó $\mathbf{Z}^2$ in [0, 1], but in practical work an image A is a map A: $D_A \longrightarrow [0, 1]$. Where D_A is a bounded subset included in $\mathbf{R}^2$ or $\mathbf{Z}^2$. The subset D_A is called the "domain" of the image A; in general different images have different domains.

Examples in this paper use digital images. The domain of a digital image will be rectangular in shape and contain a finite number of elements. In such a case, a digital image will be represented in a manner similar to a matrix $A = \{a_{ij}\}$ where $a_{ij} = A(i, j)$ with $(i, j) \in D_A$. In all examples we assume that the element of the matrix corresponding to the (0, 0) lattice point of the xy coordinate system is the left inferior vertex of D_A. This is not restrictive because, as we have seen, erosion and dilation are invariant with respect translation.

We give now some graphics, to illustrate the effect of our dilation-erosion operators in images. It is know that Binary Mathematical Morphology dilation expands the image and erosion shrinks it. Erosion yields a "smaller" image than the original and dilation the opposite, this idea can be extended to grey level imagery.

A point to notice here, is the fact that erosion and dilation, basic operations in binary imagery, can be extended to grey level. If the image and structuring element are binary (zero and one), binary operators, erosion-dilation hold the same effect that the fuzzy erosion-dilation. But in the case of fuzzy image and/or fuzzy structuring element, dilation-erosion operation results in overlapping effects of expands/shrink images and the modification of the contrast of image. The function λ and fuzzification of the structuring element are the responsible of the contrast effect.

Given the image Figure 1, the Figures 2-3 are output images of the fuzzy erosion and dilation respectively, they show the shrink-expand effect and modification of contrast of the image. The structuring element B used here, is an array 5x5, whose values are one. The function λ used is defined by $\lambda(p) = -0.5p + 1$ if $p \in [0, 0.5]$ and $\lambda(p) = -0.5p + 0.5$ if $p \in (0.5, 1]$.

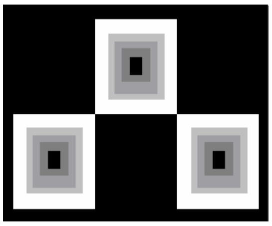

Figure 1, Initial image l.

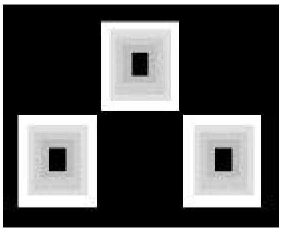

Figure 2, Erosion image

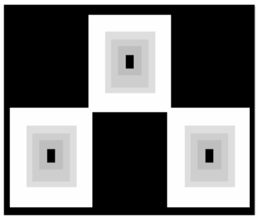

Figure 3, Dilation image.

Fuzzy erosion-dilation effect can also be seen in Figure 4 and 5. Figure 1 is the original image, the Figure 4 and Figure 5 show the erosion and the dilation of Figure 1 respectively. These images are obtained by using the structuring element B = {0,0,0,0,1; 0,0,0,1,0; 0,0,1,0,0; 0,1,0,0,0; 1,0,0,0,0} and λ function defined by $\lambda(p) = -0.5p+1$ if $p \in [0, 0.5]$ y $\lambda(p) = -0.5p+0.5$ if $p \in (0.5, 1]$. These figures illustrate as the selection of structuring element can contribute to creation of deformations of image shades.

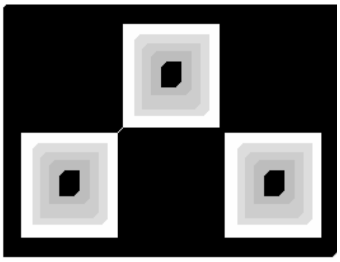

Figure 4, Erosion Image.

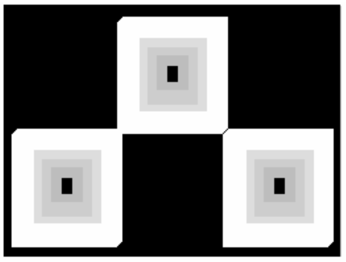

Figure 5, Dilation image

If the structuring element is the subset B = {1}, then $\forall A \in [0, 1]^U$, $\xi(A, B)(z) = \lambda(1-A(z))$, we have that, when we apply it to the initial image, produces a new image with only grey level changes (contrast enhancement) and it depends on the λ function used in the inclusion grade operator, $R(A, B) = \inf_{x \in \mathbf{U}} \{\min(1, \lambda(A(x)) + \lambda(1-B(x)))\}$.

6 Opening and closing: definition and properties.

Let us consider the opening and closing definition such as the have been described by Sinha and Dougherty [16-19]. We present now, some theorems and properties.

Definition 15: The opening (closing) of a fuzzy subset A of **F**(U) by another B, is a fuzzy subset denoted by @(A, B) (ζ(A, B)), and defined by @(A, B) = $\mathcal{D}$(ξ(A, B), B) and ζ(A, B) = ξ($\mathcal{D}$(A, -B), -B) respectively.

If A and B are fuzzy subsets of **U**, then the opening (closing) fuzzy operators, whereas the λ functions characterised in Table 2, verify the following properties: (see [11]).

- Opening and closing are dual operations: $z(A^C, B)^C$ = @(A, B)
- Opening and closing are invariant in translations
- Opening and closing are increasing with respect to the first variable

Sinha and Dougherty stated these same properties for opening (closing) fuzzy operators by using λ functions characterised in Table 1.

Theorem 16: Let A and fuzzy subset of **U** and λ a continuous functions of [0, 1] in [0, 1], then:

1) $@(A, B) = \bigcup_{\lambda(B^C)_y \Diamond (\alpha - 1) \subseteq \lambda(A)^C} \{\lambda(B_y)^C \Diamond (-\lambda(\alpha))\}$, $\alpha \in [0, 1]$

2) $\zeta(A, B) = \bigcap_{\lambda(B^c)_x \Diamond (\alpha - 1) \subseteq \lambda(A)} \{\lambda(B)_x \Diamond \lambda(\alpha)\}$, $\alpha \in [0, 1]$.

The Figures 7-8 are output images of the fuzzy opening-closing operators for image Figure 6, they shows modification in its shape and the contrast image. The structuring element B used here, is an array 5x5, B = {1,1,1,1,1; 1,1,1,1,1; 1,1,1,1,1; 1,1,1,1,1; 1,1,1,1,1}. The function λ used is defined by λ(p) = -0.5p + 1 if p∈[0, 0.5] and λ(p) = -0.5p + 0.5 if p∈(0.5, 1].

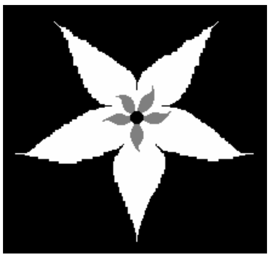

Figure 6, Initial image

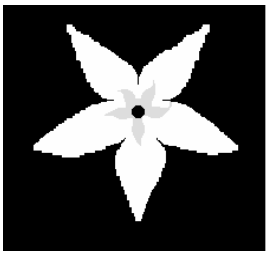

Figure 7, Opening image.

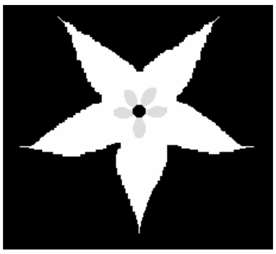

Figure 8, Closing image.

Note the resulting images when the former operators are used. Figures 7 and 8 show the small flower inside with attenuated grey-level, as a flower that becomes lighter than initial image. Considering that white and black denote 0 and 1 respectively. Note the effects in some corners of the image, differences between Figure 7 (opening) and Figure 8 (closing), we can observe that @(A, B) $\not\subset$ A and that A $\subseteq$ ζ(A, B).

Figures 9-14 illustrate effects of opening and closing operator. The function used is $\lambda(p) = -0.5p+1$ if $p \leq 0.5$ and $\lambda(p) = -0.5p+0.5$ if $p > 0.5$, $p \in [0, 1]$, and the structuring element, the array B = {1,1,1; 1,1,1; 1,1,1}. Figures 10-11 show the opening and closing of the initial image of the Figure 9. Figures 13-14 show the opening and closing of initial image of the Figure 12, with the same structuring element and λ function. See the modification to smoother grey and note that all outward pointing corners were rounded, whereas inward pointing corners were no affected, in other words, say that image transformation can vary using different

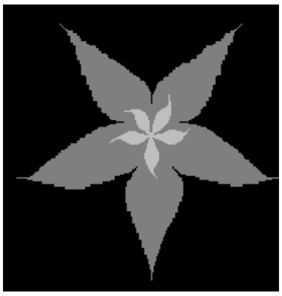

Figure 9, Initial image.

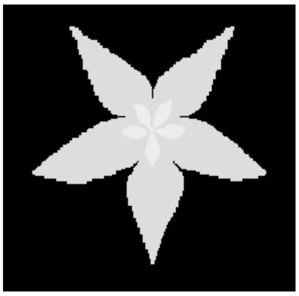

Figure 10, Opening image.

Figure 11, Closing image.

structuring elements.

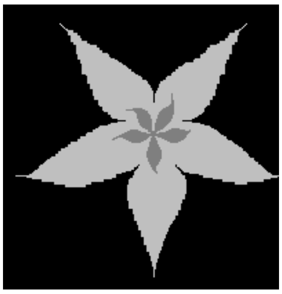

Figure 12, Initial image.

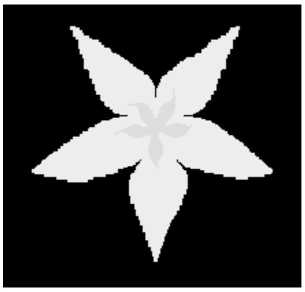

Figure 13, Opening image.

Figure 14, Closing image.

In the present study, one of our goals is to formulate, the most important theoretical difference, respect to the results obtained by Sinha and Dougherty:

The opening (closing) operator does not satisfy antiextensive (extensive) and idempotent properties

To end this subsection, let us note that: if we use the operators defined in this paper only with a binary image and a binary structuring element, the output images are equal to output images of binary morphology operators. In this case opening (closing) fuzzy operator verifies the antiextensive (extensive) and idempotent properties to similar way to binary case. Indeed, given A and B crips subsets of U, we have that:

- For the fuzzy erosion is verified that $\forall z \in \mathbf{U}$ $\xi(A, B)(z) = R(B_z, A) = \begin{cases} 0 & \text{if} \quad B_Z \not\subset A \\ 1 & \text{if} \quad B_Z \subseteq A \end{cases}$ and for the binary erosion $\xi(A, B) = A \ominus (-B) = \{z \mid B_Z \subseteq A\}$ (B_Z is equivalent to B+z), in fact $x \in B_Z \Leftrightarrow B(x\text{-}z)=1 \Leftrightarrow \exists b \in B \ni b=x\text{-}z \Leftrightarrow \exists b \in B \ni x=b+z \Leftrightarrow x \in B+z$). We have that $z \in \xi(A, B) \Leftrightarrow B_Z \subseteq A \Leftrightarrow \xi(A, B)(z) = 1$ y $z \notin \xi(A, B) \Leftrightarrow B_Z \not\subset A \Leftrightarrow \xi(A, B)(z) = 0$.

- For the fuzzy dilation is verified that $\forall z \in \mathbf{U}$ $\mathcal{D}(A, B)(z) = 1\text{-}R((\text{-}B)_z, A^C) = \begin{cases} 1 & \text{if} \quad R((\text{-}B)_Z, A^C) = 0 \\ 0 & \text{if} \quad R((\text{-}B)_Z, A^C) = 1 \end{cases}$, but , $R((\text{-}B)_z, A^C) = 0 \Leftrightarrow \exists$ $y \in \mathbf{U} \ni (\text{-}B)_Z(y) = 1$ y $A^C(\mathbf{y}) = 0 \Leftrightarrow$ $(\text{-}B)(y\text{-}z) = 1$ y $A(y) = 1 \Leftrightarrow B(z\text{-}y) = 1$ y $A(y) = 1$, and for the binary dilation is $\mathcal{D}(A, B) = A \oplus B = \{x \mid (\text{-}B)+z \cap A \neq \varnothing\}$.

We have that $z \in \mathcal{D}(A, B) \Leftrightarrow (\text{-}B)+z \cap A \neq \varnothing \Leftrightarrow \exists$ $y \in U \ni y \in (\text{-}B)+z$ e $y \in A \Leftrightarrow$ $\exists$ $y \in U \ni y = \text{-}b + z$, $b \in B$ e $y \in A \Leftrightarrow \exists$ $y \in U \ni z\text{-}y \in B$ e $y \in A \Leftrightarrow \exists$ $y \in U \ni B(z\text{-}y)=1$ e $A(y) = 1 \Leftrightarrow R((\text{-}B)z, AC) = 0 \Leftrightarrow \mathcal{D}(A, B)(z) = 1$.

- If we consider that opening and closing operations are given by:

$$@(A, B) = \mathcal{D}(\xi(A, B), B) \text{ and } \zeta(A, B) = \xi(\mathcal{D}(A, \text{-}B), \text{-}B)$$

Then, we can write that, in case of binary images and binary structuring elements, the output image of this operator is equal to the output image with binary Euclidean Morphology operators.

In any case, fuzzy operator opening (closing) used in binary imagery holds antiextensive (extensive) and idempotent properties.
When the λ function is defined by $\lambda(p)=1\text{-}p$ $\forall p \in [0, 1]$, opening (closing) operator's properties do not differ from the traditional properties of morphology binary as it has been mentioned in (Sinha - Dougerty [16-19], and Frago [13]).

We postulate the following properties for the opening (closing) operator when λ is, $\lambda(p) \geq 1\text{-}p$: Opening is antiextensive and idempotent, closing is extensive and idempotent. Only when $\lambda(p) = 1\text{-}p$, Table 2´s conditions are satisfied.

7 Concluding remarks

By considering both, the intuitive concept of the set inclusion and the demands of important applications in fuzzy set theory, a collection of inclusion grade operators have been proposed. Their characterisation has been achieved by means of the Lukasiewicz generalised implications and functions λ of [0, 1] in [0, 1] that verifies all the requirements summarised in Table 4 of this paper.

The proposed family of inclusion grade operators, it is shown to be a good extension of the Zadeh inclusion for fuzzy sets.

As an example of potential applications, these new inclusion grade operators have been used in the processing of the images, where they help to define the fundamental operations of erosion, dilatation, opening and closing thus having the way to the task of determining a Fuzzy Mathematical Morphology.

A second objective of the present paper has been to construct a family of operators: erosion (dilation), opening (closing), that satisfy most of the binary mathematical morphology properties.

In this paper, an attempt has also been made by examining in figures the effect of these operators in fuzzy images, which may be translated to image processing and techniques with images grey-scale, used in medical imaging, etc.

For binary images and binary structuring elements, erosions and dilations, as defined here, produce the same results as the analogous operators in ordinary mathematical Morphology.

We can finish these conclusions, by saying that the morphological operators proposed in this work, extend the correspondent concepts of binary operators. How these operators act on fuzzy images is of interest, for instance, in the computerised treatment of sonography or radiology pictures, where grey, black and white levels are apt to appear.

References

[1] C. Alsina and E.Trillas, On almost distributive Lukasiewicz triplets, Fuzzy Sets and Systems, 50(1992) 175-178.

[2] W. Bandler and L. Kohout, Fuzzy Power sets and Fuzzy Implication Operator. Fuzzy Sets and Systems, 4(1980) 13-30.

[3] Bloch, I. and Maître, H., Constructing a Fuzzy Mathematical Morphology: Alternative ways, Telecom Paris 92 C 002, September 1992.

[4] B. De Baets, Fuzzy morphology: a logical approach, Uncertainty Analysis in Engineering and Sciences: Fuzzy logic, Statistics and neural Network Approach (B. Ayyub and M. Gupta, eds.) Kluwer Academic Publishers, 1997, pp. 53-67.

[5] P. Burillo, n. Frago, R. Fuentes, Inclusión Grade and Fuzzy Implication operators. Fuzzy Sets and Systems, 114(2000)417-429-

[6] P. Burillo, n. Frago, R. Fuentes, Generation of Fuzzy Mathematical Morphologies. Mathware & Soft Computing 8(2001) 31-46.

[7] S. Divyendu and E. R. Dougherty, Fuzzification of Set inclusion, SPIE Vol. 1708 Applications of Artificial Intelligence X: Machine Vision and Robotics (1992) 440-449.

[8] S. Divyendu and E. R. Dougherty, Fuzzification of set inclusion: Theory and applications, Fuzzy Sets and Systems 55(1993) 15-42.

[9] D. Dubois and H. Prade, Fuzzy Logic and the generalised modus ponens revised, Cybernetics and Systems: An International Journal, 15 (1984) 293-331.

[10] D. Dubois and H. Prade, Fuzzy Sets and Systems: Theory and Applications, Mathematics in Science and Engineering, (Academic Press, New York 1980).

[11] D. Dubois and H. Prade, Fuzzy sets in approximate reasoning, Part 1: Inference with possibility distributions, Fuzzy Sets and Systems 40 (1991) 143-202.

[12] D. Dubois, L. Jérôme and H. Prade, Fuzzy Sets in approximate reasoning, Part 2: Logical approaches, Fuzzy Sets and Systems 40 (1991) 203-244.

[13] N. Frago, Morfología Matemática Borrosa basada en operadores generalizados de Lukasiewicz: Procesamiento de imágenes, Ph.D.Thesis, Universidad Pública de Navarra 1996.

[14] E. Kerre, A comparative study of the behaviour of some popular fuzzy implication operators on the generalised modus ponens. Fuzzy Logic for The Management of Uncertainty (Zadeh L. A., Kacprzyk J. eds, Wiley J., New York, 1992), 281-295.

[15] B. Kosko, Neural Networks and Fuzzy Systems: A Dynamical Approach to Machine Intelligence, (Prentice-Hall, New Yersey 1992).

[16] N.N. Morsi, Hiperspace fuzzy binary relations, Fuzzy Sets and Systems 67 (1994) 221-237.

[17] N. Rescher., Many-Valued Logic, (McGraw-Hill, New York, 1969)

[18] D. Ruan and E. E. Kerre, Fuzzy implication operators and generalised fuzzy method of cases, Fuzzy Sets and Systems 54 (1993) 23-27.

[19] D. Ruan, E. E. Kerre, G. De Cooman, B. Capelle and F.Vanmassenhove, Influence of the Fuzzy implication operator on the method of cases inference regla, Int. J. Approximate Reasoning, 4 (1990) 307-318.

[20] Serra, J., Image Analysis and Mathematical Morphology, Vol. 1 y 2, Academic Press, London 1982.

[21] P. Smets and P. Magrez, Implications in Fuzzy Logic. International Journal of Approximate Reasoning, 1 (1987) 327-347.

[22] E. Trillas and L. Valverde, On implication and indistinguishability in the setting of Fuzzy logic, in: J. Kacprzyk, R.R. Yager, Eds., Management Decision Support Systems Using Fuzzy Sets and Possibility Theory (Verlag TÜV Rheiland, Köln, 1985), 198-212.

[23] E. Trillas and L. Valverde, On inference in Fuzzy Logic, Proc. 2nd IFSA Congress, Tokyo, Japan, (1987) 294-297.

[24] E. Trillas and L. Valverde, On mode and implication in approximate reasoning, in: Approximate Reasoning in Expert System (M. M Gupta et al., Eds.), North-Holland, Amsterdam, (1985) 157-166.

[25] R. Willmott, On the transitivity of Containment and Equivalence in Fuzzy Power Set Theory. Journal of Mathematical Analysis and Applications 120 (1986) 384-396.

[26] Xu , J. and Giardina, C. R. ,The algebraic structure of the generalisation of binary Morphology using Fuzzy set concept. Inst. Graphic Commun Waltham, MA, USA, 1988, 292-297.