Mathware & Soft Computing 8 (2001) 71-82

$(\top, \perp, N)$ Fuzzy Logic

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Abstract

To investigate more reasonable fuzzy reasoning model in expert systems as well as more effective logical circuit in fuzzy control, a $(\top, \perp, N)$ fuzzy logic is proposed in this paper by using $\top$-norm, $\perp$-norm and pseudo-complement N as the logical connectives. Two aspects are discussed: (1) some concepts of $(\top, \perp, N)$ fuzzy logic are introduced and some properties of $(\top, \perp, N)$ fuzzy logical formulae are discussed. (2) G-fuzzy truth (falsity) of $(\top, \perp, N)$ fuzzy logical formulae are investigated and also the relation between the Boolean truth (falsity) of $\perp$-normal forms ($\top$-normal forms) and the G-fuzzy truth (falsity) of them are analyzed.

Keywords: $(\top, \perp, N)$ fuzzy logic, $\top$-phrase, $\perp$-sentence, $\top(\perp)$-normal form, G-fuzzy truth (falsity)

1 Introduction

Fuzzy logical systems constituted by the logical operators triple $(\wedge, \vee, ^-)$ have been discussed by many researchers and also have been applied into uncertain reasoning of expert systems as well as logical circuit designing of fuzzy control. However, since "$\wedge$" and "$\vee$" are defined as *min* and *max* operators respectively, and "$^-$" is defined as $\bar{a} = 1 - a$ for any $a \in [0, 1]$, these logical operators are so rough that there exists much useful information being lost in the application. Hence, it is necessary to investigate the fuzzy logical system constituted by more general logical operators. As we knew that $\top$-norm, $\perp$-norm and pseudo-complement N are the generalization of $\wedge$, $\vee$ and $^-$ respectively, so it is suitable to extend the $(\wedge, \vee, ^-)$ fuzzy logical systems to $(\top, \perp, N)$ fuzzy logical systems. This paper will focus on establishing a fuzzy logical system with $\top$, $\perp$, and N as the logical connectives and discussing its some fundamental properties.

2 $(\top, \bot, N)$ Fuzzy Logical Formulae

In $(\wedge, \vee, ^-)$ fuzzy logical systems, the compositional fuzzy propositions [2, 17] by formalizing them into the corresponding $(\wedge, \vee, ^-)$ fuzzy logical formulae were studied. Similarly, in $(\top, \bot, N)$ fuzzy logical system, the compositional fuzzy propositions by using the corresponding $(\top, \bot, N)$ fuzzy logical formulae will be investigated. In the following, some basic concepts of $(\top, \bot, N)$ fuzzy logical formulae are firstly introduced.

Definition 2.1 *If the domain of definition of x is the closed interval $[0,1]$, then x is called a fuzzy variable.*

In this paper, we take $\{x_1, x_2, \ldots, x_n\}$ as the set of fuzzy variables.

Definition 2.2 *A mapping*

$$F : [0,1]^n \rightarrow [0,1]$$

is called a $(\top, \bot, N)$ fuzzy logical formula, if F is made up of $\{x_1, x_2, \ldots, x_n, 0, 1\}$ by using the finite $\top, \bot, N$ operators and the parentheses.

We write the set of all $(\top, \bot, N)$ fuzzy logical formulae as $\mathcal{F}^*$. From Definition 2.2, we have

(1) $0, 1, x_i \ (i = 1, 2, \ldots, n) \in \mathcal{F}^*$;

(2) If $F, F_1, F_2 \in \mathcal{F}^*$, then $F_1 \top F_2, F_1 \bot F_2, N(F) \in \mathcal{F}^*$, where $F_1 \top F_2, F_1 \bot F_2$ and $N(F)$ are called the $\top$-conjunction, the $\bot$-disjunction of F_1 and F_2 as well as the N-complement of F respectively.

Definition 2.3 *Let $A = (a_1, a_2, \ldots, a_n) \in [0,1]^n, F, F_1, F_2 \in \mathcal{F}^*$. We define:*

$$0(A) \triangleq 0, 1(A) \triangleq 1, x_i(A) \triangleq a_i (i = 1, 2, \ldots, n),$$

$$F_1 \top F_2(A) \triangleq F_1(A) \top F_2(A),$$

$$F_1 \bot F_2(A) \triangleq F_1(A) \bot F_2(A),$$

$$(N(F))(A) \triangleq N(F(A)).$$

Definition 2.4 *Let $F_1, F_2 \in \mathcal{F}^*$.*

$$F_1 \leq F_2 \overset{\triangle}{\Leftrightarrow} \forall A \in [0,1]^n, F_1(A) \leq F_2(A);$$

$$F_1 = F_2 \overset{\triangle}{\Leftrightarrow} F_1 \leq F_2 \text{ and } F_2 \leq F_1.$$

Definition 2.5 *Let $F \in \mathcal{F}^*$.*

$$F \text{ is called } G\text{-fuzzy true} \overset{\triangle}{\Leftrightarrow} \forall A \in [0,1]^n, F(A) \geq 0.5 \overset{\triangle}{\Leftrightarrow} F \geq 0.5;$$
$$F \text{ is called } G\text{-fuzzy false} \overset{\triangle}{\Leftrightarrow} \forall A \in [0,1]^n, F(A) \leq 0.5 \overset{\triangle}{\Leftrightarrow} F \leq 0.5.$$

Definition 2.6 *The fuzzy variable x or its N-complement $N(x)$ is called literal. The $\top$-conjunction P of finite literals is called a $\top$-phrase, denoted by $(\top)P$; The $\bot$-disjunction C of finite literals is called a $\bot$-sentence, denoted by $(\bot)C$; x and $N(x)$ are called a complementary pair; If the maximal number of times that the same literals occur in $(\top)P$ is K, then it is called a K-time $\top$-phrase; If the maximal number of times that the same literals occur in $(\bot)C$ is K, then it is called a K-time $\bot$-sentence.*

Definition 2.7 *Let $(\top)P_i (i = 1, 2, \dots, m)$ be $\top$-phrases. The $(\top, \bot, N)$ fuzzy logical formula in the form*

$$F_1 = (\top)P_1 \bot (\top)P_2 \bot \cdots \bot (\top)P_m$$

is called a $\bot$-normal form. Let $(\bot)C_i (i = 1, 2, \dots, m)$ be $\bot$-sentences. The $(\top, \bot, N)$ fuzzy logical formula in the form

$$F_2 = (\bot)C_1 \top (\bot)C_2 \top \cdots \top (\bot)C_m$$

is called a $\top$-normal form.

Definition 2.8 (1) *The pseudo-complement N is called regular $\overset{\triangle}{\Leftrightarrow} N(0.5) = 0.5$;*
(2) *$\bot$ is called regular $\overset{\triangle}{\Leftrightarrow} \forall a, b \in [0, 1), a \bot b < 1$;*
(3) *$\top$ is called regular $\overset{\triangle}{\Leftrightarrow} \forall a, b \in (0, 1], a \top b > 0$.*

Corollary 2.1 (1) *$\bot$ is regular $\Leftrightarrow$ if $a, b \in [0, 1]$ and $a \bot b = 1$, then $a = 1$ or $b = 1$;*
(2) *$\top$ is regular $\Leftrightarrow$ if $a, b \in [0, 1]$ and $a \top b = 0$, then $a = 0$ or $b = 0$;*

Proof. (1) ($\Leftarrow$) obviously. ($\Rightarrow$) If $a, b \in [0, 1]$, $a \neq 1$ and $b \neq 1$, then $a \bot b < 1$, which is a contradiction. (2) can be proved in the same way.

Obviously, the following properties hold:

(a) The special pseudo-complement "$-$" is regular.

(b) $\forall a, b \in [0, 1]$, let

$$a \top_0 b = a \wedge b, a \top_1 b = ab, a \top^{(\lambda)} b = ab/[\lambda + (1 - \lambda)(a + b - ab)] (\lambda > 0).$$

Then $\top_0, \top_1, \top^{(\lambda)}$ are all regular $\top$-norms.

(c) $\forall a, b \in [0, 1]$, let

$$a \bot_0 b = a \wedge b, a \bot_1 b = a + b - ab, a \bot^{(\lambda)} b = [a + b + (\lambda - 2)ab]/[1 + (\lambda - 1)ab] (\lambda > 0).$$

Then $\bot_0, \bot_1, \bot^{(\lambda)}$ are all regular $\bot$-norms.

Definition 2.9 *Set $(\top)\mathcal{F}^1$ as the set of all 1-time $\top$-phrases together with 1.*

$$(\top)P_1 \nabla (\top)P_2 \overset{\triangle}{=} (\top)P_3 (\in (\top)\mathcal{F}^1),$$

$$(\top)P_1\triangle(\top)P_2 \triangleq (\top)P_4(\in (\top)\mathcal{F}^1),$$

for any $(\top)P_1, (\top)P_2 \in (\top)\mathcal{F}^1$*, where, the set of literals occurring in* P_1 *is* L_1*, the set of literals occurring in* P_2 *is* L_2*.* $(\top)P_3$ *is a* $\top$*-phrase constituted from the literals in* $L_1 \cap L_2$ *by using* $\top$*,* $(\top)P_4$ *is a* $\top$*-phrase constituted from the literals in* $L_1 \cup L_2$ *by using* $\top$*.*

Stipulation: If $L = \emptyset$, then the $\top$-phrase corresponding to L is defined as 1.

Definition 2.10 *Set* $(\bot)\mathcal{F}^1$ *as the set of all* 1*-time* $\bot$*-sentences together with* 0*.*

$$(\bot)C_1 \sqcup (\bot)C_2 \triangleq (\bot)C_3(\in (\bot)\mathcal{F}^1),$$

$$(\bot)C_1 \sqcap (\bot)C_2 \triangleq (\bot)C_4(\in (\bot)\mathcal{F}^1),$$

for any $(\bot)C_1, (\bot)C_2 \in (\bot)\mathcal{F}^1$*, where, the set of literals occurring in* C_1 *and* C_2 *are* L_1 *and* L_2 *respectively.* $(\bot)C_3$ *is a* $\bot$*-sentence constituted from the literals in* $L_1 \cup L_2$ *by using* $\bot$*,* $(\bot)C_4$ *is a* $\bot$*-sentence constituted from the literals in* $L_1 \cap L_2$ *by using* $\bot$*.*

Stipulation: If $L = \emptyset$, then the $\bot$-sentence corresponding to L is defined as 0.

3 Properties of $(\top, \bot, N)$ fuzzy logical formulae

Theorem 3.1 *Let* $F_1, F_2 \in \mathcal{F}^*$*. Then*

$$F_1 \top F_2 \leq F_1 \wedge F_2 \leq F_1 \vee F2 \leq F_1 \bot F_2$$

Proof. It follows from the Definition 2.3 and Definition 2.4.

Theorem 3.2 *Let* $F, F_1, F_2, F3 \in \mathcal{F}^*$*. Then*

(1) $F_1 \top F_2 = F_2 \top F_1, F_1 \bot F_2 = F_2 \bot F_1$*;*
(2) $F_1 \bot (F_2 \bot F_3) = (F_1 \bot F_2) \bot F_3, F_1 \top (F_2 \top F_3) = (F_1 \top F_2) \top F_3$*;*
(3) $N(N(F)) = F$*;*
(4) *If* $F \leq F_1, F_2 \leq F_3$*, then* $F \bot F_2 \leq F_1 \bot F3, F \top F_2 \leq F_1 \top F_3$*;*
(5) $N(F_1 \bot F_2) \leq N(F_1) \wedge N(F_2)$*,* $N(F_1 \top F_2) \geq N(F_1) \vee N(F_2)$*;*
(6) $N(F_1 \bot F_2) = N(F_1) \top N(F_2) \Leftrightarrow N(F_1 \top F_2) = N(F_1) \bot N(F_2)$*;*
(7) $N(F_1 \vee F_2) = N(F_1) \wedge N(F_2) \Leftrightarrow N(F_1 \wedge F_2) = N(F_1) \vee N(F_2)$*;*
(8) *If* $\bot$ *is a* $\bot$*-norm [13] induced by* $\top$*,* N *is "* $-$ *", then*

$$N(F_1 \bot F_2) = N(F_1) \top N(F_2), N(F_1 \top F_2) = N(F_1) \bot N(F_2)$$

Proof. (1), (2), (3) and (4) follow from the definition of $\top$ and $\bot$. (7) follows from Theorem 2.1.7 in [10] and Definition 2.3 in Section 2. (5) can be proved from Theorem 3.1, the definition of N and (7). (8) follows from Theorem 2.3.3 in [10] and Definition 2.3 in Section 2. As for (6), we only prove ($\Rightarrow$), i.e.,

$$N(F_1 \bot F_2) = N(N(N(F_1)) \top N(N(F_1))) = N(N(N(F_1) \top N(F_2))) = N(F_1) \bot N(F_2).$$

($\Leftarrow$) can be proved in the same way.

Theorem 3.3 *If* $\forall F_1, F_2, F_3 \in \mathcal{F}^*, N(F_1 \perp F_2) = N(F_1) \top N(F_2)$. *Then the following statements are equivalent:*

(1) $F_1 \top (F_2 \perp F_3) = (F_1 \top F_2) \perp (F_1 \top F_3), F_1 \perp (F_2 \top F_3) = (F_1 \perp F_2) \top (F_1 \perp F_3)$;
(2) $F_1 \top F_1 = F_1, F_1 \perp F_1 = F_1$;
(3) $F_1 \perp (F_1 \top F_2) = F_1, F_1 \top (F_1 \perp F_2) = F_1$;
(4) $T = \wedge$ *and* $\perp = \vee$.

Proof. It follows from Theorem 2.3.5 in [10].

Theorem 3.4 (1) *In* $(\top, \perp, N)$ *fuzzy logic, we have*

$$(\top)P_1 \vee (\top)P_2 \vee \cdots \vee (\top)P_m \leq (\top)P_1 \perp (\top)P_2 \perp \cdots \perp (\top)P_m \leq (\wedge)P_1 \perp (\wedge)P_2 \perp \cdots \perp (\wedge)P_m;$$

$$(\perp)C_1 \wedge (\perp)C_2 \wedge \cdots \wedge (\perp)C_m \geq (\perp)C_1 \top (\perp)C_2 \top \cdots \top (\perp)C_m \geq (\vee)C_1 \top (\vee)C_2 \top \cdots \top (\vee)C_m.$$

(2) *In Boolean logic, we have*

$$\begin{aligned}(\top)P_1 \vee (\top)P_2 \vee \cdots \vee (\top)P_m &= (\top)P_1 \perp (\top)P_2 \perp \cdots \perp (\top)P_m \\ &= (\wedge)P_1 \perp (\wedge)P_2 \perp \cdots \perp (\wedge)P_m \\ &= (\wedge)P_1 \vee (\wedge)P_2 \vee \cdots \vee (\wedge)P_m; \\ (\perp)C_1 \wedge (\perp)C_2 \wedge \cdots \wedge (\perp)C_m &= (\perp)C_1 \top (\perp)C_2 \top \cdots \top (\perp)C_m \\ &= (\vee)C_1 \top (\vee)C_2 \top \cdots \top (\vee)C_m \\ &= (\vee)C_1 \wedge (\vee)C_2 \wedge \cdots \wedge (\vee)C_m.\end{aligned}$$

Proof. (1) It follows from Theorem 3.1. (2) It follows from the fact that $\forall a, b \in \{0, 1\}, a \top b = a \wedge b$ and $a \perp b = a \vee b$.

Theorem 3.5 (1) *If* $\top$ *is regular and* $(\top)P_1, (\top)P_2 \in (\top)\mathcal{F}^1$, *then*

$$(\top)P_1 \leq (\top)P_2 \Leftrightarrow L_2 \subseteq L_1,$$

where L_1 *and* L_2 *are the literal set of* P_1 *and* P_2 *respectively.*
(2) *If* $\perp$ *is regular and* $(\perp)C_1, (\perp)C_2 \in (\perp)\mathcal{F}^1$, *then*

$$(\perp)C_1 \leq (\perp)C_2 \Leftrightarrow L_1 \subseteq L_2,$$

where L_1 *and* L_2 *are the literal set of* C_1 *and* C_2 *respectively.*
(3) *If* $\top$ *and* $\perp$ *are regular,* $(\top)P_i, (\perp)C_i$ *are the* $\top$*-phrases constituted from the literals in* L_i *by using* $\top$ *and the* $\perp$*-sentences constituted from the literals in* L_i *by using* $\perp$ *respectively* ($i = 1, 2$)*, then*

$$(\top)P_1 \leq (\top)P_2 \Leftrightarrow (\perp)C_2 \leq (\perp)C_1.$$

(4) *Let De-Morgan law holds in* $(\top, \perp, N)$ *fuzzy logic,* L_i *be the literal set. Set*

$$N(L_i) = \{N(y); y \in L_i\},$$

and suppose $(\top)P_i, (\bot)C_i$ *are the* $\top$*-phrases constituted from the literals in* L_i *by using* $\top$ *and the* $\bot$*-sentences constituted from the literals in* L_i *by using* $\bot$ *respectively* $(i = 1, 2)$*, then*

$$(\top)P_1 \le (\top)P_2 \Leftrightarrow (\bot)C_2 \le (\bot)C_1.$$

Proof. (1) ($\Leftarrow$) Note that $(\top)P_1, (\top)P_2$ are 1-time $\top$-phrases. ($\Rightarrow$) If $y \in L_2, y \bar{\in} L_1$, then $\exists A = \{a_1, a_2, \ldots, a_n\} \in [0,1]^n$ such that

$$a_i = \begin{cases} 0, & \text{if } y = x_i, \\ 1, & \text{if } y = N(x_i), \\ 0.5, & \text{otherwise.} \end{cases}$$

So $(\top)P_2(A) = 0$. Note that $N(1) = 0, N(0) = 1$, hence $N(0.5) \ne 0$ and $N(0.5) \ne 1$ (in fact, if $N(0.5) \ne 0$, then $0.5 = N(N(0.5)) = N(0) = 1$, which is a contradiction; If $N(0.5) = 1$, then $0.5 = N(N(0.5)) = N(1) = 0$, which is also a contradiction). Therefore, it follows from the regularity of $\top$ that $(\top)P_1(A) > 0$, which is contradict to $(\top)P_1 \le (\top)P_2$. Consequently, $L_2 \subseteq L_1$.

(2) can be proved in the same way as (1).

(3) follows from (1) and (2).

Now we will prove (4):

($\Rightarrow$) $(\bot)C_2 = N(N(\bot)(C_2)) = N((\top)P_2) \le N((\top)P_1) = (\bot)C_1$.

($\Leftarrow$) $(\top)P_1 = N(N(\top)(P_1)) = N((\bot)C_1) \le N((\bot)C_2) = (\top)P_2$.

Theorem 3.6 $((\top)\mathcal{F}^1, \triangledown, \vartriangle)$ *is a lattice.*

Proof. (1) It follows from the definition of $\triangledown$ and $\vartriangle$ that they are closed.
(2) It follows from the communicative law and associative law of $\cap$ and $\cup$ in set that $\triangledown$ and $\vartriangle$ also satisfy these two laws.
(3) Let $(\top)P_1, (\top)P_2 \in (\top)\mathcal{F}^1$. We will prove

$$(\top)P_1 \triangledown ((\top)P_1 \vartriangle (\top)P_2) = (\top)P_1 \quad (3.1)$$
$$(\top)P_1 \vartriangle ((\top)P_1 \triangledown (\top)P_2) = (\top)P_1 \quad (3.2)$$

As for Eq. (3.1), we suppose that the literal set of P_1 and P_2 are L_1 and L_2 respectively. Set

$$(\top)P_1 \vartriangle (\top)P_2 = (\top)P.$$

Then the literal set of P is $L_1 \cup L_2$. Hence, the corresponding literal set of $(\top)P_1 \triangledown (\top)P_2$ is $L_1 \cap (L_1 \cup L_2) = L_1$. So

$$(\top)P_1 \triangledown (\top)P = (\top)P_1 \triangledown ((\top)P_1 \vartriangle (\top)P_2) = (\top)P_1.$$

Similarly, Eq.(3.2) can be proved. Consequently, the proof completes from (1), (2) and (3).

In the following, we set

$$\mathcal{L} = \mathcal{P}(x_1, x_2, \ldots, x_n, N(x_1), N(x_2), \ldots, N(x_n)).$$

Theorem 3.7 $((\top)\mathcal{F}^1, \bigtriangledown, \bigtriangleup) \cong (\mathcal{L}, \cap, \cup)$.

Proof. Let

$$\varphi : (\top)\mathcal{F}^1 \to \mathcal{L},$$
$$(\top)P \mapsto L,$$

where L is the literal set of P. Obviously, φ is a bijection. We need to prove:

$$\varphi((\top)P_1 \bigtriangledown (\top)P_2) = \varphi((\top)P_1) \cap \varphi((\top)P_2) \tag{3.3}$$
$$\varphi((\top)P_1 \bigtriangleup (\top)P_2) = \varphi((\top)P_1) \cup \varphi((\top)P_2) \tag{3.4}$$

hold $\forall (\top)P_1, (\top)P_2 \in (\top)\mathcal{F}^1$. As for Eq.(3.3), we suppose that the literal set of P_1 and P_2 are L_1 and L_2 respectively. Then the corresponding literal set of $(\top)P_1 \bigtriangleup (\top)P_2$ is $L_1 \cap L_2$. Hence

$$\varphi((\top)P_1 \bigtriangledown (\top)P_2) = L_1 \cap L_2 = \varphi((\top)P_1) \cap \varphi((\top)P_2).$$

Eq.(3.4) can be proved in the same way.

Remarks: In Theorem 3.7, $\bigtriangledown$ corresponds to $\cap$ and $\bigtriangleup$ corresponds to $\cup$.

Similar to Theorem 3.6 and 3.7, we have

Theorem 3.8 $((\perp)\mathcal{F}^1, \sqcup, \sqcap)$ *is a lattice.*

Theorem 3.9 $((\perp)\mathcal{F}^1, \sqcup, \sqcap) \cong (\mathcal{L}, \cup, \cap)$.

Remarks: In Theorem 3.9, $\sqcup$ corresponds to $\cup$ and $\sqcap$ corresponds to $\cap$.

Corollary 3.1 $((\top)\mathcal{F}^1, \bigtriangledown, \bigtriangleup)$ *is anti-isomorphic to* $((\perp)\mathcal{F}^1, \sqcup, \sqcap)$.

4 G-Fuzzy truth (falsity) of $(\top, \perp, N)$ fuzzy logical formulae

Theorem 4.1 *Let N be regular, $F \in \mathcal{F}^*$. Then*
(1) $F \perp N(F) \geq 0.5$;
(2) $F \top N(F) \leq 0.5$.

Proof. (1) $\forall A \in [0,1]^n, F \perp N(F)(A) = F(A) \perp N(F)(A) \geq F(A) \vee N(F)(A)$. If $F(A) \leq 0.5$, then $N(F)(A) \geq 0.5$. Hence, $F \perp N(F)(A) \geq 0.5$.

Corollary 4.1 *Let N be regular. Then*
(1) $(x_1 \perp N(x_1)) \wedge (x_2 \perp N(x_2)) \wedge \cdots \wedge (x_n \perp N(x_n)) \geq 0.5$
(2) $(x_1 \top N(x_1)) \vee (x_2 \top N(x_2)) \vee \cdots \vee (x_n \top N(x_n)) \leq 0.5$.

Theorem 4.2 *Let N be regular. Then*
(1) $\perp$*-sentence* $(\perp)C \geq 0.5 \Leftrightarrow$ *There exists the complementary pair in* C.
(2) $\top$*-phrase* $(\perp)P \leq 0.5 \Leftrightarrow$ *There exists the complementary pair in* P.

Proof. (1) ($\Leftarrow$) Let x_i and $N(x_i)$ occur in P. Then it follows from Theorem 3.10 in [15] that

$$(\perp)C \geq x_i \perp N(x_i) \geq 0.5.$$

($\Rightarrow$) If C does not contain any complementary pair, then $\exists A = \{a_1, a_2, \ldots, a_n\} \in [0,1]^n$ such that

$$a_i = \begin{cases} 1, & \text{if } N(x_i) \text{ occurs in } C, \\ 0, & \text{if } x_i \text{ occurs in } C \text{ or, neither } x_i \text{ nor } N(x_i) \text{ occurs in } C. \end{cases}$$

And note that $N(1) = 0$, hence $0.5 \leq ((\perp)C)(A) = 0$, which is a contradiction.

(2) can be proved in the same way.

Theorem 4.3 (1) *Suppose* $F = (\top)P_1 \perp (\top)P_2 \perp \cdots \perp (\top)P_m$ *is a* $\perp$*-normal form. Then*

a. $F \leq 0.5 \Rightarrow (\top)P_i \leq 0.5 \ (i = 1, 2, \ldots, m)$,

b. $\exists (\top)P_i, (\top)P_i \geq 0.5 \Rightarrow F \geq 0.5$.

(2) *Suppose* $F = (\perp)C_1 \top (\perp)C_2 \top \cdots \top (\perp)C_m$ *is a* $\top$*-normal form. Then*

a. $F \geq 0.5 \Rightarrow (\perp)C_i \geq 0.5 \ (i = 1, 2, \ldots, m)$,

b. $\exists (\perp)C_i, (\perp)C_i \leq 0.5 \Rightarrow F \leq 0.5$.

Proof. (1) It can be easy to see from $F \geq (\top)P_i (i = 1, 2, \ldots, m)$. (2) It can be easy to see from $F \leq (\perp)C_i (i = 1, 2, \ldots, m)$.

Theorem 4.4 *Let* N *be regular.*
(1) *Set*

$$F = (\perp)C_1 \wedge (\perp)C_2 \wedge \cdots \wedge (\perp)C_m.$$

Then

$$F \geq 0.5 \Leftrightarrow F \geq (x_1 \perp N(x_1)) \wedge (x_2 \perp N(x_2)) \wedge \cdots \wedge (x_n \perp N(x_n)).$$

(2) *Set*

$$F = (\top)P_1 \vee (\top)P_2 \vee \cdots \vee (\top)P_m.$$

Then

$$F \leq 0.5 \Leftrightarrow F \leq (x_1 \top N(x_1)) \vee (x_2 \top N(x_2)) \vee \cdots \vee (x_n \top N(x_n)).$$

Proof. (1) ($\Leftarrow$) It follows from Corollary 4.1(1). ($\Rightarrow$) $F \geq 0.5$ (from Theorem 4.3(2))
$\Rightarrow (\perp)C_i \geq 0.5 (i = 1, 2, \ldots, m)$ (from Theorem 4.2(1)) $\Rightarrow C_i$ contains the complementary pair $(i = 1, 2, \ldots, m)$. Moreover, suppose that there exists the complementary pair x_{ik} and $N(x_{ik})$ in $C_i (i = 1, 2, \ldots, m)$. Then

$$(\perp)C_i \geq x_{ik} \perp N(x_{ik}) \ (i = 1, 2, \ldots, m).$$

Hence

$$F = (\perp)C_1 \wedge (\perp)C_2 \wedge \cdots \wedge (\perp)C_m$$

$$\geq (x_{1k} \perp N(x_{1k})) \wedge (x_{2k} \perp N(x_{2k})) \wedge \cdots \wedge (x_{mk} \perp N(x_{mk}))$$

$$\geq (x_1 \perp N(x_1)) \wedge (x_2 \perp N(x_2)) \wedge \cdots \wedge (x_n \perp N(x_n)).$$

(2) can be proved similarly.

From Theorem 3.4, Theorem 4.2 in this paper as well as Theorem 5.2.3 in [2], we have

Theorem 4.5 *Let N be regular. Then*

(1) $(\top)P_1 \perp (\top)P_2 \perp \cdots \perp (\top)P_m$ *is a Boolean tautology* $\Leftrightarrow (\wedge)P_1 \vee (\wedge)P_2 \vee \cdots \vee (\wedge)P_m \geq 0.5$
(2) $(\top)P_1 \perp (\top)P_2 \perp \cdots \perp (\top)P_m$ *is Boolean inconsistent* $\Leftrightarrow (\wedge)P_1 \vee (\wedge)P_2 \vee \cdots \vee (\wedge)P_m \leq 0.5$
(3) $(\perp)C_1 \top (\perp)C_2 \top \cdots \top (\perp)C_m$ *is a Boolean tautology* $\Leftrightarrow (\vee)C_1 \wedge (\vee)C_2 \wedge \cdots \wedge (\vee)C_m \geq 0.5$
(4) $(\perp)C_1 \top (\perp)C_2 \top \cdots \top (\perp)C_m$ *is Boolean inconsistent* $\Leftrightarrow (\vee)C_1 \wedge (\vee)C_2 \wedge \cdots \wedge (\vee)C_m \leq 0.5$

Theorem 4.6 *Let N be regular.*

(1) *Set*

$$F_1 = (\wedge)P_1 \perp (\wedge)P_2 \perp \cdots \perp (\wedge)P_m.$$

Then

$$F_1 \text{ is a Boolean tautology} \Leftrightarrow F_1 \geq 0.5.$$

(2) *Set*

$$F_2 = (\perp)C_1 \wedge (\perp)C_2 \wedge \cdots \wedge (\perp)C_m.$$

Then

$$F_2 \text{ is a Boolean tautology} \Leftrightarrow F_2 \geq 0.5.$$

(3) *Set*

$$F_3 = (\top)P_1 \vee (\top)P_2 \vee \cdots \vee (\top)P_m.$$

Then

$$F_3 \text{ is Boolean inconsistent} \Leftrightarrow F_3 \leq 0.5.$$

(4) *Set*

$$F_4 = (\vee)C_1 \top (\vee)C_2 \top \cdots \top (\vee)C_m.$$

Then

$$F_4 \text{ is Boolean inconsistent} \Leftrightarrow F_4 \leq 0.5.$$

Proof. (1) ($\Leftarrow$) Obviously. ($\Rightarrow$) It follows from Theorem 4.5(1) that

$$(\wedge)P_1 \vee (\wedge)P_2 \vee \cdots \vee (\wedge)P_m \geq 0.5.$$

Moreover, it follows from Theorem 3.4(1) that $F_1 \geq 0.5$. (2),(3) and (4) can be proved similarly.

Corollary 4.2 *Let N be regular. Then*

(1) $(\wedge)P_1 \perp (\wedge)P_2 \perp \cdots \perp (\wedge)P_m \geq 0.5 \Leftrightarrow (\wedge)P_{i1} \vee (\wedge)P_{i2} \vee \cdots \vee (\wedge)P_{ik}$ *is a Boolean tautology*
$\Leftrightarrow \forall (y_1, y_2, \ldots, y_k) \in L_{i1} \times L_{i2} \times \cdots \times L_{ik}$*, there exist the complementary pairs in* $y_1, y_2, \ldots, y_k$*, where,*

$$(\wedge)P_{i1} \vee (\wedge)P_{i2} \vee \cdots \vee (\wedge)P_{ik}$$

is a $\wedge$-phrase without the complementary pairs in $(\wedge)P_1 \perp (\wedge)P_2 \perp \cdots \perp (\wedge)P_m$*,* L_{ij} *is the literal set of* P_{ij} $(j = 1, 2, \ldots, k)$*, and* $(\wedge)P_{ij} \not\leq (\wedge)P_{it}$ $(j, t = 1, 2, \ldots, k, j \neq t)$*.*

(2) $((\vee)C_1 T (\vee)C_2 \top \cdots \top (\vee)C_m \leq 0.5 \Leftrightarrow (\vee)C_{i1} \wedge (\vee)C_{i2} \wedge \cdots \wedge (\vee)C_{ik}$ *is Boolean inconsistent*
$\Leftrightarrow \forall (y_1, y_2, \ldots, y_k) \in L_{i1} \times L_{i2} \times \cdots \times L_{ik}$*, there exist the complementary pairs in* $y_1, y_2, \ldots, y_k$*, where,*

$$(\vee)C_{i1} \wedge (\vee)C_{i2} \wedge \cdots \wedge (\vee)C_{ik}$$

is $\vee$-sentence without the complementary pairs in $(\vee)C_1 \top (\vee)C_2 \top \cdots \top (\vee)C_m$*,* L_{ij} *is the literal set of* C_{ij} $(j = 1, 2, \ldots, k)$*, and* $(\vee)C_{ij} \not\leq (\vee)C_{it}$ $(j, t = 1, 2, \ldots, k, j \neq t)$*.*

Proof. (1) $(\wedge)P_1 \perp (\wedge)P_2 \perp \cdots \perp (\wedge)P_m \geq 0.5 \Leftrightarrow (\wedge)P_1 \vee (\wedge)P_2 \vee \cdots \vee (\wedge)P_m$ is a Boolean tautology. Note that

$$(\wedge)P_{i1} \vee (\wedge)P_{i2} \vee \cdots \vee (\wedge)P_{ik}$$

$$= \bigvee_{(y_1, y_2, \ldots, y_k) \in L_{i1} \times L_{i2} \times \cdots, \times L_{ik}} (y_1 \vee y_2 \vee \cdots \vee y_k)$$

Hence, $(\wedge)P_{i1} \vee (\wedge)P_{i2} \vee \cdots \vee (\wedge)P_{ik}$ is a Boolean tautology $\Leftrightarrow \forall (y_1, y_2, \ldots, y_k) \in L_{i1} \times L_{i2} \times \cdots \times L_{ik}, y_1 \vee y_2 \vee \cdots \vee y_k$ is a Boolean tautology $\Leftrightarrow \forall (y_1, y_2, \ldots, y_k) \in L_{i1} \times L_{i2} \times \cdots \times L_{ik}$, there exist the complementary pairs in $y_1, y_2, \ldots, y_k$. (2) can be proved similarly.

5 Conclusion

In this paper, some elementary concepts and properties of $(\top, \perp, N)$ fuzzy logic were discussed in details. As for the simplification of $(\top, \perp, N)$ fuzzy logical formulae, $(\top, \perp, N)$ fuzzy logical circuit, $(\top, \perp, N)$ fuzzy reasoning etc., further investigations will be carried on in the near future. Moreover, first-order $(\top, \perp, N)$ fuzzy logic and the soundness and completeness of $(\top, \perp, N)$ fuzzy logic such that $(\top, \perp, N)$

fuzzy logic become a complete logical system will be discussed.

Acknowledgements

The work was partly supported by the National Natural Science Foundation of P.R. China (Grant No. 60074014)

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