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Averaging Premises*

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Abstract

This paper deals with the sets of strict conjectures and consequences of a given collection P of premises. The set of Averaging Functions is introduced on lattices and some properties of these functions are shown. Averaging Functions allow to interpret "restricted consequences" as averages of premises. The subset of consequences $C_g^*(P)$ and the subset of conjectures $\Phi_g^*(P)$ defined by means of the averaging function g are introduced, and their properties are studied. This sets allow to give decomposition theorems for the restricted consequences and for the strict conjectures.
Key words: consequences, conjectures, averaging functions.

1 Introduction

Let it be L a complete orthocomplemented lattice (see [1] and [4]) with its three operations of intersection, union and complementation represented, respectively, by $\cdot$, $+$ and $'$, the least element represented by 0, and the greatest element by 1. $\mathbb{P}_0(L)$ designates the subset of $\mathbb{P}(L)$ whose elements P verify $\text{Inf}(P) \neq 0$. From now on and if there is no confusion, we will write $x_\wedge = x_1 \cdot x_2 \cdots x_k$ and $x_\vee = x_1 + x_2 + \cdots + x_k$ and denote $\mathbb{P}_{0f}(L)$ the set of those $P \in \mathbb{P}_0(L)$ that are finite subsets of L; that is $P \in \mathbb{P}_{0f}(L)$ if and only if $P = \{p_1, \ldots, p_k\}$ with $p_\wedge = \text{Inf}\, P \neq 0$. Let's write $\{p_1, \ldots, p_k\} = P_k$.

Given a complete orthocomplemented lattice L and a collection of premises $P \in \mathbb{P}_0(L)$, in [4] where introduced the sets:

- $\Phi_\wedge(P) = \{x \in L;\ p_\wedge \not\leq x'\}$, of strict conjectures of P

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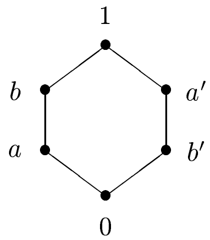
- $C_\wedge(P) = \{x \in L;\, p_\wedge \le x\}$, of consequences of P,

and it was proven that for any finite set P_k in $\mathbb{P}_{0f}(L)$ it is:

$$\Phi_\wedge(P_k) = \bigcap_{g \in G_k(L)} \Phi_g(P_k), \text{ and } C_\wedge(P_k) = \bigcup_{g \in G_k(L)} C_g(P_k),$$

where $G_k(L) = \{g : L^k \to L; \wedge \le g\}$, $\Phi_g(P_k) = \{x \in L; g(p_1, \dots, p_k) \not\le x'\}$, $C_g(P_k) = \{x \in L; g(p_1, \dots, p_k) \le x\}$, with $\wedge$ the function $\wedge(x_1, \dots, x_k) = x_1 \cdot x_2 \cdots x_k$. It was also proven that $C_\wedge(P_k) = \{g(p_1, \dots, p_k); g \in G_k(L)\}$ and, consequently, when the set of premises is finite, a complete characterization of consequences by means of functions belonging to $G_k(L)$ was obtained.

For example, let L be the hexagonal orthocomplemented lattice shown in the following figure. If $P = \{b\}$, for each $g \in G_1$ it is $C_g(P) = \{x \in L; g(b) \le x\} = \{1, b\}$, since $g(b) \ge p_\wedge = b$.



Hence, $C_\wedge(P) = \bigcup_{g \in G_1} C_g(P) = \{1, b\}$. With regard to conjectures of P, as $g(b) \ge b$, the following cases are possible:

- if $g(b) = b$ it is $\Phi_g(P) = \{1, a, a', b\}$,
- if $g(b) = 1$ it is $\Phi_g(P) = L_0$;

hence $\Phi_\wedge(P) = \bigcap_{g \in G_1} \Phi_g(P) = \{1, a, a', b\}$.

2 Averaging functions in lattices

In what follows we will consider the subset $A_k(L) = \{g : L^k \to L; \wedge \le g \le \vee\}$ of $G_k(L)$, where $\vee$ denotes the function $\vee(x_1, \dots, x_k) = x_1 + \cdots + x_k$. The elements of $A_k(L)$ are k-dimensional Averaging Functions of L or, for short, Averaging Functions. The sets $A_k(L)$ and $G_k(L)$ can be defined on any not orthocomplemented lattice. For example, if $L = [0, 1]$, with $\cdot$ = Min, and $+$ = Max, the elements of $A_k([0, 1])$ are the well known k-dimensional means (see [2] and [3])

For any $g \in A_k(L)$ and for each $x \in L$ it is $g(x, \dots, x) = x$; in particular is $g(0, \dots, 0) = 0$ and $g(1, \dots, 1) = 1$. Of course, it is $A_1 = \{id_L\}$ as if $g \in A_1$

it is $x \leq g(x) \leq x$. Note that functions $g \in A_k(L)$ can not be decreasing (that is, if $x_i \leq y_i$, for all $i \in \{1, \ldots, k\}$, then $g(x_1, \ldots, x_k) \geq g(y_1, \ldots, y_k)$) since $g(0, \ldots, 0) = 0$ and $g(1, \ldots, 1) = 1$.

Functions $g : L^k \to L$ such that $g(x_1, \ldots, x_k) = \bigwedge_{j \in J} \bigvee_{i \in I_j} x_i$, or $g(x_1, \ldots, x_k) = \bigvee_{j \in J} \bigwedge_{i \in I_j} x_i$ with $I_j \subset \{1, 2, \ldots, k\}$ and $J \subset \mathbb{N}$ finite, belong to $A_k(L)$ and are non-decreasing, that is, if $x_i \leq y_i$ $(1 \leq i \leq k)$ then $g(x_1, \ldots, x_k) \leq g(y_1, \ldots, y_k)$.

Functions $g : L^k \to L$ with terms in which there is the complement x'_j of some variable x_j, either can belong or not to $A_k(L)$; for example, if $k = 3$, $g(x_1, x_2, x_3) = x'_1 \cdot x_2 + x_3$ belongs to $A_3(L)$, but $g(x_1, x_2, x_3) = x'_1 + x_2 \cdot x'_3$ and $g(x_1, x_2, x_3) = x'_1 \cdot x_2 + x'_3$ do not belong to $A_3(L)$ as in both cases it is $g(0, 0, 0) = 1$. Furthermore, these functions can be or not non-decreasing; for example, if L is the hexagonal orthocomplemented lattice mentioned above, the function given by $g_1(x_1, x_2) = x_1 + x'_1 \cdot x_2$ belongs to $A_2(L)$ and is non-decreasing as it can be checked, while the function defined as $g_2(x_1, x_2) = x_1 \cdot x'_2 + x_1 \cdot x_2$ belongs to $A_2(L)$ and is not non-decreasing since $g(b, 0) = b$ and $g(b, a) = a$. Nevertheless, both functions are non-decreasing if L is any Boolean Algebra; in fact, the first one is $g_1(x_1, x_2) = x_1 + x'_1 \cdot x_2 = x_1 + x_1 \cdot x_2 + x'_1 \cdot x_2 = x_1 + (x_1 + x'_1) \cdot x_2 = x_1 + x_2$, and the second one is $g_2(x_1, x_2) = x_1 \cdot x'_2 + x_1 \cdot x_2 = x_1(x'_2 + x_2) = x_1$. Note that $g_1(x_1, x_2) \neq x_1 + x_2$ in the hexagonal lattice (for example $g_1(a, b) = a < b = a + b$).

For each function $g : L^k \to L$ its dual $g^* : L^k \to L$ is defined by $g^*(x_1, \ldots, x_k) = g(x'_1, \ldots, x'_k)'$, and it is immediate that "$g \in A_k(L)$ if and only if $g^* \in A_k(L)$". If we denote, as it is usual, by π_i the k functions (projections) given by $\pi_i(x_1, \ldots, x_k) = x_i$, as $x_1 \cdots x_k \leq x_i \leq x_1 + \cdots + x_k$, it is $\pi_i \in A_k(L)$ for any $1 \leq i \leq k$, and it is also obvious that $\pi_i = \pi_i^*$.

If $s \in \mathcal{S}(k)$ is a permutation of $\{1, 2, \ldots, k\}$, to each $g : L^k \to L$ we can associate the function $g^s : L \to L$ given by $g^s(x_1, \ldots, x_k) = g(x_{s(1)}, \ldots, x_{s(k)})$. For example, it is $\pi_i^s = \pi_{s(i)}$, for any $s \in \mathcal{S}(k)$. A function $g : L^k \to L$ is symmetrical if for any $s \in \mathcal{S}(k)$ it is $g = g^s$. Obviously, $\wedge^s = \wedge$ and $\vee^s = \vee$ for any $s \in \mathcal{S}(k)$; furthermore, functions of $A_k(L)$ defined as $g(x_1, \ldots, x_k) = \sum_{\substack{i,j=1 \\ i \neq j}}^{k} x_i \cdot x_j$ and $g(x_1, \ldots, x_k) = \sum_{\substack{i,j,h=1 \\ i \neq j, i \neq h, j \neq h}}^{k} x_i \cdot x_j \cdot x_h$ are symmetrical too. Functions with terms in which there is the complement x'_j of some variable x_j can be also symmetrical, for example $g \in A_3(L)$ given by $g(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x'_2 \cdot x'_3 + x'_1 \cdot x_2 \cdot x'_3 + x'_1 \cdot x'_2 \cdot x_3$.

For any $P \in \mathbb{P}_0(L)$ it was also considered in [4] the set of **restricted** consequences, $C(P) = \{x \in L; p_\wedge \leq x \leq p_\vee\} \subset C_\wedge(P)$, with $p_\vee = \operatorname{Sup} P$, and the first goal of this paper is to study the relationship of restricted consequences and strict conjectures with, respectively, the sets $C_g(P_k)$ and $\Phi_g(P_k)$ when g belongs to the before mentioned subset $A_k(L)$ of $G_k(L)$.

3 Restricted consequences are averages of premises

For each $P_k \in \mathbb{P}_0(L)$ let's consider $A_k(L)$ and $C_{A_k}(P_k) = \{g(p_1, \dots, p_k); g \in A_k(L)\}$. As $A_k(L) \subset G_k(L)$ it is $C_{A_k}(P_k) \subset C_{G_k}(P_k) = \{g(p_1, \dots, p_k); g \in G_k(L)\} = C_\wedge(P_k)$ (see [4]): all the elements of $C_{A_k}(P_k)$ are consequences of P_k. As $p_\wedge \leq g(p_1, \dots, p_k) \leq p_\vee$, it is $C_{A_k}(P_k) \subset C(P_k)$: all the elements of $C_{A_k}(P_k)$ are restricted consequences of P_k (see [4]). More again,

Theorem 3.1. For any $P_k \in \mathbb{P}_0(L)$ it is

$$C_{A_k}(P_k) = C(P_k)$$

Proof. For each $q \in C(P_k)$ let's consider the function $g_q : L^k \to L$ defined by

$$g_q(x_1, \dots, x_k) = \begin{cases} q & \text{if } x_1 = p_1, \dots, x_k = p_k \\ x_\wedge & \text{otherwise.} \end{cases}$$

As it is obvious that $g_q \in A_k(L)$ and that $g_q(p_1, \dots, p_k) = q$, it follows $q \in C_{A_k}(P_k)$. Then $C(P_k) \subset C_{A_k}(P_k)$ ■ Consequently, **the restricted consequences of $\mathbf{P_k}$ are exactly those consequences of $\mathbf{P_k}$ that are averages of the k premises**.

Examples. Let $E = \{1, 2 \dots, 5, 6\}$ and let $L = \mathbb{P}(E)$ be the corresponding Boolean Algebra with 2^6 elements.

1. If $P = \{p_1, p_2, p_3\}$ with $p_1 = \{2, 4, 6\}$, $p_2 = \{1, 2, 3\}$ and $p_3 = \{2, 5\}$, it is $p_\wedge = \{2\} \neq \emptyset$; the function $g \in A_3$ defined by $g(x_1, x_2, x_3) = x_1' \cdot x_2 + x_3$ allows to obtain the element $x = \{1, 2, 3, 5\}$ as a consequence of P.

2. If $P = \{p_1, p_2\}$ with $p_1 = \{1, 2, 3, 4\}$ and $p_2 = \{4, 5, 6\}$, it is $p_1 \cdot p_2 = \{4\}$ and $x = \{3, 4, 5\} \in C_{A_2}(P)$ is obtained from function

$$g(x_1, x_2) = \begin{cases} x_1 \cdot x_2 & \text{if } (x_1, x_2) \neq (p_1, p_2), \\ x & \text{if } (x_1, x_2) = (p_1, p_2). \end{cases}$$

which belongs to A_2 since $x_1 \cdot x_2 \leq g(x_1, x_2) \leq x_1 + x_2$. However, there is not any function $g(x_1, x_2) = a \cdot x_1 \cdot x_2 + b \cdot x_1' \cdot x_2 + c \cdot x_1 \cdot x_2' + d \cdot x_1' \cdot x_2'$ with $a, b, c, d \in \{0, 1\}$ (that is, a boolean function) such that $g(p_1, p_2) = x$. Hence, **if functions considered are just boolean** functions, in general, **restricted consequences do not belong to $C_{A_2}(P)$**.

Theorem 3.2. (a) For any $P_k \in \mathbb{P}_0(L)$, $P_k \subset C_{A_k}(P_k)$.

(b) If $P_n, Q_m \in \mathbb{P}_0(L)$ and $P_n \subset Q_m$, then $C_{A_n}(P_n) \subset C_{A_m}(Q_m)$.

(c) For any $P_k \in P_0(L)$ it is $\text{Inf}\, C_{A_k}(P_k) = p_\wedge \neq 0$.

(d) For each $P_k \in \mathbb{P}_0(L)$ such that $C_{A_k}(P_k)$ is finite with m elements, it is

$$C_{A_k}(P_k) \supset C_{A_m}(C_{A_k}(P_k)).$$

Proof. Notwithstanding the proofs can directly follow from theorem 3.1 and the results obtained in [4], it is illustrative to show different ones by using averaging functions.

(a) As for each $p_i \in P_k$ it is $\pi_i(p_1, \ldots, p_k) = p_i$, it follows $p_i \in C_{A_k}(P_k)$.

(b) As $n \leq m$, let's suppose $m = n + r$. For each $g(p_1, \ldots, p_n) \in C_{A_n}(P_n)$ it is: $p_1 \cdots p_n \cdot p_{n+1} \cdots p_{n+r} \leq p_1 \cdots p_n \leq g(p_1, \ldots, p_n) \leq p_1 + \cdots + p_n \leq p_1 + \cdots + p_n + p_{n+1} + \cdots + p_{n+r}$. The function $f_g : L^{n+r} \to L$ given by $f_g(x_1, \ldots, x_{n+r}) = g(x_1, \ldots, x_n) + 0 \cdot x_{n+1} + \cdots + 0 \cdot x_{n+r}$ obviously verifies $x_1 \cdots x_{n+r} \leq x_1 \cdots x_n \leq g(x_1, \ldots, x_n) = f_g(x_1, \ldots, x_{n+r}) \leq x_1 + \cdots + x_n \leq x_1 + \cdots + x_{n+r}$, and then $f_g \in A_{n+r} = A_m$. Consequently, $g(p_1, \ldots, p_n) = f_g(p_1, \ldots, p_m) \in C_{A_m}(Q_m)$.

(c) $\text{Inf}\, C_{A_k}(P_k) = \text{Inf}\{g(p_1, \ldots, p_k); g \in A_k(L)\} = \text{Inf}\{g(p_1, \ldots, p_k); p_\wedge \leq g(p_1, \ldots, p_k) \leq p_\vee\} = p_\wedge$. Consequently, $C_{A_k}(P_k) \in \mathbb{P}_0(L)$.

(d) As $C_{A_k}(P_k)$ is finite, let it be $C_{A_k}(P_k) = \{g_1(p_1, \ldots, p_k), \ldots g_m(p_1, \ldots, p_k)\}$ and $g_i \in A_k(L)$ $(1 \leq i \leq k)$ with $m \geq k$. Then, any $q \in C_{A_m}(C_{A_k}(P_k))$ can be written as $q = G(g_1(p_1, \ldots, p_k), \ldots, g_m(p_1, \ldots, p_k))$ with $G \in A_m$. Now for each $G \in A_m$, let's consider the function $f_G : L^k \to L$ defined by:

$$f_G(x_1, \ldots, x_k) = G(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k)).$$

It is $f_G \in A_k(L)$; in fact, for any $i \in \{1, \ldots, m\}$ and for each $(x_1, \ldots, x_k) \in L^k$ it is

$$x_1 \cdots x_k \leq \bigwedge_{i=1}^{m} g_i(x_1, \ldots, x_k) \text{ and } \bigvee_{i=1}^{m} g_i(x_1, \ldots, x_k) \leq x_1 + \cdots + x_k,$$

and because of $y_1 \cdots y_m \leq G(y_1, \ldots, y_m) \leq y_1 + \cdots + y_m$ for each $(y_1, \ldots, y_m) \in L^m$, it suffices to take $y_i = g_i(x_1, \ldots, x_k)$ for each $1 \leq i \leq m$ to have

$$x_1 \cdots x_k \leq G(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k)) \leq x_1 + \cdots + x_k,\ \forall (x_1, \ldots, x_k) \in L^k.$$

That is, $x_1 \cdots x_k \leq f_G(x_1, \ldots, x_k) \leq x_1 + \cdots + x_k$, or $f_G \in A_k(L)$, and then

$$G(g_1(p_1, \ldots, p_k), \ldots, g_m(p_1, \ldots, p_k)) = f_G(p_1, \ldots, p_k) \in C_{A_k}(P_k).$$

Finally, $C_{A_m}(C_{A_k}(P_k)) \subset C_{A_k}(P_k)$. ∎ Consequently, if for each $P_k \in \mathbb{P}_0(L)$ we define $C_A(P_k) = C_{A_k}(P_k)$, then

Corollary 3.3. If L is finite, function $C_A : \mathbb{P}_0(L) \to \mathbb{P}_0(L)$, defined as $C_A(P_k) = C_{A_k}(P_k)$ is a Tarski's Consequences Operator.

Theorem 3.4. If $P \subset L$, either finite or not, belongs $\mathbb{P}_0(L)$, for each finite $P_k \subset P$ it is $C_A(P_k) \subset C(P)$.

Proof. Obvious because of $C_A(P_k) = C_{A_k}(P_k) = C(P_k) \subset C(P)$. ■ Consequently, by means of the operator C_A defined on $\mathbb{P}_0(L)$ some restricted consequences of any $P \in \mathbb{P}_0(L)$ can be obtained. In particular for any non-empty $F \subset A_k(L)$ it is $\{g(p_1 \ldots, p_k); g \in F\} \subset C(P)$.

4 Consequences and averages of premises

Let's consider the set $C_g^*(P_k) = \{x \in L; g(p_1, \ldots, p_k) \leq x\}$ for each $g \in A_k(L)$. Of course, as $A_k(L) \subset G_k(L)$ it is $C_g(P_k) = C_g^*(P_k)$ if $g \in A_k(L)$ and, then, it is $C_g^*(P_k) \subset C_\wedge(P_k)$. That is, any $x \in C_g^*(P_k)$ is a consequence of P_k. Again, $p_\wedge = \operatorname{Inf} C_\wedge(P_k) \leq \operatorname{Inf} C_g^*(P_k) = g(p_1, \ldots, p_k)$ implies $\operatorname{Inf} C_g^*(P_k) \neq 0$ and $C_g^*(P_k) \in \mathbb{P}_0(L)$. Obviously, if $g_1 \leq g_2$ (pointwise) for both $g_1, g_2 \in A_k(L)$ it is $C_{g_2}^*(P_k) \subset C_{g_1}^*(P_k)$; then, as it is $\wedge \leq g \leq \vee$ for any $g \in A_k(L)$, it follows $C_\vee(P_k) \subset C_g^*(P_k) \subset C_\wedge(P_k)$.

Theorem 4.1. For any $P_k \in \mathbb{P}_0(L)$, $C_\wedge(P_k) = \bigcup_{g \in A_k(L)} C_g^*(P_k)$.

Proof. As $C_g^*(P_k) \subset C_\wedge(P_k)$ it follows $\bigcup_{g \in A_k(L)} C_g^*(P_k) \subset C_\wedge(P_k)$. As $\wedge \in A_k(L)$ it also follows $C_\wedge(P_k) = C_\wedge^*(P_k) \subset \bigcup_{g \in A_k(L)} C_g^*(P_k)$. ■

Corollary 4.2. $x \in C_\wedge(P_k)$ if and only if **it exists some** $g \in A_k(L)$ **for which it is** $g(p_1, \ldots, p_k) \leq x$.

Theorem 4.3. For any $P_k \in \mathbb{P}_0(L)$, it is $P_k \subset C_g^*(P_k)$ if and only if $g(p_1, \ldots, p_k) = p_1 \cdots p_k$.

Proof. Of course, if $g(p_1, \ldots, p_k) = p_1 \cdots p_k$ it is $g(p_1, \ldots, p_k) \leq p_i$ for each i, and then $P_k \subset C_g^*(P_k)$. Let's suppose $P_k \subset C_g^*(P_k)$; that implies $g(p_1, \ldots, p_k) \leq p_i$ $(1 \leq i \leq k)$ and consequently $g(p_1, \ldots, p_k) = p_1 \cdots p_k$. ■ Then, if $p_1 \cdots p_k < g(p_1, \ldots, p_k)$, P_k is not included in $C_g^*(P_k)$. The theorem holds if in particular it is $g = \wedge$, in which case $C_\wedge^*(P_k) = \{x \in L; p_\wedge \leq x\}$.

Theorem 4.4. If $P_k, Q_{k+r} \in \mathbb{P}_0(L)$, and $P_k \subset Q_{k+r}$, then $C_g^*(P_k) \subset C_{\widetilde{g}}^*(Q_{k+r})$ with $\widetilde{g} \in A_{k+r}$ defined as $\widetilde{g}(x_1, \ldots, x_{k+r}) = g(x_1, \ldots, x_k) + 0 \cdot x_{k+1} + \ldots + 0 \cdot x_{k+r}$.

Proof. From $x_1 \cdots x_{k+r} \leq x_1 \cdots x_k \leq g(x_1, \ldots, x_k) \leq x_1 + \cdots + x_k \leq x_1 + \cdots + x_{k+r}$, it follows $x_1 \cdots x_{k+r} \leq \widetilde{g}(x_1, \ldots, x_{k+r}) \leq x_1 + \ldots + x_{k+r}$. Then if $x \in C_g^*(P_k)$, from $g(p_1, \ldots, p_k) \leq x$ it follows $\widetilde{g}(p_1, \ldots, p_{k+r}) \leq x$, or $x \in C_{\widetilde{g}}^*(Q_{k+r})$. ■ Clearly the theorem can be proven by means of any $\widetilde{g} \in A_{k+r}$ whose restriction to L^k coincides with g.

Corollary 4.5. If $P_k \in \mathbb{P}_0(L)$, $g \in A_k(L)$ is such that $g(p_1, \ldots, p_k) = p_1 \cdots p_k$, and $C_g^*(P_k)$ is finite, then $C_g^*(P_k) = C_{\widetilde{g}}^*(C_g^*(P_k))$, with $\widetilde{g}$ as in theorem 4.4.

Proof. It is $P_k \subset C_g^*(P_k)$ and if $C_g^*(P_k) = \{p_1, \ldots, p_k, q_1, \ldots, q_r\}$ it follows, from theorem 4.4, that $C_g^*(P_k) \subset C_{\widetilde{g}}^*(C_g^*(P_k))$ with $\widetilde{g}(x_1, \ldots, x_{k+r}) = g(x_1, \ldots, x_k) + 0 \cdot x_{k+1} + \ldots + 0 \cdot x_{k+r}$ and $\widetilde{g} \in A_{k+r}$. Reciprocally, if $y \in C_{\widetilde{g}}^*(C_g^*(P_k))$, or $\widetilde{g}(p_1, \ldots, p_k, q_1, \ldots, q_r) \leq y$, it is $g(p_1, \ldots, p_k) \leq y$, and $y \in C_g^*(P_k)$. ■

For $P_k = \{p_1, \ldots, p_k\}$ and $Q_k = \{q_1, \ldots, q_k\}$ let's define $P_k \leq Q_k$ whenever $p_1 \leq q_1, \ldots, p_k \leq q_k$ for these orderings of the elements of P_k and Q_k.

Theorem 4.6. If $g \in A_k(L)$ is non-decreasing, $P_k \leq Q_k$ and $P_k \in \mathbb{P}_0(L)$, then $Q_k \in \mathbb{P}_0(L)$ and $C_g^*(Q_k) \subset C_g^*(P_k)$.

Proof. If $x \in C_g^*(Q_k)$, from $g(q_1, \ldots, q_k) \leq x$ it follows $g(p_1, \ldots, p_k) \leq x$, as $p_1 \leq q_1, \ldots, p_k \leq q_k$. Then $x \in C_g^*(P_k)$. ■

Remark If g_1 and g_2 are in $A_k(L)$, $g_1 \leq g_2$ (pointwise) and $P_k \in \mathbb{P}_0(L)$, the set $C_{g_1,g_2}(P_k) = \{x \in L; g_1(p_1, \ldots, p_k) \leq x \leq g_2(p_1, \ldots, p_k)\}$ verifies $C_{g_1,g_2}(P_k) \subset C(P_k)$ as $p_1 \cdots p_k \leq g_1(p_1, \ldots, p_k) \leq x \leq g_2(p_1, \ldots, p_k) \leq p_1 + \ldots + p_k$. Obviously, $C(P_k) = C_{\wedge\vee}(P_k)$ is the particular case in which $\wedge = g_1$, $\vee = g_2$.

5 Strict Conjectures and averages of premises

For each $P_k \in \mathbb{P}_0(L)$ and each $g \in A_k(L)$ it is defined

$$\Phi_g^*(P_k) = \{x \in L; g(p_1, \ldots, p_k) \not\leq x'\}.$$

If $g_1, g_2 \in A_k(L)$ and $g_1 \leq g_2$ (pointwise) it is obvious that $\Phi_{g_1}^*(P_k) \subset \Phi_{g_2}^*(P_k)$. Then, as it is $\wedge \leq g \leq \vee$ for any $g \in A_k(L)$ it follows $\Phi_\wedge(P_k) \subset \Phi_g^*(P_k) \subset \Phi_\vee(P_k)$ and then:

$$\Phi_\wedge(P_k) = \bigcap_{g \in A_k(L)} \Phi_g^*(P_k) \text{ and } \Phi_\vee(P_k) = \bigcup_{g \in A_k(L)} \Phi_g^*(P_k).$$

Consequently, $x \in \Phi_\wedge(P_k)$ if and only if **for all** $g \in A_k(L)$ **it is** $g(p_1, \ldots, p_k) \not\leq x'$, and $x \in \Phi_\vee(P_k)$ if and only if **it exists some** $g \in A_k(L)$ **for which it is** $g(p_1, \ldots, p_k) \not\leq x'$.

Theorem 5.1. For each $g \in A_k(L)$, it is $P_k \subset \Phi_g^*(P_k)$.

Proof. If for some $p_i \in P_k$ it is $g(p_1, \ldots, p_k) \leq p_i'$ it will follow $p_1 \cdot \ldots \cdot p_k \leq p_i'$ and hence $p_\wedge = 0$ that is absurd. ■

Theorem 5.2. Given a family $\{g_k \; ; \; k \in \mathbb{N}, k \geq 2\}$ of functions $g_k : L^k \to L$ such that for each $k \geq 2$, each $r \geq 1$ and each $(x_1, \ldots, x_{k+r}) \in L^{k+r}$ verify

$$g_{k+r}(x_1, \ldots, x_{k+r}) \leq g_k(x_1, \ldots, x_k),$$

it is $\Phi_{g_{k+r}}(P_{k+r}) \subset \Phi_{g_k}(P_k)$, for any P_k, $P_{k+r} \in \mathbb{P}_0(L)$ such that $P_k \subset P_{k+r}$.

Proof. If $x \in \Phi_{g_{k+r}}(P_{k+r})$ it is $g_{k+r}(p_1, \ldots, p_{k+r}) \not\leq x'$ and $g_k(p_1, \ldots, p_k) \not\leq x'$. Then $x \in \Phi_{g_k}(P_k)$. ■ In particular, when $g_k(x_1, \ldots, x_k) = \bigwedge_{i=1}^{k} x_i$ it results $\Phi_\wedge(P_{k+r}) \subset \Phi_\wedge(P_k)$.

An example of such a family $\{g_k\}$ is given by $g_k(x_1, \ldots, x_k) = x_1 \cdot x_2 \cdots x_k + x_1' \cdot x_2 \cdots x_k + x_1 \cdot x_2' \cdots x_k + \cdots + x_1 \cdots x_{k-1} \cdot x_k'$. This is a family of functions that, if L is a Boolean Algebra, follows the law of recurrence: $g_2(x_1, x_2) = x_1 \cdot x_2 + x_1' \cdot x_2 + x_1 \cdot x_2'$ and if $k > 2$, $g_k(x_1, \ldots, x_k) = x_k \cdot g_{k-1}(x_1, \ldots, x_{k-1}) + x_1 \cdots x_{k-1} \cdot x_k'$.

Theorem 5.3. Let it be a family $\{g_k\}$ of functions $g_k : L^k \to L$ such that for each $k \geq 2$, each $r \geq 1$ and each $(x_1, \ldots, x_{k+r}) \in L^{k+r}$, verify:

$$g_k(x_1, \ldots, x_k) \leq g_{k+r}(x_1, \ldots, x_{k+r}).$$

It holds

$$\Phi_{g_k}(P_k) \subset \Phi_{g_{k+r}}(P_{k+r}),$$

for any couple P_k, $P_{k+r} \in P_0(L)$ such that $P_k \subset P_{k+r}$.

Proof. Like that of theorem 5.2. ■ If $g_k(x_1, \ldots, x_k) = x_1 + \cdots + x_k$ it is $\Phi_\vee(P_k) \subset \Phi_\vee(P_{k+r})$.

Theorem 5.4. If $g \in A_k(L)$ is non-decreasing, $P_k \leq Q_k$ and $P_k \in P_{0,f}(L)$, then $Q_k \in P_0(L)$ and $\Phi_g^*(P_k) \subset \Phi_g^*(Q_k)$.

Proof. If $x \in \Phi_g^*(P_k)$ and $g(q_1, \ldots, q_k) \leq x'$, as $g(p_1, \ldots, p_k) \leq g(q_1, \ldots, q_k)$ it will follow $g(p_1, \ldots, p_k) \leq x'$, that is absurd. Then $x \in \Phi_g^*(Q_k)$ ■.

Example. Let's consider the hexagonal orthocomplemented lattice of the first example. With $P = \{a, b\}$, then $p_\wedge = a$. If $g \in A_2$ it is $a \leq g(a, b) \leq b$, and the following cases are possible:

- if $g(a, b) = a$ it is $C_g^*(P) = \{1, a, b\}$ and $\Phi_g(P) = \{1, a, b\}$,
- if $g(a, b) = b$ it is $C_g^*(P) = \{1, b\}$ and $\Phi_g(P) = \{1, a, a', b\}$;

hence $C_\wedge(P) = \bigcup_{g \in A_2} C_g^*(P) = \{1, a, b\}$, $\Phi_\wedge(P) = \bigcap_{g \in A_2} \Phi_g(P) = \{1, a, b\}$ and $\Phi_\vee(P) = \bigcup_{g \in A_2} \Phi_g(P) = \{1, a, a', b\}$.

Conclusions

This work continues that initiated in [4], where structure theorems for both strict conjectures and restricted consequences were given by using functions in $G_k(L) = \{g : L^k \to L; \wedge \leq g\}$. This paper is restricted to functions in $A_k(L) = \{g : L^k \to L; \wedge \leq g \leq \vee\}$, the set of averaging functions.
It is shown that restricted consequences are just averages of the premises, and that the operator assigning to a family of premises P_k the set of their averages $C_{A_k}(P_k)$, is a Tarski's Consequences Operator. Moreover, sets of conjectures $\Phi^*_g(P_k) = \{x \in L; g(p_1, \ldots, p_k) \not\leq x'\}$ with $g \in A_k(L)$ are defined and initially studied.

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