Mathware & Soft Computing 8 (2001) 227-237

On Dispersion Measures

J. Martín, G. Mayor and J. Suñer
Dept. of Mathematics and Computer Science.
University of the Balearic Islands
07071-Palma de Mallorca
e-mail: {javier.martin,gmayor,jaume.sunyer}@uib.es

Abstract

In this paper a new framework for the study of measures of dispersion for a class of n-dimensional lists is proposed. The concept of monotonicity with respect to a "sharpened"-type order is introduced. This type of monotonicity, together with other well known conditions, allows to create a reasonable and general ambit where the notion of dispersion measure can be studied. Some properties are analized and relations with other approaches carried out by different authors on this subject are established.

Key words: Entropy, uncertainty measures, "sharpened" order, dispersion, Schur concavity, quasi-concavity.

1 Introduction

Shannon defined in 1948 ([5]) the first probabilistic uncertainty measure in the framework of communication theory. According to this author, the uncertainty (or entropy) of a random experiment can be measured by means of $H(P) = -\sum_{i=1}^{n} p_i \log p_i$, where the values p_i are the probabilities of the possible results of the experiment. Since then there have been many works which have been motivated in some sense by this idea from Shannon: some of them deal with the characterization of Shannon measure by means of a small set of conditions, other studies try to generalize this measure by considering families of functions which, in particular, contain Shannon measure. Recent papers of Morales, Pardo and Vajda ([4]), Couso and Gil ([1]) deal with both aspects appearing, among others, conditions of concavity and Schur concavity. In different fields from the stochastic (discrete) systems, concentration measures have been studied (in Economics) by using families of functions which contain, in particular, the ones of the type $S(x) = \gamma \sum_{i=1}^{n} x_i \log x_i$ ($\gamma > 0$, const.) (see [3]); in this paper by Gehrig (1984) some conditions related with the concavity (Schur concavity, quasi-concavity) are introduced. On the other hand, DeLuca and Termini introduce in 1972 a non probabilistic definition of entropy ([2]) by considering the concept of measure of "fuzziness" for fuzzy sets using a condition of monotonicity with respect to a partial order defined on the class of fuzzy sets (sharpened order). In Yager ([7]) we can find Shannon entropy measures as well as measures of the type $1 - \max[x_i]$ for weighting lists corresponding to OWA operators. O'Hagan (1987) studies the problem of the optimization of the Shannon entropy (maximum dispersion) to determine weighting lists with a fixed degree of optimism ("orness"). We plan in this paper a definition of dispersion measure applied to a class of n-dimensional lists which can be understood as probabilistic distributions, weighting lists, etc. We use, besides the usual conditions of symmetry and extremal values, a condition of monotonicity with respect to a partial order of "sharpened" type which shows a priori "a list x of less dispersion than another list y". Following this idea (see [6]) and after giving in the next section the necessary background, these measures are studied, establishing relations with other families defined through some concavity condition.

2 Preliminaries

Let us consider, for each $n \geq 2$, the following set L formed by n-dimensional lists:

$$L = \{x = (x_1, \ldots, x_n) \in [0,1]^n : \sum_{i=1}^{n} x_i = 1\}.$$

Observe that L is a convex subset of $[0,1]^n$; more precisely, it is the convex closure of the lists $\delta_1 = (1,0,\ldots,0), \delta_2 = (0,1,0,\ldots,0), \ldots, \delta_n = (0,\ldots,0,1)$ written as $L = <\delta_1, \ldots, \delta_n>$.

The sharpened order is defined as follows:

Definition 1 *Given two lists $x, y \in L$, we will say that $x \leq_d y$ if, and only if, for each $i = 1, \ldots, n$ we have $x_i \leq y_i \leq \frac{1}{n}$ or $x_i \geq y_i \geq \frac{1}{n}$.*

We obtain immediately:

Proposition 1 *$\leq_d$ is a partial order on L.*

In figure 1 we represent the set L for the case $n = 3$,which is the triangle of vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. The three striped zones correspond to the points which are greater than each one of these vertices, with respect to the order $\leq_d$. We represent as well a generic list x and its coordinates x_1, x_2 and x_3.

Observe that $\leq_d$ has a maximum element which is the list $(\frac{1}{n}, \ldots, \frac{1}{n})$ and that, if we call $weight(x) = |\{i : x_i \neq 0\}|$, for each $x \in L$, then x is minimal if, and only if, $weight(x) < n$ and $x_i > \frac{1}{n}$ for each $x_i \neq 0$.

Given $x \in L$ such that $x_1 \leq \ldots \leq x_n$, let $p_x = \max\{i : x_i \leq \frac{1}{n}\}$. Then $1 \leq p_x \leq n$, and $p_x = n$ if, and only if, $x = (\frac{1}{n}, \ldots, \frac{1}{n})$.

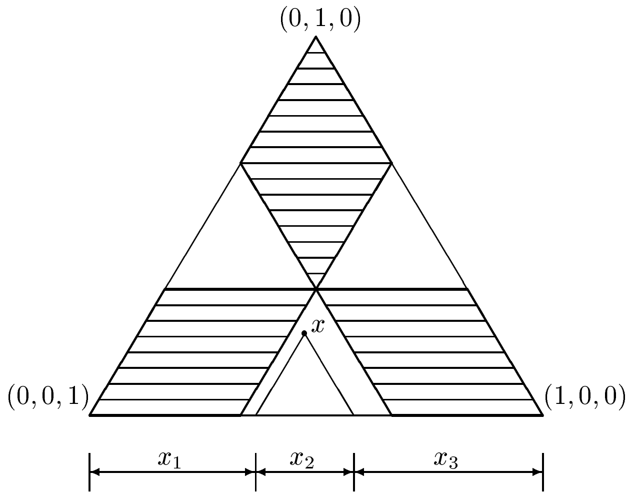

Figure 1: Order $\leq_d$ for the case $n = 3$

Given $x \in L$, we will indicate the list formed by the elements of x written in ascending order by $x^* = (x_{(1)}, \dots, x_{(n)})$, where, for each $i = 1, \dots, n$,

$$x_{(i)} = \min\Big\{\max\{x_{j_1}, \dots, x_{j_i}\} : 1 \leq j_1 \leq \cdots \leq j_i \leq n\Big\}.$$

Proposition 2 *The following properties hold:*

1) If $x \leq_d y$ then $x^* \leq_d y^*$.

2) For lists written in ascending order, it holds:
$x \leq_d y$ if, and only if, $p_x \leq p_y$ and

$$(x_1, \dots, x_{p_x}) \leq_\pi (y_1, \dots, y_{p_x})$$

and

$$(y_{p_x+1}, \dots, y_n) \leq_\pi (x_{p_x+1}, \dots, x_n),$$

where $\leq_\pi$ is the usual product order in $[0,1]^{p_x}$ and $[0,1]^{n-p_x}$, respectively.

3) $x \leq_d y \Longleftrightarrow (1-a)x + ay \leq_d (1-b)x + by$ for all $0 \leq a \leq b \leq 1$.

Proof:
The properties *2)* and *3)* are easy to prove.
To prove *1)*, we must show two things:

a) If $x_{(i)} \leq y_{(i)}$, then $y_{(i)} \leq \frac{1}{n}$.

b) If $x_{(i)} \geq y_{(i)}$, then $y_{(i)} \geq \frac{1}{n}$.

Let us put $x_{(i)} = \max\{x_{p_1}, \ldots, x_{p_i}\} = x_{p_r}$ and $y_{(i)} = \max\{y_{q_1}, \ldots, y_{q_i}\} = y_{q_s}$. Then, for all $1 \leq j_1 \leq \cdots \leq j_i \leq n$, $x_{p_r} \leq \max\{x_{j_1}, \ldots, x_{j_i}\}$ and $y_{p_r} \leq \max\{y_{j_1}, \ldots, y_{j_i}\}$.

a) Let us suppose now that $x_{p_r} \leq y_{q_s}$. Then, if we call $y_{p_t} = \max\{y_{p_1}, \ldots, y_{p_i}\}$, we will have $x_{p_t} \leq x_{p_r} \leq y_{q_s} \leq y_{p_t}$, thus $y_{p_t} \leq \frac{1}{n}$ and $y_{q_s} \leq \frac{1}{n}$.

b) If $x_{p_r} \geq y_{q_s}$, a similar reasoning shows that $y_{q_s} \geq \frac{1}{n}$.

$\diamondsuit$

In this paper we are interested in the study of a class of functions $D : L \longrightarrow I\!R$. In [4] we can find a list of interesting properties for this type of functions to represent an "entropy". In this sense, we would like to recall the following concepts:

Definition 2 *a) Given two lists $x, y \in L$, we will say that x dominates y, indicated by $x \succ y$, if there exists a doubly stochastic $n \times n$ matrix A such that $y = x\,A$.*

b) We will say that a function $D : L \longrightarrow I\!R$ is Schur concave if $D(x) \leq D(y)$ whenever $x \succ y$.

c) Given two lists $x, y \in L$, we will say that y is a smoothing of x, written $y = Sm(x)$, if there exist j, k with $1 \leq j \neq k \leq n$, such that $y_i = x_i \quad \forall i \in \{1, \ldots, n\} \setminus \{j, k\}$ and $|x_j - x_k| \geq |y_j - y_k|$.

d) We will say that a function $D : L \longrightarrow I\!R$ is weakly monotone if $D(x) \leq D(y)$ whenever $y = Sm(x)$.

Let us recall that a square matrix is doubly stochastic if the sum of the elements of each row and each column equals 1.

It is interesting to observe that none of the previous correspondences between lists (a and c) is an order relation.

We have the following result (see [4]):

Proposition 3 *D is Schur concave if, and only if, it is weakly monotone.*

An interesting property for our study is the following:

Definition 3 *A function $D : L \longrightarrow I\!R$ is quasi-concave if*

$$D((1-a)x + ay) \geq \min(D(x), D(y))$$

for all $x, y \in L$ and $a \in [0, 1]$,

Remarks: Let us consider $D : L \longrightarrow I\!R$.

1) If D is concave (that is, $D((1-a)x + ay) \geq (1-a)D(x) + aD(y)$ for all $a \in [0, 1]$), then D is quasi-concave.

2) Let us consider $L_c = \{x \in L : D(x) \geq c\}$ (the c-cut of D) for each $c \in I\!R$. Then:

a) If $c \leq c'$, then $L_c \supset L_{c'}$,

b) L_c is a convex set for each $c \in I\!R$ if, and only if, D is quasi-concave.

The concept of c-cut provides an interesting result to check whether a function $D : L \longrightarrow I\!R$ is Schur concave. Given $x = (x_1, \ldots, x_n) \in L$, we will write $\Pi(x)$ to indicate the convex closure of $(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for any permutation π of $1, \ldots, n$. We need previously two results to make the dominance relation and the order $\leq_d$ easier to use.

Proposition 4 *If $x \leq_d y$, then $y \in \Pi(x)$.*

Proof:
We will divide the proof into two parts:

1) Let us consider $x = (x_1, \ldots, x_{p_x} | x_{p_x+1}, \ldots, x_n)$. Let us take $y \geq x$ of the form $y = (x_1, \ldots, x_r + \delta, \ldots, x_{p_x} | x_{p_x+1}, \ldots, x_s - \delta, \ldots, x_n)$, where $\delta > 0$, $0 < x_r + \delta < \dfrac{1}{n}$ and $\dfrac{1}{n} < x_s - \delta < 1$. Then y is a linear convex combination of x and $x' = (x_1, \ldots, x_s, \ldots, x_r, \ldots, x_n)$; more precisely, $y = \lambda x + (1 - \lambda) y$ with $\lambda = 1 - \dfrac{\delta}{x_s - x_r} \in [0, 1]$.

2) In general, given $y = (y_1, \ldots, y_{p_y} | y_{p_y+1}, \ldots y_n)$, let us consider $x \leq_d y$. Then we can write

$$x = (y_1 - \delta_1, \ldots, y_{p_y} - \delta_{p_y} | y_{p_y+1} + \delta_{p_y+1}, \ldots, y_n + \delta_n),$$

where $\delta_i > 0 \;\; \forall i = 1, \ldots, n, \;\; \sum_{i=1}^{p_y} \delta_y = \sum_{i=p_y+1}^{n} \delta_i$ and, obviously,

$$y_i - \delta_i < \frac{1}{n} \;\; \forall i = 1, \ldots, p_y,$$

and

$$y_i + \delta_i > \frac{1}{n} \;\; \forall i = p_y + 1, \ldots, n.$$

The procedure followed is to elliminate the δ_i one by one by applying the way shown in the first part of the proof, ending with x as a linear convex combination of y and all its permutations.

Let $\delta_r = \min\{\delta_1, \ldots, \delta_n\}$. We can consider two cases:

a) If $r \leq p$, let $\delta_s = \max\{\delta_{p_y+1}, \ldots, \delta_n\}$.

Then, if $x_1 = (y_1 - \delta_1, \ldots, y_r, \ldots, y_s + \delta_s - \delta_r, \ldots, y_n + \delta_n)$ and $x' = (y_1 - \delta_1, \ldots, y_s + \delta_s, \ldots, y_r - \delta_r, \ldots, y_n + \delta_n)$, we will have that x_1 is a linear convex combination of x and x'.

Let us put now $\delta_s := \delta_s - \delta_r$ and, thus, $x_1 = (y_1 - \delta_1, \ldots, y_r, \ldots, y_{p_y} - \delta_{p_y} | y_{p_y+1} + \delta_{p_y+1}, \ldots, y_s + \delta_s, \ldots, y_n + \delta_n)$.

b) If $r > p$, let $\delta_s = \max\{\delta_1, \ldots, \delta_{p_y}\}$.

Then, if $x_1 = (y_1 - \delta_1, \ldots, y_s - \delta_s + \delta_r, \ldots, y_r, \ldots, y_n + \delta_n)$ and $x' = (y_1 - \delta_1, \ldots, y_r + \delta_r, \ldots, y_s - \delta_s, \ldots, y_n + \delta_n)$, we have that x_1 is a linear convex combination of x and x'.

Let us put now $\delta_s := \delta_s - \delta_r$ and, thus, $x_1 = (y_1 - \delta_1, \ldots, y_s - \delta_s, \ldots, y_{p_y} - \delta_{p_y} | y_{p_y+1} + \delta_{p_y+1}, \ldots, y_r, \ldots, y_n + \delta_n)$.

We repeat now the previous procedure with x_1. This procedure will finish with $\delta_i = 0$ for all i, that is, we will obtain y and we will have proved that y is an element of $\Pi(x)$. ♢

Proposition 5 *Given $x, y \in L$, we have:*

$$x \succ y \Longleftrightarrow y \in \Pi(x).$$

Proof:
Let us suppose first that $x \succ y$. Then there exists a doubly stochastic matrix A such that $y = x\,A$. But we know that $A = \sum\limits_i \lambda_i A_i$, where $\sum\limits_i \lambda_i = 1$, $\lambda_i \geq 0$, and A_i are the matrices of the permutations. Then $y = x\left(\sum\limits_i \lambda_i A_i\right) = \sum\limits_i \lambda_i x\,A_i = \sum\limits_i \lambda_i \pi_i(x)$, where π_i indicate all the permutations of $1, \ldots, n$. Thus $y \in \Pi(x)$.

To prove the reciprocal, it is sufficient to repeat the previous steps but in reverse order, observing that $A = \sum\limits_i \lambda_i A_i$ is a doubly stochastic matrix. ♢

Now we can state the proposition which relates Schur concavity with the c-cuts.

Proposition 6 *A function $D : L \longrightarrow \mathbb{R}$ is Schur concave if, and only if, $\Pi(x) \subset L_{D(x)}$ for any $x \in L$.*

Proof:
Let us suppose first that D is Schur concave and let us take $x \in L$. Now let $y \in \Pi(x)$. Then we have $x \succ y$ by proposition 5 and $D(x) \leq D(y)$, thus $y \in L_{D(x)}$ because D is Schur concave.

Reciprocally, let us suppose that $x \succ y$ and we will prove that $D(x) \leq D(y)$. But, again by proposition 5, we will have $y \in \Pi(x)$ and, by hypothesis $y \in L_{D(x)}$. Then $D(y) \geq D(x)$ and D is Schur concave. ♢

3 Dispersion measures

Definition 4 *A function $D : L \longrightarrow \mathbb{R}$ is a dispersion measure if it satisfies the following conditions:*

1) *Symmetry: $D(x_1, \ldots, x_n) = D(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for each $x \in L$ and for every permutation π of $1, \ldots, n$.*

2) Extremal values:

$$D(1,0,\ldots,0) \leq D(x_1,\ldots,x_n) \leq D(\frac{1}{n},\ldots,\frac{1}{n})$$

for any $(x_1,\ldots,x_n) \in L$.

3) Monotonicity: if $x \leq_d y$*, then* $D(x) \leq D(y)$.

Next we present a set of examples of dispersion measures, showing in each case their range. The first eight dispersions are quasi-concaves. The dispersion measure of the example nine is neither quasi-concave nor Schur concave.

Examples:

1) $D(x_1,\ldots,x_n) = 1 - \bigvee_{i=1}^{n} x_i$. In this case, range $= [0, \frac{n-1}{n}]$.

2) $D(x_1,\ldots,x_n) = \bigwedge_{i=1}^{n} x_i$. Range $= [0, \frac{1}{n}]$.

3) $D_3(x_1,\ldots,x_n) = 1 - (\bigvee_{i=1}^{n} x_i - \bigwedge_{i=1}^{n} x_i)$. Range $= [0,1]$.

4) $D_4(x_1,\ldots,x_n) = -\sum_i x_i \log x_i$ $(0 \log 0 = 0)$ (Shannon entropy).
   Range $= [0, \log n]$.

5) $D(x_1,\ldots,x_n) = 1 - \sum_i x_i^2$. Range $= [0, \frac{n-1}{n}]$.

6) $D(x_1,\ldots,x_n) = 1 - \sum_i |x_i - \frac{1}{n}|$. Range $= [\frac{2-n}{n}, 1]$.

7) $D_a(x_1,\ldots,x_n) = \frac{1}{(1-a)} \log \sum_i x_i^a$, where $a > 0, a \neq 1$, (a-order Rényi entropy). Range $= [0, \log n]$.

8) $D(x_1,\ldots,x_n) = \log(\text{weight}(x))$ (Hartley entropy).
   Range $=\{0, \log 2, \ldots, \log n\}$.

9) $D(x_1,\ldots,x_n) = x_{(1)} + \ldots + x_{(p_x)}$. Range $= [0,1]$.
   Observe that D is neither quasi-concave nor Schur concave: If we take $x = (0.1, 0.1, 0.1, 0.3, 0.4)$ and $y = (0.05, 0.05, 0.05, 0.2, 0.65)$, then $D(\frac{1}{2}x + \frac{1}{2}y) = 0.225 < \min\{D(x), D(y)\} = 0.3$ and the function is not quasi-concave. To prove that it is not Schur concave, it is sufficient to take $x = (0.1, 0.3, 0.6)$ and its "smoothed" $y = (0.1, 0.4, 0.5)$.

On the other hand, it is well known that the Shannon entropy is the limit when $a \to 0$ of the Rényi entropy, whereas the Hartley entropy is the limit when $a \to 1$.

Next proposition shows an interesting inequality relation between the dispersion measure $D_3(x_1,\ldots,x_n) = 1 - (\bigvee_{i=1}^{n} x_i - \bigwedge_{i=1}^{n} x_i)$ and the Shannon entropy $D_4(x_1,\ldots,x_n) = -\sum_i x_i \log x_i$ $(0 \log 0 = 0)$.

Proposition 7 *Let $x \in L$. Then:*

a) If $weight(x) = 1$, $D_3(x) = D_4(x) = 0$.

b) If $weight(x) = 2$, $D_3(x) \leq \frac{1}{\log 2} D_4(x)$.

c) If $weight(x) \geq 3$, $D_3(x) < D_4(x)$.

Proof:
a) Obvious.

b) Let us consider two cases:
Case 1: $n = 2$.
Then $x = (a, 1-a)$ and $D_3(x) = 2\min\{a, 1-a\}$. The concavity of D_4 and the fact that $D_3(\frac{1}{2}, \frac{1}{2}) = D_4(\frac{1}{2}, \frac{1}{2}) = 1$ assure the inequality.
Case 2: $n \geq 3$.
In this case, $D_3(0, \ldots, a, \ldots, 1-a, \ldots, 0) = \min\{a, 1-a\} \leq 2\min\{a, 1-a\} = D_3(a, 1-a) \leq \frac{1}{\log 2} D_4(a, 1-a) = \frac{1}{\log 2} D_4(0, \ldots, a, \ldots, 1-a, \ldots, 0)$.

c) Observe previously that

- If $x \in [0, \frac{1}{e}]$, then $x \leq -x \log n$.
- If $x \in [\frac{1}{e}, 1]$, then $\frac{1-x}{e-1} \leq -x \log n$.

Let us consider now three cases, depending on whether the number of components greater than $\frac{1}{e}$ are 0, 1 or 2 (it can not be greater than 2 as $3\frac{1}{e} > 1$).
Let us put $a = \bigwedge x_i$ and $b = \bigvee x_i$.
Case 1: If all the components are less than or equal to $\frac{1}{e}$, we have $D_4(x) = -\sum_i x_i \log x_i \geq \sum_{i=1}^{n} x_i = 1 \geq 1-(b-a) = D_3(x)$ because $x_i \in [0, \frac{1}{e}]$, $\forall i = 1, \ldots, n$.
Case 2: If only one component is greater than $\frac{1}{e}$ (the maximum b), since $D_3(x) = 1-(b-a)$ and $D_4(x) = -\sum_i x_i \log x_i = -\sum_{x_i \neq b} x_i - b\log b \geq \sum_{x_i \neq b} x_i + \frac{1-b}{e-1} = 1 - b + \frac{1-b}{e-1} = \frac{e-eb}{e-1}$, we have to prove that $1-(b-a) \leq \frac{e-eb}{e-1}$ or, equivalently, $a(e-1) + b \leq 1$, and this is true because $a(n-1) + b \leq \sum_i x_i = 1$ and $n \geq 3$.

Case 3: If two components are greater than $\frac{1}{e}$, $b \geq b' > \frac{1}{e}$, since $D_3(x) = 1-(b-a)$ and $D_4(x) = -\sum_i x_i \log x_i = -\sum_{x_i \neq b, b'} x_i - b\log b - b'\log b' \geq \sum_{x_i \neq b, b'} x_i + \frac{1-b}{e-1} + \frac{1-b'}{e-1} = 1 - b - b' + \frac{2b'-b}{e-1}$, we have to prove that $1 - b + a \leq 1 - b - b' + \frac{2b'-b}{e-1}$ or, equivalently, $(e-1)a + eb' + b \leq 2$.
We know that $a \leq 1-b'-b$ and thus $(e-1)a + eb' + b \leq (e-1)(1-b'-b) + eb' + b = e - 1 + b' + (2-e)b$, which has to be less than or equal to 2. But this is true because $b' + (2-e)b \leq b + (2-e)b = (3-e)b \leq 3-e$. $\diamondsuit$
¿From proposition 1, it is immediate to prove the following result:

Proposition 8 *If $D : L \longrightarrow [0, 1]$ is symmetric and it satisfies the monotonicity condition for lists in ascending order, then it is monotone.*

Next proposition analyzes transformations of dispersion measures through increasing functions.

Proposition 9 *1) Let D be a dispersion measure and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ an increasing function. Then $D'(x) = \phi(D(x))$ is also a dispersion measure.*

2) If D is quasi-concave and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a increasing function, then $D'(x) = \phi(D(x))$ is also quasi-concave.

In this case, we will say that D' is the ϕ-transformed of D.

Remarks:

1) According to this proposition, we have that, if D is a quasi-concave dispersion measure, then its ϕ-transformed is also a quasi-concave dispersion measure.

2) Given a dispersion measure D, we can normalize it in such a way that $D(1, 0, \ldots, 0) = 0$ and $D(\frac{1}{n}, \ldots, \frac{1}{n}) = 1$, by simply defining a new dispersion measure of the form:

$$\tilde{D}(x_1, \ldots, x_n) = \frac{D(x_1, \ldots, x_n) - D(1, 0, \ldots, 0)}{D(\frac{1}{n}, \ldots, \frac{1}{n}) - D(1, 0, \ldots, 0)}$$

in the case that $D(1, 0, \ldots, 0) < D(\frac{1}{n}, \ldots, \frac{1}{n})$. Observe that $\tilde{D}$ is the ϕ-transformed of D for $\phi(x) = \dfrac{x - D(1, 0, \ldots, 0)}{D(\frac{1}{n}, \ldots, \frac{1}{n}) - D(1, 0, \ldots, 0)}$.

3) In the case $n = 2$, a dispersion measure D is simply a function $f : [0, 1] \longrightarrow \mathbb{R}$ which is symmetric with respect to $x = 1/2$ (that is, $f(x) = f(1 - x)$ for all x), increasingly monotone on $[0, 1/2]$ and decreasingly monotone on $[1/2, 1]$.

Proposition 10 *If D is Schur concave, then D is symmetric, it satisfies the extremal values condition and it is monotone.*

Remark: This result says that the uncertainty measures in the sense of [4] are dispersion measures. As proved in [3], concavity plus symmetry imply Schur concavity. We will see later that we do not need to take concave functions, it is sufficient that they are quasi-concaves (and symmetric) to obtain the Schur concavity. In particular, we will have that if we impose quasi-concavity to the dispersion measures, then we obtain a more restrictive concept than the uncertainty measures.

Proof:
The proof of the symmetry and the extremal values can be found in [4]. Let us prove now that every uncertainty measure D (that is, Schur concave) is monotone.

Let us suppose that $x \leq_d y$. Then proposition 4 asserts that $y \in \Pi(x)$ and according to proposition 5, $x \succ y$. Since D is Schur concave, we will have $D(x) \leq D(y)$. $\diamondsuit$

Next proposition proves, as mentioned above, that every symmetric, quasi-concave function is an uncertainty measure in the sense of [4].

Proposition 11 *If D is symmetric and quasi-concave, then it is Schur concave.*

Proof:
According to proposition 6, we want to prove that $\Pi(x) \subset L_{D(x)}$ for all $x \in L$. But, due to the symmetry of D, $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in L_{D(x)}$ for any permutation π of $1, \ldots, n$. Since $L_{D(x)}$ is a convex set because D is quasi-concave and $\Pi(x)$ is the minor convex set containing x and all the points $(x_{\pi(1)}, \ldots, x_{\pi(n)})$, we will have that $\Pi(x) \subset L_{D(x)}$. ◊

As a consequence of the last two propositions, we have the following result:

Proposition 12 *If D is symmetric and quasi-concave, then it is monotone with respect to the order $\leq_d$ and it satisfies the extremal values condition.*

Remark: We have then proved that every quasi-concave dispersion measure is an uncertainty measure.

Next we will give some examples which prove that the Schur concavity and the quasi-concavity are not related, in general.

1) The Schur concavity does not imply the quasi-concavity, as it can be seen for the case $n = 3$ with the following function:

$$D(x_1, x_2, x_3) = \begin{cases} 3\min\{x_1, x_2, x_3\} \\ \qquad \text{if } \ p_x \geq 2 \\ \\ \max\Big\{0, 2 - 3\max\{x_1, x_2, x_3\}\Big\} \\ \qquad \text{if } \ p_x = 1 \end{cases}$$

Observe that D is not quasi-concave, since if we take $x = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ and $y = \left(\frac{1}{6}, \frac{5}{12}, \frac{5}{12}\right)$, then $D(x) = D(y) = 0.75$ and, on the other hand, in the medium point $D\left(\frac{5}{24}, \frac{8}{24}, \frac{11}{24}\right) = 5/8 < 0.75$.

On the other hand, it is easy to prove that D is weakly monotone and thus Schur concave.

2) The quasi-concavity does not imply the Schur concavity, as it can be seen by taking any projection $p_i(x_1, \ldots, x_n) = x_i$.

3) Finally, observe that if D is symmetric and quasi-concave, it is not necessarily concave. To prove this fact, we can take the function $D(x, y) = e^{-(x^2+y^2)}$.

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