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Application of Cauchy's Equation in Combinatorics and Genetics

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Abstract

We are familiar with the combinatorial formula $\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n(n-1)\cdots(n-r+1)}{r!} =$ number of possible ways of choosing r objects out of n objects.

In section 1 of this paper we obtain $\begin{pmatrix} n \\ 2 \end{pmatrix}$ and $\begin{pmatrix} n \\ 3 \end{pmatrix}$ by using a functional equation, *the additive Cauchy equation.*

In genetics it is important to know the combinatorial function $g_r(n) =$ the number of possible ways of picking r objects at a time from n objects *allowing repetitions*, since this function describes the number of possibilities from a gene pool. Again we determine $g_2(n)$ and $g_3(n)$ with the help of the additive Cauchy equation in section 2.

Functional equations are used increasingly in diverse fields. The method of finding $\begin{pmatrix} n \\ 2 \end{pmatrix}$, $\begin{pmatrix} n \\ 3 \end{pmatrix}$, $g_2(n)$ and $g_3(n)$ (see Snow [6]) is similar to that of finding the well known sum of powers of integers $S_K(n) = 1^K + 2^K + \cdots + n^K$ (Aczél [2], Snow [5]).

Notation. Let $\mathbb{R}$ denote the set of real numbers, Z_+^* denote the set of positive integers. A function $A : \mathbb{R} \to \mathbb{R}$ is said to be *additive* (satisfies the *additive Cauchy equation*) provided A satisfies the functional equation

$$\text{(A)} \qquad A(x+y) = A(x) + A(y), \quad \text{for } x, y \in \mathbb{R}.$$

It is well known (Aczél [1]) that on Z_+^*, $A(n) = cn$, where c is a constant.

1. Combinatorial formula

Let B and D be two sets of n and m objects, respectively. Let $f_r(n) =$ number of possible ways of taking r objects from n objects. It is clear that $f_r(n+m) =$ the number of possible r objects among $m+n$ objects, satisfies the system of functional equations

$$f_r(m+n) = f_r(n) + f_r(m) + \sum_{i=1}^{r-1} f_i(n) f_{r-i}(m)\,, \tag{1}$$

for $m, n, r \in Z^*$. We will now obtain $f_2(n)$ and $f_3(n)$, using the Cauchy functional equation (A), noting that $f_2(2) = 1 = f_3(3)$, $f_1(n) = n$. For $r = 2$, (1) yields

$$\begin{aligned} f_2(m+n) &= f_2(n) + f_2(m) + f_1(m) f_1(n) \\ &= f_2(n) + f_2(m) + mn \\ &= f_2(n) + f_2(m) + \frac{1}{2}[(m+n)^2 - m^2 - n^2] \end{aligned}$$

which can be rewritten as

$$A_1(m+n) = A_1(m) + A_1(n), \text{ for, } m, n \in Z^*_+ \tag{A}$$

whose $A_1(n) = f_2(n) - \frac{1}{2}n^2, n \in Z^*_+$.

Since $A_1(n) = c_1 n$, $f_2(n) = \dfrac{1}{2}n^2 + c_1 n$. Using $f_2(2) = 1$, we get $c_1 = -1$ and

$$f_2(n) = \frac{1}{2}n^2 - \frac{1}{2}n = \frac{n(n-1)}{2!} = \binom{n}{2}.$$

To obtain $f_3(n)$, take $r = 3$ in (1) to get

$$\begin{aligned} f_3(m+n) &= f_3(n) + f_3(m) + f_1(n) f_2(m) + f_2(n) f_1(m) \\ &= f_3(n) + f_3(m) + \frac{n \cdot m(m-1)}{2} + \frac{m \cdot n(n-1)}{2} \\ &= f_3(n) + f_3(m) + \frac{1}{2}(nm^2 + n^2 m) - mn \\ &= f_3(n) + f_3(m) + \frac{1}{6}[(m+n)^3 - m^3 - n^3] - \frac{1}{2}[(m+n)^2 - m^2 - n^2] \end{aligned}$$

which goes over into

$$A_2(m+n) = A_2(m) + A_2(n) \tag{A}$$

where $A_2(n) = f_3(n) - \dfrac{1}{6}n^3 + \dfrac{1}{2}n^2$.

Since $A_2(n) = c_2 n$, $f_3(n) = \dfrac{1}{6}n^3 - \dfrac{1}{2}n^2 + c_2 n$. Using $f_3(3) = 1$, we have $c_2 = 1/3$ and

$$f_3(n) = \frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n = \frac{n(n-1)(n-2)}{3!} = \binom{n}{3}.$$

Remark 1. This method is applicable to find $f_k(n)$ for any specific k and for general k, ref [3],[4],[6].

2. Application in genetics

In genetics, it is of interest to determine the combinatorial function $g_r(n)$ = the number of ways to take r out of n objects *permitting repetitions.*

Determining $g_r(n)$ is similar to finding $f_r(n)$. We will determine $g_r(n)$ for the specific values $2, 3$ for r using the additive Cauchy equation (A). Note that $g_1(n) = n$ and $g_2(1) = 1 = g_3(1)$, that is, there is only one way one can take one object twice and one object three times. It is evident that $g_r(m + n)$ satisfy the system of functional equations

$$g_{r(m+n)} = g_r(m) + g_r(n) + \sum_{i=1}^{r-1} g_i(n)g_{r-i}(m), \tag{2}$$

for $m, n, r \in Z_+^*$, which is the same as (1).

To determine $g_2(n)$ we proceed as in section 1. Take $r = 2$ in (2) to get

$$\begin{aligned} g_2(m+n) &= g_2(m) + g_2(n) + mn \\ &= g_2(m) + g_2(n) + \frac{1}{2}[(m+n)^2 - m^2 - n^2] \end{aligned}$$

which as before reduces to

$$A_3(m+n) = A_3(m) + A_3(n), m, n \in Z_+^* \tag{A}$$

where $A_3(n) = g_2(2) - \dfrac{1}{2}n^2, n \in Z_+^*$ since $A_3(n) = c_3 n$,

$$\begin{aligned} g_2(n) &= \frac{1}{2}n^2 + c_3 n\,, \\ &= \frac{1}{2}n^2 + \frac{1}{2}n \ \ (\text{using } \ g_2(1) = 1), \\ &= \frac{1}{2!}n(n+1)\,. \end{aligned}$$

Finally to obtain $g_3(n)$, let $r = 3$ in (2) to have

$$\begin{aligned} g_3(m+n) &= g_3(n) + g_3(m) + \frac{n \cdot m(m+1)}{2} + \frac{m \cdot n(n+1)}{2} \\ &= g_3(n) + g_3(m) + \frac{1}{2}(nm^2 + n^2 m) + mn \\ &= g_3(n) + g_3(m) + \frac{1}{6}\left[(m+n)^3 - m^3 - n^3\right] + \frac{1}{2}[(m+n)^2 - m^2 - n^2] \end{aligned}$$

which can be put in the form

$$A_4(m+n) = A_4(m) + A_4(n), m, n, \in Z_+^* \tag{A}$$

where $A_4(n) = g_3(n) - \frac{1}{6}n^3 - \frac{1}{2}n^2$. Then $A_4(n) = c_4 n$,

$$\begin{aligned} g_3(n) &= \frac{1}{6}n^3 + \frac{1}{2}n^2 + c_4 n \\ &= \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n \ \text{ (using } g_3(1) = 1) \\ &= \frac{1}{6}n(n+1))(n+2) = \frac{n(n+1)(n+2)}{3!}\,. \end{aligned}$$

Remark 2. By this method, $g_k(n)$ for any specific k can be determined and for general k (see [3],[4],[6]).

Remark 3. Even though the system of equations (1) and (2) satisfied by $f_r(n)$ and $g_r(n)$ are the same, the solutions are different because of the different boundary conditions – other combinatorial formulas and the sum of powers of integers can also be determined by adopting this method.

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