Mathware & Soft Computing 10 (2003) 23-41

Combining Degrees of Impairment: The Case of the Index of Balthazard

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Abstract

Using techniques for modeling indices by means of functional equations and resources from fuzzy set theory, the classical Balthazard index used in order to combine several degrees of impairment is characterized in two natural ways and its use is criticized. In addition some hints are given on how to study better solutions than Balthazard's one for the problem of combining impairment's degrees.

Keywords: Balthazard index, degrees of impairment, combination of several degrees of impairment.

1 Introduction. The problem of the global degree of impairment.

Given a person with two (independent) impairments whose corresponding degrees of impairment are known, how can the global degree of impairment be determined? (See e.g. [3], [10], [14]). This is a problem of

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evident medical, social and legal relevance. To the best knowledge of the authors, except for Victor Balthazard (1872 - 1950) for the case of two impairments [3], there has been no approach to the problem from the mathematical point of view. This is the intention of the present paper: the critical study of mathematical formulas appropriate to combining several degrees of impairment to achieve a global degree. The interest in this problem was awaked by a juristic analysis recently made by J. Fargas [9].

Any formula intending to solve this problem should be obtained based on clearly expressed and previously accepted criteria which necessarily would lead to it. When adopting such a formula, the corresponding criteria are accepted independently of the fact that they may be known or not. In the latter case the criteria are implicitly used anytime the formula is applied to a particular case. It becomes apparent that if the conditions of that particular case do not correspond to the criteria supporting the formula, then it has been incorrectly applied or *de facto*, misused. This can generate situations which may be considered as unjust and therefore may lead, and in fact have lead (see [5], [4], [16]) to Courts of Justice pleading for a correction of the result. The decision of a Court of Justice, that is always based on arguments, against the result given by the formula in a particular case, is not necessarily a decision against the formula, but clearly a decision against the use of the formula in that particular case. A negative decision of a Court of Justice should warn the designer of such a formula to review the criteria to see whether they are clear enough not to have been misunderstood. At the same time a negative judgement represents a counterexample pointing out that there are at least doubts on the general validity of the formula. These considerations constitute a strong motivation to mathematicaly study the problem, looking for explicit mathematical principles that are equivalent to, or at least imply, the formula.

A first evident problem is that a formula adopted to calculate the combined degree of two impairments may not be valid for the case of more than two impairments, or if it were valid, but given as a table with a strong limit in precision, then *the table* is not acceptable. This is a simple practical problem and a solution will be presented in the next section. It may however have a much deeper conceptual relevance as will be discussed in section 4.

2 Balthazard's formula

Victor Balthazard, a french specialist for legal medicine introduced in [3] a method to combine degrees of impairment, which is today referred to as the *formula* or *index* of Balthazard. An analysis of the method allows the following reasoning, which may have been its basis.

If someone has only a first impairment and has been assigned p per cent as a degree of impairment, then he has $(100 - p)$ per cent free of impairments. Would this person some time later start suffering from a new impairment, that is independent from the former and would for this new impairment considered alone be assigned q per cent as degree of impairment, then the former $(100 - p)$ per cent free of impairments becomes reduced by $(100 - p) \cdot q/100$. As a consequence, the global percentage of impairment to be assigned to this person would be given by

$$B(p, q) = p + (100 - p) \cdot q/100 = p + (1 - p/100) \cdot q$$

This equation may be rewritten as

$$B(p, q) = 100 \left[\frac{p}{100} + \left(1 - \frac{p}{100}\right) \frac{q}{100} \right]$$

In order to simplify the notation, in what follows instead of working with percentages, the corresponding real values in the interval $[0, 1]$ will be used in the same way that Balthazard did in [3]. Let $a = p/100$ and $b = q/100$. The former equation then turns into

$$\widetilde{B}(a, b) := 0.01B(p, q) = a + (1 - a) \cdot b$$

The formula of Balthazard may be analyzed from several points of view.

(i) The simplified notation makes apparent that $\widetilde{B}(a, b) = a + b - ab$, and represents a well known continuous t-conorm, which is called "Prod*" [15], [12]. (Readers not familiar with t-conorms may find their characterizing properties in the Appendix). Notice that $\widetilde{B}$ satisfies the following: $\widetilde{B}(a, b) = 1$ iff $a = 1$ or $b = 1$. This is however not an exclusive property of the t-conorm $\widetilde{B}$. The continuous t-conorm $S(a, b) = (a + b)/(1 + ab)$, for instance, also satisfies this property; it is however different from $\widetilde{B}$. For a unique characterization of $\widetilde{B}$ other properties have to be specified (see Section 3).

(ii) Since $\widetilde{B}$ is a t-conorm, it is associative and commutative, i.e., to calculate the total degree of impairment for the hypothetical case of three impairments it would be enough to apply $\widetilde{B}$ to the (normalized) degree of impairment of any two of them followed by an application of $\widetilde{B}$ to the obtained result and to the (normalized) degree of the remaining impairment. However, when a table is used instead of the function $\widetilde{B}$ as in [10] and [14], and moreover both the entries a and b and the values of $\widetilde{B}(a, b)$ are given with only two digits of precision, then associativity is lost. For instance assume that $a = 0.35, b = 0.75$ and $c = 0.41$. Then

$$\begin{aligned}\widetilde{B}(a, b, c) &= \widetilde{B}(\widetilde{B}(a, b), c) = \widetilde{B}(\widetilde{B}(0.35, 0.75), 0.41) = \\ &= \widetilde{B}(0.8315, 041) = 0.904125\end{aligned}$$

However, when the table of [10] is used, the following is obtained:

- $\widetilde{B}_{table}(\widetilde{B}_{table}(0.35, 0.75), 0.41) = \widetilde{B}_{table}(0.84, 0.41) = 0.91$ (rounding by increasing)
- $\widetilde{B}_{table}(0.35, \widetilde{B}_{table}(0.75, 0.41) = \widetilde{B}_{table}(0.35, 0.85) = 0.90$ (rounding by decreasing)

It becomes apparent that in the case of calculating the combined degree of impairment with the table in [10], the result *is not independent of the order* in which the impairments are considered (see [9]). The intuitive plausible idea of calculating the cumulative degree of impairment according to the order of appearance of the corresponding impairments is not given a fair support when using a table with limited precision, as the one under discussion. Since the table is commutative, it seems adviceable to use the following procedure in the benefit of possibly affected patients::

$$\widetilde{B}_{table}(a, b, c) :=$$

$$Max[\widetilde{B}_{table}(\widetilde{B}_{table}(a, b), c), \widetilde{B}_{table}(\widetilde{B}_{table}(a, c), b), \widetilde{B}_{table}(\widetilde{B}_{table}(b, c), a)]$$

Notice that in [10] it is said that the values a, b and c may be taken in any order and in [14] the question of ordering is not considered. It is to be expected that in the prevailing times of the Information Society, tables will no longer be used and will be replaced by calculations on-line supported by an appropriate *program of the formula*, which may be processed using high precision. Roundings will still take place at the last digit; this is unavoidable; but they will be no longer relevant.

(iii) V. Balthazard, as mentioned above, introduced the formula $\widetilde{B}(a,b) = a + b - ab$ for impairments that successively affect different functions of a patient. The formula does not seem to be in general applicable when a second impairment may be some kind of consequence of the first one. Thus in [5] a court decision states that due to the fact of a second impairment being a consequence of the first, the combined degree of impairment should be $Min(1, a+b)$ instead of $a+b-ab$. (By the way, $Min(1, a+b)$ is known as the t-conorm of Łukasiewicz and is given the symbolic expression $W^*(a,b)$.) For similar reasons, in [4] the combined degree of impairment based on $a+b-ab$ is not approved and for two particular cases, combined degrees of impairment are given, that correspond to results that would have been obtained by applying the t-conorms $(a+b)/(1+ab)$ and Max, respectively.

(iv) A probabilistic analysis allows to show, at least indirectly, the non-applicability of the formula of Balthazard when the second impairment is not independent of the first one. Let A and B be two discrete random variables, whose probabilities correspond to the degrees of two impairments respectively. Then

$$Prob[(A = \alpha) or (B = \beta)] =$$

$$Prob[(A = \alpha)] + Prob[(B = \beta)] - Prob[(A = \alpha) and (B = \beta)].$$

In the case that the variables A and B may be considered statistically independent, it follows

$$Prob[(A = \alpha) or (B = \beta)] =$$

$$Prob[(A = \alpha)] + Prob[(B = \beta)] - Prob[(A = \alpha)].Prob[(B = \beta)].$$

Thus in this statistical context, the product of probabilities may be subtracted from the sum, when the random variables A and B (representing the first and second impairments, respectively) may be considered as statistically independent. Notice that by letting $Prob[(A = \alpha)] = a$ and $Prob[(B = \beta)] = b$, the formula $a + b - ab$ represents the probability of a disjunction. This aspect will be considered again below.

(v) Notice that the (normalized) degrees of impairment may be considered to be values of membership functions of fuzzy sets on a given population [13], [17]. In this context, if H_1 denotes the first impairment (as a predicate), $\mathbf{H_1}$ is its corresponding fuzzy set and i denotes an individual with that impairment, then the value of a equals $\mu_{\mathsf{H_1}}(i)$. Similarly, if H_2 denotes the second impairment and $\mathbf{H_2}$ is its fuzzy set, then $b = \mu_{\mathsf{H_2}}(i)$. It follows that for the individual i,

$$\widetilde{B}(a, b) = \mu_{\mathsf{H_1}}(i) + \mu_{\mathsf{H_2}}(i) - \mu_{\mathsf{H_1}1}(i) \cdot \mu_{\mathsf{H_2}}(i).$$

This expression however, corresponds to $\mu_{\mathsf{H_1} \cup \mathsf{H_2}}(i)$, where the union $\cup$ of the fuzzy sets $\mathbf{H_1}$ and $\mathbf{H_2}$ is calculated with the t-conorm Prod*. It may be concluded that in this context, the formula of Balthazard expresses something that does not seem to be related to the problem of calculating a global degree of impairment. It expresses the degree with which an individual i has a degree of impairment with respect to the first impairment or a degree of impairment with respect to the second one. It expresses through "i is in $\mathbf{H_1}$ or i is in $\mathbf{H_2}$ ", the degree of the disjunction of two gradable statements "i is H_1" (i has H_1) and "i is H_2" (i has H_2).

(vi) Balthazard's reasoning as presented earlier in this section, accepts still another interpretation, different from the former one (v). The original (normalized) formula $\widetilde{B}(a, b) = a + (1 - a) \cdot b$ may be given the following expression:

$$\widetilde{B}(a, b) = Min[1, a + (1 - a) \cdot b],$$

which under the notation for fuzzy sets and the t-conorm of Łukasiewicz becomes

$$\widetilde{B}(a,b) = W^*[\mu_{\mathsf{H_1}}(i), (\mu'_{\mathsf{H_1}}(i))\cdot\mu_{\mathsf{H_2}}(i)].$$

With this expression, $\widetilde{B}(a,b)$ denotes the degree of membership to the fuzzy set $\mathbf{H_1}\cup(\mathbf{H_1}'\cap\mathbf{H_2})$, when the union $\cup$ is calculated by means of W^*, the t-conorm of Łukasiewicz, the intersection $\cap$ is calculated with the t-norm product and the pseudo-complement ' is calculated with the negation function $1 - id_{[0,1]}$ [13], [17]. Hence if i belongs to a population X, in the non-dual theory of fuzzy sets
$([0,1]^X, Product, W^*, 1 - id_{[0,1]})$ the global degree of impairment of Balthazard $\widetilde{B}(a,b)$ corresponds to the membership degree of i to the fuzzy set $\mathbf{H_1} \cup (\mathbf{H_1}' \cap \mathbf{H_2})$, meaning the degree of (acceptance of) the statement consisting of the disjunction of two gradable assessments: 'i has H_1' and ('i has H_2' and 'i does not have H_1'). This interpretation does not appear to be "unnatural", as the previous one in (v) and is consistent with the reasoning in [3].

3 Characterization of the formula of Balthazard

In this section, two sets of principles will be studied, which are mathematically equivalent to those implied by the formula of Balthazard $\widetilde{B} = Prod*$ to calculate the combined degree of impairment of two independent impairments. Obviously, the main focus of interest falls on theoretical principles which are interpretable in the context of the problem under discussion and may contribute to characterize Balthazard's solution. This will clarify criteria which are implicitly accepted when using Balthazard's formula; particularly by responsible authorities, as in the case of the Spanish Administration [14] or the American Medical Association [10]. This will also contribute to progress in the critical analysis of this solution as well as to suggest alternative possibilities, as it is done in the French legislation [7] (which considers the index of Balthazard only as a reference) or in Italy [18], where other possible formulas are mentioned.

Let $S : [0,1] \times [0,1] \to [0,1]$ be a function which given two degrees of impairment a and b, determines the global impairment $S(a,b)$. It seems natural to require that S should satisfy principles or algebraic properties easily understandable. For instance:

(P1) Principle of immutability under inexistence of new impairments: $S(0,a) = S(a,0) = a$.

(P2) Principle of incremental coherence with respect to changes in the degrees of impairment: For any $a,b,c \in [0,1]$ with $a+c < 1$, $S(a+c,b) - S(a,b) = S(c,b) - S(0,b)$. This condition states that the increment of the index when a degree of impairment grows, directly depends on this growth.

(P3) Principle of invariance of the index with respect to the order the impairments: $S(a,b) = S(b,a)$.

(P4) Principle of total impairment: $S(1,1) = 1$.

The four principles given above are independent from each other. To show this, it is sufficient to give four functions such that each one of them satisfies a different set of three principles, but does not satisfy the fourth one.

- 1st function:$S(x,y) = x \cdot y$ satisfies (P2), (P3) and (P4), but does not satisfy (P1).

- 2nd function: $S(x,y) = Min(1, x+y)$ satisfies (P1), (P3) and (P4), but does not satisfy (P2). (See numerical example in Appendix).

- 3rd function: $S(x,y) = y$ if $x,y \in (0,1)$, otherwise $S(x,y) = Max(x,y)$. This function satisfies (P1), (P2) and (P4), but does not satisfy (P3).

- 4th function: $S(x,y) = x + y - 2xy$ satisfies (P1), (P2) and (P3), but does not satisfy (P4).

The following theorem shows that the index of Balthazard is characterized by the four given principles.

Theorem 3.1 *A function* $S : [0,1] \times [0,1] \to [0,1]$ *satisfies (P1), (P2), (P3) and (P4)* **iff** $S(a,b) = a + b - ab$ *for all* a, b *in* $[0,1]$.

Proof. From (P1) and (P2) follows that for all a, b and x in $[0,1]$,

$$S(a + x, b) = S(a, b) + S(x, b) - b$$

Then:

$$S(a + x, b) - b = S(a, b) - b + S(x, b) - b.$$

Let b be fixed and define $f : [0,1] \to \mathbb{R}$, such that $f(z) = S(z, b) - b$. The function f is positive, bounded by 1 and satisfies the classical equation of Cauchy $f(a + x) = f(a) + f(x)$ for all a, x in $[0,1]$ and $a + x \cdot \ 1$. Therefore [1], there exists a constant K, more properly $K(b)$, whose value depends on the selected fixed value of b, such that $f(z) = K(b) \cdot z$, from where

$$S(a, b) = K(b) \cdot a + b$$

since $K(b) \cdot a = S(a, b) - b$.
According to (P3), the principle of commutativity, the following holds:

$$K(b) \cdot a + b = K(a) \cdot b + a$$

and by substituting $a = 1$,

$$K(b) = (K(1) - 1)b + 1.$$

Then,

$$S(a, b) = K(b) \cdot a + b = [(K(1) - 1)b + 1]a + b = a + b + (K(1) - 1)ab.$$

Finally from (P4), $S(1, 1) = 1$, follows that $K(1) = 0$, leading to

$$S(a, b) = a + b - ab$$

which is the formula of Balthazard.

The reciprocal is quite simple, since (P1), (P3) and (P4) are properties of any t-conorm. (P2) is obtained by construction. For all a, b and c in $[0,1]$ with $a + c < 1$.

$$S(a+c,b) = (a+c) + b - (a+c)b = a + c + b - ab - cb =$$

$$(a + b - ab) + (b + c - cb) - b = S(a,b) + S(c,b) - S(0,b).$$

The assertion follows. □

Three alternative principles are discussed below.

(P1) Principle of immutability under inexistence of new impairments: $S(0,a) = S(a,0) = a$ for all a in $[0,1]$. (The same first principle of the former group).

(P2*) Principle of incremental linearity with respect to changes in the degrees of impairment: For any $a, b, c \in [0,1]$ with $a + c \cdot \; 1$, $S(a+c,b) - S(a,b) = \alpha c + \beta$, where $\alpha = \alpha(a,b)$ and $\beta = \beta(a,b)$ are arbitrary functions on a and b with range $[0,1]$.

(P4*) Principle of absorption of the largest degree of impairment: $S(1,b) = 1$ for all b in $[0,1]$.

These three principles are equivalent to the solution $S = Prod*$. Similarly as in the former case, it is simple to show that these principles are independent of eachother.

Theorem 3.2 *A function* $S : [0,1] \times [0,1] \to [0,1]$ *satisfies (P1), (P2*) and (P4*)* **iff** $S = Prod*$.

Proof. It is obvious that $S = Prod^*$ satisfies (P1) and (P4*) since it is a t-conorm. With respect to (P2*),

$$S(a+c,b) - S(a,b) = a + c + b - (a+c)b - [a + b - ab] =$$

$$a + c + b - ab - cb - a - b + ab = c - cb = (1-b)c$$

i.e., (P2*) is satisfied with $a = (1 - b)$ and $b = 0$.

For the reciprocal case, it is assumed that S satisfies the three principles. From (P2*) with $c = 0$ follows that $b = 0$ and with both $a = 0$ and $c = 1$, (i.e. $a + c = 1$) follows that $S(1, b) - S(0, b) = \alpha \cdot 1 - 0 = \alpha$. With (P1) and (P4*), $\alpha = 1 - b$. Therefore with (P2*) and any a and b in $[0, 1]$ it is enough to take $c = 1 - a$ (which of course is in $[0, 1]$) to obtain

$$S(a + c, b) - S(a, b) = S(1, b) - S(a, b) = \alpha c + \beta = \alpha c = \alpha(1 - a)$$

and again with (P4*),

$$S(a, b) = S(1, b) - \alpha(1 - a) = 1 - \alpha(1 - a)$$

and since $\alpha = 1 - b$, then $S(a, b) = a + b - ab$. □

Corollary 3.3 *For functions $S : [0, 1] \times [0, 1] \to [0, 1]$ the four principles (P1), (P2), (P3) and (P4) and the three principles (P1), (P2*) and (P4*) are equivalent.*

Notice that by setting $b = 1$ it becomes obvious that (P4*) implies (P4). Furthermore (P2*) implies (P2). (P2) requires $S(a + c, b) - S(a, b) = S(c, b) - S(0, b)$. The left hand side, with (P2*) equals $(1-b)c$. The right hand side can be written as $S(0 + c, b) - S(0, b)$, which with (P2*) also gives $(1 - b)c$.

4 Comments beyond Balthazard's solution $S = Prod^*$

Recall that both sets of principles equivalent to the solution $S = Prod^*$ lead to $S(a, b) = 1$ if and only if $a = 1$ or $b = 1$. This means that, in terms of percentages, a combined value of 100% can only be obtained if at least one of values to be combined is already 100%. This might eventually be considered unfair in the context of combining impairments. In this case, (P2) and (P2*) could be replaced by another principle, like

(P2**) $S(a, b) = 1$ if and only if $(a + b) > 1$.

The set of principles (P1), (P2**), (P3) and (P4) turns out to be equivalent to the solution $S(a, b) = Min(1, (a + b))$. In [7] this solution is called "additive" and, as mentioned earlier, it corresponds to the t-conorm of Łukasiewicz and is denoted by W^*. It is well known [2], [12] that for all a, b in $[0, 1]$ $Prod^*(a, b) \cdot W^*(a, b)$. Since $Prod^*(a, b) = W^*(a, b)$ only in the case that $a = b = 1$ or $a{\cdot}b = 0$, then except for these cases, it is always $Prod^*(a, b) < W^*(a, b)$. Therefore a table based on $W^*(a, b)$ would provide higher values than in the table in [10], [14]. For instance from the combination of 35% and 60%, the following would be obtained:

- $Prod^*(0.35, 0.60) = 0.74$, which corresponds to 74%
- $W^*(0.35, 0.60) = 0.95$, which corresponds to 95%

The new table would represent the function

$$B(p, q) = 100{\cdot}Min(1, (p + q)/100) = Min(100, p + q)$$

for any p, q in $[0, 100]$. The values obtained with the table could be in many cases strongly larger than those obtained with the original table [10] (see [9]) and moreover, associativity could be lost due to the limited precision bounded by two digits.

If the values obtained with W^* would be considered to be too high, other continuous t-conorms greater than $Prod^*$ but smaller than W^* could be considered. One such t-conorm is

$$S(a, b) = (a + b)/(1 + ab)$$

which for the former case gives $S(0.35, 0.60) = 0.95/1.21 = 0.79$ (rounded to two digits). However it is fair to mention that $S(a, b) = 1$ if and only if $Prod^*(a, b) = 1$, hence, if and only if $a = 1$ or $b = 1$.

It should be noticed that some experts consider that 100% -(or normalized 1)- should only correspond either to death or a state of complete lethargy (Coma) [7], [18], [9]. Therefore to comply with this position, S should satisfy the following requirement:

$$S(a, b) = 1 \text{ if and only if } Max(a, b) = 1.$$

From a more general point of view, it seems adequate to make a critical appreciation of the use of t-conorms to combine degrees of impairment. As shown in section 2 (v), degrees of impairment may be considered as values of fuzzy sets and t-conorms are the operations which realize the linguistic disjunction ”or” (i.e., the union of fuzzy sets [17]). Therefore, if the degrees of impairment are combined with a t-conorm S, then $S(a, b)$ gives a numerical value to the fact that ”an individual i has an impairment with degree a **or** another one with degree b” (for a given disjunction, which may or may not be intuitively understandable). This clearly does not seem to be the intention of the corresponding authorities, neither should be the numerical interpretation that ”an individual i has an impairment with degree a **and** another one with degree b” (for a given conjunction). The problem is neither of disjunction nor of conjunction: it is a problem of aggregation according to some previously accepted principles.

Therefore assuming that the corresponding authorities intend to make explicit the related principles, then these should not be like (P1), (P2), (P3) and (P4) or (P1), (P2*) and (P4*) since they necessarily lead to the t-conorm $Prod^*$. Neither should they be any other criteria leading to some other t-conorm. The reasons were discussed above. The authorities, however, may choose from a huge family of aggregation functions [6], that may well help to solve the problem, starting from previously accepted criteria. In fact (see [18]), some authors plead in favour of a possibility they call the ”salomonic solution”, given by $S(a, b) = Min(1, a + b - ab/2)$. This function satisfies (P1), (P3) and (P4*) but neither (P2) nor (P2*) and it is not associative. Therefore it is not a t-conorm, even though the boundaries $Prod^* < S < W^*$ are valid. Since S is not associative it has the draw back that it cannot be directly defined for more than two degrees of impairment. On the other hand, it has the advantage that $S(a, b) = 1$ if and only if $ab/2 \cdot \; a + b - 1$.

The principles that lead to $Prod^*$ or W^* as solutions are really too simple. The problem of finding the combined value of several degrees of impairment should follow the basic principle explained below:

(P0) There exists a list of m impairments, of which an individual may effectively suffer from $n \cdot \; m$ of them, in a given order.

Then for every natural number $n \cdot \; m$, there are functions $A_n : [0, 1]^n \to [0, 1]$ of n variables, each one in $[0, 1]$, and representing in decimal form percentages of impairment. Furthermore these functions should satisfy a set of principles previously accepted and should be aggregation functions, i.e., they should exhibit the following properties:

(A1) If for all $1 \cdot \; i \cdot \; n, x_i = 1$, then $A_n(1, 1, , 1) = 1$

(A2) If for all $1 \cdot \; i \cdot \; n, x_i = 0$, then $A_n(0, 0, , 0) = 0$

(A3) A_n grows with all its variables: from $x_1 \cdot \; y_1, \dots, x_n \cdot \; y_n$ follows that
$A_n(x_1, x_2, \dots, x_n) \cdot \; A_n(y_1, y_2, \dots, y_n)$

(A4) $Max(x_1, x_2, \dots, x_n) \cdot \; A_n(x_1, x_2, \dots, x_n)$ for any $x_j \in [0, 1]$, $1 \cdot \; j \cdot \; n$, where $Max(x_1, x_2, \dots, x_n) = A_n(x_1, x_2, \dots, x_n)$ should not always be the case.

Remark: (A4) is not a general requirement for aggregations, but is stated in [14] as a requirement for a formula to combine degrees of impairment.

It seems that the following three principles should belong to the set of principles that should constitute the basis to solve the problem:

(P1***) Continuous functions should be used to combine several degrees of impairment. If not, a small variation in one degree may eventually lead to a relatively high variation of the combined value

(P2***) If for some j, $1 \cdot \; j \cdot \; n, x_j = 1$, then $A_n(x_1, x_2, \dots, x_j, \dots, x_n)$ should take the value 1, i.e, $A_n(x_1, x_2, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n) = 1$

(P3***) If for some j, $1 \cdot \; j \cdot \; n$, $x_j = 0$, then the value of $A_n(x_1, x_2, \dots, x_j, \dots, x_n)$ equals that of the function A_{n-1} without x_j, i.e., $A_n(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = A_{n-1}(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$

It should be noticed, that (A4) implies (P2***). From $Max(x_1, x_2, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n) \cdot A_n(x_1, x_2, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n)$ follows that $A_n(x_1, x_2, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n) = 1$, since $Max(x_1, x_2, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_n) = 1$. The reciprocal is however not necessarily true.

In the case of principles related to "commutativity" (as (P3) in section 3) and to the global increment by a growing variation of a degree of impairment (similar to (P2) and (P2*) in section 3), it becomes fairly obvious that their meaning related to the problem to be solved should be thoroughly discussed before accepting types of principles like, for instance:

(P4***) $A_n(x_1, x_2, \ldots, x_j, \ldots, x_n) = A_n(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(j)}, \ldots, x_{\sigma(n)})$, for any permutation σ of the indices $1, 2, \ldots, n$, e.g., $A_n(x_1, x_2, \ldots, x_j, \ldots, x_n) = A_n(x_j, x_2, \ldots, x_1, \ldots, x_n)$

(P5***) $A_n(x_1 + h, x_2, \ldots, x_j, \ldots, x_n) - A_n(x_1, x_2, \ldots, x_j, \ldots, x_n) = A_n(h, x_2, \ldots, x_j, \ldots, x_n) - A_n(0, x_2, \ldots, x_j, \ldots, x_n)$, with $x_1 + h \cdot 1$

Since $Prod^*$ and W^* are associative, it is simple to consistently define $Prod * (x_1, x_2, \ldots, x_j, \ldots, x_n)$ and $W^*(x_1, x_2, \ldots, x_j, \ldots, x_n)$, which satisfy (A1) through (A3), thus being aggregations. Furthermore they satisfy (A4), (P4***) and, with adequate constraints (in the case of W*), (P5***). Hence they give a solution to the general problem of combining degrees of impairment. However (P4***) is a matter of high controversy, since the order of appearance of impairments may be of great relevance (see e.g. [5]).

5 Conclusions

The index of Balthazard, which has been used to develop (restricted precision) tables to combine degrees of impairment [10], [14], is equivalent to sets of well defined principles as shown in section 3. Due to their mathematical equivalence with the index, such principles are implicitly accepted when using the index, even if the principles are not explicitly declared.

The only principle explicited in [14] refers to the requirement that the combined degree of two impairments a and b should be strictly greater than the Max of these degrees. However, the fact that $\widetilde{B}(a, b) > Max(a, b)$ follows, on the one side, from the principles that lead to $\widetilde{B}$ being a t-conorm (as is the case with $\widetilde{B} = Prod*$) and, on the other side, represents such a general property, that allows for many other possible solutions and leads non necessarily to the index of Balthazard. Similarly, the property $S(a,b) = 1$ if and only if $Max(a,b) = 1$ does not lead to the index $\widetilde{B}$.

The non-associativity of the table in [10] -reproduced in [14] without further explanations- contributes uncertainty in the case of patients with more than two impairments. It seems mandatory, that the instructions to operate with the table illustrate the kind of rounding to be used in the case of more than two degrees of impairment, or that a solution as the one disclosed in section 2 (ii), (see also [9]), is adopted. (Recall, however, the closing remark of section 2 (ii)).)

Even though the extension via associativity of a t-conorm S to a function $S(a_1, \ldots, a_n)$ with more than two variables is an aggregation function (as shown in section 4) and furthermore is strictly larger than the function $Max(a_1, \ldots, a_n)$, there are many other n-dimensional aggregation functions $A_n : [0,1]^n \to [0,1]$ such that $A_n(a_1, \ldots, a_n) > Max(a_1, \ldots, a_n)$, which are available to the corresponding authorities. Because of this, and of the fact that neither commutativity nor associativity seem to be properties naturally bounded to the combination or aggregation of degrees of impairment, a review of the underlying principles is adviceable. Such a review may or may not confirm the sets {(P1), (P2), (P3), (P4)} or {(P1), (P2*), (P4*)} and should indeed be done previously to adopting an aggregation method or a formula to combine degrees of impairment, be it either a general one or a personalized one.

It is scientifically non-acceptable, technically risky, legally uncertain and therefore a source of conflicts, to assign a computing procedure which is not clearly based on explicit principles, that allow understanding what is obtained in each case, i.e., understanding the meaning of $A_n(a_1, \ldots, a_n)$. It is not a surprise that the many court trials originated by the application of the index of Balthazard (see [4], [5], [16] as minimal sample of the many examples available on the Web) have lead to partial reconsiderations, as in [18], [9] or to the rejection of any *general* mathematical procedure, as in Germany [4].

It may finally be concluded, that there seem to be enough reasons for authorities, medical associations or insurance companies that might have rather uncritically adopted the formula of Balthazard and, particularly, its table form, to take proper actions allowing experts in the process of aggregation of numerical information to reconsider the theoretical problem of calculating the combined grade of several degrees of impairment. This seems possible to be done, as discussed in section 4. It remains however an interesting open problem, which according to the authors is typical of what is called "fuzzy modeling" [11].

6 Acknowledgements

The authors thank Josep Fargas (University Pompeu Fabra, Barcelona), Julio López de la Paz (Health Institute Carlos III, Madrid), Eloy Renedo and Sergio Guadarrama (Technical University of Madrid) both for their effective assistance in the bibliographic search for this work and for valuable discussions on the subject.

This work was partially supported by the project TIC-2000-1420, CICYT (Spain). The work of C. Moraga was partially supported by the Spanish State Secretary of Education and Universities of the Ministry of Education, Culture and Sports (Grant SAB2000-0048), and by the Social Fund of the European Community.

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7 Appendix

Definition: A function $S : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if it is associative, commutative, continuous and monotone increasing in both arguments, and furthermore has two special elements:

- Neutral element 0. $S(0,a) = a$, for any $a \in [0,1]$
- Absorbent element 1. $S(1,a) = 1$, for any $a \in [0,1]$

A more compact definition may be given [2]. A function $S : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if it is associative, continuous and monotone increasing in both arguments, and has an absorbent element 1.

Illustration that $Min(1, x+y)$ does not satisfy (P2) for all $x, y \in [0,1]$.
(P2) states that for any $a, b, c \in [0,1]$ with $a + c \cdot 1$, $S(a+c, b) - S(a,b) = S(c,b) - S(0,b)$.
Let $z \in (0,1]$ be such that $x + z \cdot 1, y + z \cdot 1$ but $x + y \geq 1$. Then in the case of $Min(1, x+y)$, (P2) would read

$$Min(1, x+y+z) - Min(1, x+y) = Min(1, z+y) - Min(1, 0+y).$$

The left hand side reduces to $1 - 1 = 0$ meanwhile the right hand side gives $z + y - y = z$, i.e., $z = 0$ would be the condition for (P2) to be satisfied; but since z was chosen to be different from 0 the contradiction shows that (P2) does not hold for wide ranges of x, y and z.