

Hausdorffness in Intuitionistic Fuzzy Topological Spaces

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Abstract

The basic concepts of the theory of intuitionistic fuzzy topological spaces have been defined by D. Çoker and co-workers. In this paper, we define new notions of Hausdorffness in the intuitionistic fuzzy sense, and obtain some new properties, in particular on convergence.

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The introduction of "intuitionistic fuzzy sets" is due to K.T. Atanassov [1], and this theory has been developed by many authors [2-4]. In particular D. Çoker has defined the intuitionistic fuzzy topological spaces, and several authors have studied this category [5-14].

Nevertheless, separation in intuitionistic fuzzy topological spaces is not studied. Only there exists a definition due to D. Çoker:

Definition 1 [5] *An IFTS (X, τ) is called Hausdorff iff $x_1, x_2 \in X$ and $x_1 \neq x_2$ imply that there exist $G_1 = \langle x, \mu_{G_1}, \gamma_{G_1} \rangle$, $G_2 = \langle x, \mu_{G_2}, \gamma_{G_2} \rangle \in \tau$ with*

$$\begin{aligned} \mu_{G_1}(x_1) = 1, \quad \gamma_{G_1}(x_1) = 0 \\ \mu_{G_2}(x_2) = 1, \quad \gamma_{G_2}(x_2) = 0 \quad \text{and} \quad G_1 \cap G_2 = \emptyset. \end{aligned}$$

But this definition is not suitable for to characterize net convergence in intuitionistic fuzzy topological spaces. Then, in this paper, we will define two new kinds of intuitionistic fuzzy topological spaces and obtain various properties.

Firstly, we list some previous definitions:

Definition 2 [7] *Let X be a nonempty set and $c \in X$ a fixed element in X . If $\alpha \in (0, 1]$ and $\beta \in [0, 1)$ are two fixed real numbers such that $\alpha + \beta = 1$, then the IFS*

$$c(\alpha, \beta) = \langle x, c_\alpha, 1 - c_{1-\beta} \rangle$$

is called an intuitionistic fuzzy point (IFP for short) in X .

If $\beta \in [0, 1)$ is a fixed real number, then the IFS

$$c(\beta) = \langle x, 0, 1 - c_{1-\beta} \rangle$$

is called a vanishing intuitionistic fuzzy point (VIFP for short) in X .

Definition 3 [7]

(a) *Let $c(\alpha, \beta)$ be an IFP in X such that $\alpha, \beta \in (0, 1)$ and $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFS in X . $c(\alpha, \beta)$ is said to be properly contained in A ($c(\alpha, \beta) \in A$ for short) if $\alpha < \mu_A(c)$ and $\beta > \gamma_A(c)$.*

(b) *Let $c(\beta)$ be a VIFP in X such that $\beta \in (0, 1)$ and $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFS in X . $c(\beta)$ is said to be properly contained in A ($c(\beta) \in A$ for short) if $\mu_A(c) = 0$ and $\beta > \gamma_A(c)$.*

Definition 4 [7]. *Let (X, τ) be an IFTS on X and N be an IFS in X . N is said to be an ε -neighborhood of an IFP $c(\alpha, \beta)$ in X if there exists an IFOS G in X such that $c(\alpha, \beta) \in G \subseteq N$. N is said to be an ε -neighborhood of a VIFP $c(\beta)$ in X if $\mu_N(c) = 0$ and there exists an IFOS G in X such that $c(\beta) \in G \subseteq N$.*

Definition 5 [13]. *Let X be a non-empty set, \mathcal{P} the set of all IFPs and VIFPs of X and D a directed set. An intuitionistic fuzzy net is a map $s : D \rightarrow \mathcal{P}$. We denote $s_d = s(d)$ (for $d \in D$), $s = (s_d)_{d \in D}$.*

Definition 6 [13] *Let (X, τ) be an IFTS and s be an intuitionistic fuzzy net in X . s converges to an IFP (or a VIFP) p in (X, τ) if for every ε -neighborhood N of p there exists $d_0 \in D$ such that $s_d \in N$ for all $d \geq d_0$.*

Definition 7 [13] *Let X be a non-empty set and \mathcal{F} a non-empty family of IFSSs $\neq 0_{\sim}$. \mathcal{F} is an intuitionistic filter in X if*

- a) *for all $F_1, F_2 \in \mathcal{F}$ is $F_1 \cap F_2 \in \mathcal{F}$.*
- b) *for all $F \in \mathcal{F}$ and each IFSS F' such that $F \subseteq F'$ is $F' \in \mathcal{F}$.*

Definition 8 [13] *Let (X, τ) be an IFTS and \mathcal{F} be an intuitionistic filter in X . \mathcal{F} converges to an IFP p in (X, τ) if every ε -neighborhood of p is member of \mathcal{F} .*

Remark. *If $p = c(\alpha, \beta)$ is an IFP in an IFTS (X, τ) and M is an IFOS in X such that $\mu_M(c) = 1, \gamma_M(c) = 0$ it does not imply that M is an ε -neighborhood of p (let, for example $\alpha = 1, \beta = 0$). This fact motives that the Çoker's definition of Hausdorffness intuitionistic fuzzy topological spaces is not suitable for to characterize convergence in intuitionistic fuzzy topological spaces.*

We will introduce some new kinds of Hausdorffness in intuitionistic fuzzy topological spaces.

Definition 9 *An IFTS (X, τ) will be called T_2 if for every p, q IFPs or VIFPs in X such that $p \neq q$ there exists ε -neighborhoods M and N of p and q respectively such that $M \cap N = 0_{\sim}$.*

Clearly, for $c(\alpha, \beta)$ we discard $\alpha = 1, \beta = 0$, and for $c(\beta)$ the case $\beta = 0$.

Proposition 1. *Let (X, τ) be an IFTS. Then (X, τ) is T_2 if and only if every convergent intuitionistic fuzzy net in (X, τ) has an only limit.*

Proof. If $(s_d)_{d \in D}$ converges to p and q , two IFPs or VIFPs such that $p \neq q$, then for every ε -neighborhoods M and N of p and q respectively there are d_M, d_N such that $s_d \in M$ for $d \geq d_M$ and $s_d \in N$ for $d \geq d_N$. Thus for some $d_0 \geq d_M, d_N$ is $s_d \in M \cap N$ for all $d \geq d_0$, and $M \cap N \neq 0_{\sim}$ [13, Proposition 1].

Conversely, if (X, τ) is not T_2 there exist two IFPs or VIFPs p and q such that $p \neq q$ and, for all ε -neighborhoods M, N of p and q respectively is $M \cap N \neq 0_{\sim}$. Then, there is either an IFP or VIFP $s_{(M,N)} \in M \cap N$ [13, Proposition 1]. Let $D = \{(M, N) | M \text{ } \varepsilon\text{-neighborhood of } p \text{ and } N \text{ } \varepsilon\text{-neighborhood of } q\}$ directed by \subseteq . Then the intuitionistic fuzzy net $(s_{(M,N)})_{(M,N) \in D}$ converges to p and q . ■

Definition 10 (15) A fuzzy topological space (X, T) is said to be fuzzy Hausdorff if for any two fuzzy points x_r, y_s with distinct supports there exist disjoint $\mu, \nu \in T$ with $x_r \in \mu$ and $x_s \in \nu$

(In this definition the value of any fuzzy point x_r is such that $0 < r < 1$).

Proposition 2. Let (X, τ) be an IFTS. If (X, τ) is T_2 , then (X, τ_1) is a fuzzy Hausdorff fts (where $\tau_1 = \{\mu_G | G \in \tau\}$).

Proof. For any two fuzzy points x_r, y_s with distinct supports and $0 < r, s < 1$, we have that $p = x(r, 1 - r)$, $q = y(s, 1 - s)$ are two distinct IFPs. Then, there exist ε -neighborhoods M and N of p and q respectively such that $M \cap N = 0_\sim$. This implies that $r < \mu_M(x)$, $s < \mu_N(y)$ and $x_r \in \mu_M, y_s \in \mu_N$ which are fuzzy neighborhoods with $\mu_M \wedge \mu_N = \emptyset$. ■

Proposition 3. Let (X, τ) be an IFTS. If (X, τ) is T_2 every convergent intuitionistic filter in (X, τ) has an only limit.

Proof. If \mathcal{F} converges to p and q two distinct IFPs, then the intuitionistic fuzzy net associated to \mathcal{F} , $s_{\mathcal{F}}$ converges to p and q [13, Theorem] and (X, τ) is not T_2 .

(The converse is not true because the convergence of intuitionistic filters is defined only to IFPs). ■

Definition 11 An IFTS (X, τ) will be called $q-T_2$ if for every distinct IFPs or VIFPs p and q in X there exist ε -neighborhoods M and N of p and q respectively such that $\mu_M \quad \mu'_N$ and $\gamma_M \geq \gamma'_N$.

Proposition 4. Every T_2 IFTS is $q-T_2$.

Proof. Let p, q be two distinct IFPs nor VIFPs then, there exist ε -neighborhoods M, N of p and q respectively such that $M \cap N = 0_\sim$. Since we have that $\mu_M \wedge \mu_N = \emptyset, \gamma_M \vee \gamma_N = 1$, this implies that $\mu_M \quad \mu'_N$ and $\gamma_M \geq \gamma'_N$. ■

Definition 12 A fuzzy topological space (X, T) will be called q -fuzzy Hausdorff if for any two fuzzy points x_r, y_s with distinct supports there exist $\mu, \nu \in T$ with $x_r \in \mu, y_s \in \nu$ and $\mu \quad 1 - \nu$.

(In this definition also the value of any fuzzy point x_r is such that $0 < r < 1$).

Proposition 5. *Let (X, τ) be an IFTS. If (X, τ) is $q-T_2$ then (X, τ_1) is a q -fuzzy Hausdorff fts.*

Proof. For any two points x_r, y_s with distinct supports and $0 < r, s < 1$, we have that $p = x(r, 1 - r), q = y(s, 1 - s)$ are two distinct IFPs. Then, there exist ε -neighborhoods M and N of p and q respectively such that $\mu_M \cap \mu'_N = \emptyset$ and $\gamma_M \geq \gamma'_N$. This implies that $x_r \in \mu_M, y_s \in \mu'_N$ which are fuzzy neighborhoods with $\mu_M \cap \mu'_N = \emptyset$. ■

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