

Topological Automorphism Groups of Chains

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Abstract

It is shown that any set-open topology on the automorphism group $A(X)$ of a chain X coincides with the pointwise topology and that $A(X)$ is a topological group with respect to this topology. Topological properties of connectedness and compactness in $A(X)$ are investigated. In particular, it is shown that the automorphism group of a doubly homogeneous chain is generated by any neighborhood of the identity element.

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1 Introduction

Let X be a chain and $A(X)$ denotes the automorphism group of X , i.e., the set of all order-preserving bijections from X onto X . $A(X)$ is also a lattice-ordered group with respect to meet and join operations defined, respectively by

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \quad \text{and} \quad (f \vee g)(x) = \max\{f(x), g(x)\}$$

for all $f, g \in A(X)$ and $x \in X$ [5].

The usual interval topology on X makes it completely normal Hausdorff space [1]. Thus $A(X)$ is a function space. In Section 2 we show that the pointwise topology is the smallest admissible topology on $A(X)$ and compatible with the group structure on $A(X)$. We also prove that this topology coincides with any set-open topology and, consequently, with the compact-open topology. The pointwise topology is also compatible with the lattice structure on $A(X)$. Thus $A(X)$ is a *topological lattice-ordered group*.

Homogeneity properties of X affect topological properties of $A(X)$. In Section 3 we establish some geometric and topological properties of doubly homogeneous chains that will be used in the succeeding sections.

A connected topological group is generated by any neighborhood of the identity element [2]. This fact motivates our studies of connectedness properties of $A(X)$

in Section 4. We show that $A(X)$ is connected (resp. totally disconnected) if and only if X is connected (resp. totally disconnected). We also prove that the automorphism group of a doubly homogeneous chain is generated by any neighborhood of the identity.

In Section 5, we establish three criteria for a subset to be compact in $A(X)$.

Finally, in Section 6, we use the techniques developed in this paper to develop an approach to robustness of aggregation procedures.

2 Topologies on automorphism groups

Let X be a chain and $A(X)$ its automorphism group. Since X is also a topological space with the usual interval topology, $A(X)$ is a function space.

For each pairs of sets $A \subseteq X$ and $B \subseteq X$, (A, B) denotes the set of all functions $f \in A(X)$ such that $f(A) \subseteq B$.

Definition 2.1. *The pointwise topology (p -topology) \mathfrak{T}_p on $A(X)$ is that having as subbasis all sets $(\{x\}, V)$ where $x \in X$ and V belongs to the subbasis of the interval topology on X consisting of open rays.*

Definition 2.2. *A topology \mathfrak{T} on $A(X)$ is admissible (or jointly continuous [6]) if the evaluation mapping $\mathcal{E} : A(X) \times X \rightarrow X$ defined by*

$$\mathcal{E}(f, x) = f(x)$$

for all $f \in A(X)$ and $x \in X$ is continuous.

Theorem 2.1. *\mathfrak{T}_p is the smallest admissible topology on $A(X)$.*

Proof. (a) First we show that \mathfrak{T}_p is admissible. It suffices to show that the inverse image W of an open ray in X is an open set in $A(X) \times X$. Consider an open ray (a, \rightarrow) (the case of open rays in the form (\leftarrow, a) is treated similarly). Then

$$W = \mathcal{E}^{-1}((a, \rightarrow)) = \{ (f, x) : f(x) > a \}$$

Let $(f_0, x_0) \in W$. Then $f_0(x_0) > a$ or, equivalently, $f_0^{-1}(a) < x_0$. Let us consider two cases.

(i) There exists b such that $f_0^{-1}(a) < b < x_0$ or, equivalently, $a < f_0(b) < x_0$. Then

$$x_0 \in V = (b, \rightarrow) \quad \text{and} \quad f_0 \in U = \{ f : f(b) > a \}.$$

Clearly, $U \times V$ is an open neighborhood of (f_0, x_0) . Suppose $(f, x) \in U \times V$. Then $f(x) > f(b) > a$ implying $U \times V \subseteq W$. Hence W is an open set.

(ii) Suppose that x_0 covers $f_0^{-1}(a)$. Then $f_0(x_0)$ covers a . We define

$$V = (f_0^{-1}(a), \rightarrow) \quad \text{and} \quad U = \{ f : f(x_0) > a \}.$$

Then $U \times V$ is an open neighborhood of (f_0, x_0) and

$$V = [x_0, \rightarrow) \quad \text{and} \quad U = \{ f : f(x_0) \geq f_0(x_0) \}.$$

Suppose $(f, x) \in U \times V$. Then

$$f(x) \geq f(x_0) \geq f_0(x_0) > a.$$

Thus $U \times V \subseteq W$ implying that W is an open set.

(b) We prove now that \mathfrak{T}_p is the smallest admissible topology on $A(X)$. Let \mathfrak{T} be an admissible topology. For any given $x \in X$, the mapping $\mathcal{E}_x : A(X) \rightarrow X$ defined by $f \mapsto f(x)$ is continuous, since \mathcal{E} is a continuous mapping. Let U be an open set in X . Then

$$\mathcal{E}_x^{-1}(U) = \{f : f(x) \in U\}.$$

These sets are open in \mathfrak{T} and form a subbasis for \mathfrak{T}_p . Thus $\mathfrak{T}_p \subseteq \mathfrak{T}$. □

Since any topology containing an admissible topology is admissible, we can reformulate the previous theorem as follows.

Theorem 2.2. *A topology \mathfrak{T} on $A(X)$ is admissible if and only if $\mathfrak{T} \supseteq \mathfrak{T}_p$.*

We now prove that group operations are continuous in p -topology on $A(X)$.

Theorem 2.3. *$A(X)$ endowed with the p -topology is a topological group.*

Proof. First we prove that $f \mapsto f^{-1}$ is a continuous mapping of $A(X)$ onto itself. Let

$$(\{a\}, (b, \rightarrow)) = \{f : f(a) > b\}$$

be an element of subbasis for \mathfrak{T}_p . The inverse image of this set is given by

$$\{f : f^{-1}(a) > b\} = \{f : f(b) < a\} = (\{b\}, (\leftarrow, a))$$

which is an element of the same subbasis. Similarly, the inverse image of $(\{a\}, (\leftarrow, b))$ is $(\{b\}, (a, \rightarrow))$. Thus $f \mapsto f^{-1}$ is continuous.

Now we prove that the binary group operation in $A(X)$ is continuous. Let $W = (\{a\}, (b, \rightarrow))$ be an element of the subbasis in $A(X)$ (the case of elements in the form $(\{a\}, (\leftarrow, b))$ is treated similarly) and $h_0 = f_0 g_0$ be an element of W . Then $f_0(g_0(a)) > b$. To prove continuity of the composition operation in $A(X)$ it suffices to find open neighborhoods U and V of f_0 and g_0 , respectively, such that for any $f \in U$ and $g \in V$, $fg \in W$. Consider two cases.

(i) There is $c \in X$ such that

$$b < c < f_0(g_0(a)).$$

Then

$$f_0^{-1}(b) < f_0^{-1}(c) < g_0(a).$$

Let $V = (\{a\}, (d, \rightarrow))$ and $U = (\{d\}, (b, \rightarrow))$, where $d = f_0^{-1}(c)$. We have $f_0 \in U$, since $f_0(d) = c > b$, and $g_0 \in V$, since $g_0(a) > d$. Thus U and V are neighborhoods of f_0 and g_0 , respectively. Let $f \in U$ and $g \in V$. Then

$$f(g(a)) > f(d) > b$$

implying $fg \in W$.

(ii) Suppose now that $f_0(g_0)$ covers b . Then $g_0(a)$ covers $c = f_0^{-1}(b)$. We define $V = (\{a\}, (c, \rightarrow))$ and $U = (\{g_0(a)\}, (b, \rightarrow))$. Then $g_0 \in V$, since $g_0(a) > f_0^{-1}(b) = c$, and $f_0 \in U$, since $f_0(g_0(a)) > b$. Since $f_0(g_0(a))$ covers b and $g_0(a)$ covers $f_0^{-1}(b)$, we have

$$V = (\{a\}, [g_0(a), \rightarrow)) \quad \text{and} \quad U = (\{g_0(a)\}, [f_0(g_0(a)), \rightarrow)).$$

Let f and g be any elements of U and V , respectively. Then

$$f(g(a)) \geq f(g_0(a)) \geq f_0(g_0(a)) > b$$

implying $fg \in W$. □

We proved that the p -topology is the smallest admissible topology on $A(X)$ and that $A(X)$ is a topological group with respect to \mathfrak{T}_p . Our next theorem shows that some other topologies that play an important role in the theory of transformation groups coincide with the p -topology in the case of automorphism groups. First we introduce the following definition.

Definition 2.3. Let \mathcal{S} be the set of all subsets A of X satisfying the following conditions:

- (a) If $\inf A$ exists, then it belongs to A .
- (b) If $\sup A$ exists, then it belongs to A .

The \mathcal{S} -topology $\mathfrak{T}_{\mathcal{S}}$ on $A(X)$ is defined by its subbasis which consists of sets (A, U) where $A \in \mathcal{S}$ and U is an open ray in X .

Clearly, \mathcal{S} contains all closed subsets of X . Thus $\mathfrak{T}_{\mathcal{S}}$ contains the largest set-open topology [8] and, consequently, compact-open and pointwise topologies.

The following theorem shows that the \mathcal{S} -topology coincides with the pointwise topology in the case of automorphism groups.

Theorem 2.4. $\mathfrak{T}_{\mathcal{S}} = \mathfrak{T}_p$ on $A(X)$.

Proof. Let $W = (A, (a, \rightarrow))$ be a nonempty element of the subbasis for $\mathfrak{T}_{\mathcal{S}}$ (the case of elements in the form $(A, (\leftarrow, a))$ is treated similarly). If $b = \inf A$ exists, then $b \in A$ and

$$(A, (a, \rightarrow)) = (\{b\}, (a, \rightarrow)),$$

since all functions in $A(X)$ are strictly increasing. Suppose $\inf A$ does not exist. Since $(A, (a, \rightarrow))$ is not empty, there is $f \in A(X)$ such that $f(x) > a$ for all $x \in X$, or, equivalently, $f^{-1}(a) < x$ for all $x \in X$. Thus the set L of all lower bounds of A is not empty. Since $\inf A$ does not exist, for any $x \in L$ there is $y \in L$ such that $y > x$. Then $x \in (\leftarrow, y) \subseteq L$ implying that L is open. Consider $V = (\{a\}, L)$. We have

$$\begin{aligned} f \in V &\Leftrightarrow f(a) \in L \Leftrightarrow f(a) < x \quad (\forall x \in A) \\ &\Leftrightarrow f^{-1}(x) > a \quad (\forall x \in A) \Leftrightarrow f^{-1} \in W. \end{aligned}$$

Thus $W = V^{-1}$. Since V is open in the p -topology, we conclude that W is also open in this topology. □

Let X be a chain and X' be the set of all *inner* elements of X , i.e.,

$$X' = \{x \in X : u < x < v, \text{ for some } u, v \in X\}.$$

Thus $a \in X \setminus X'$ if and only if a is a maximal or minimal element in X . If f is an automorphism of X , then its restriction on X' is an automorphism of the subchain X' , since maximal and minimal elements are fixed points of automorphisms. Moreover, any automorphism of the subchain X' can be uniquely extended to an automorphism of X . Clearly, $A(X)$ and $A(X')$ are isomorphic algebraic groups. They are also isomorphic as topological groups endowed with the p -topology. Indeed, any element of the subbasis of \mathfrak{X}_p in $A(X)$ in the form $(\{a\}, U)$, where a is a maximal or minimal element, is either empty or coincides with $A(X)$.

In what follows, we consider only chains without minimal and maximal elements and assume that automorphism groups are endowed with the pointwise topology.

3 Properties of chains

The purpose of this section is to introduce some basic geometric and topological properties of chains.

Let (a, b) be an open interval in X . Consider all automorphisms of X that coincide with the identity automorphism on the complement of (a, b) in X . These automorphisms form a subgroup of $A(X)$ which is isomorphic to $A((a, b))$. We shall often identify $A((a, b))$ with its isomorphic image in $A(X)$.

First we introduce notions of transitivity and homogeneity [5].

Definition 3.1. *Let X be a chain and $A(X)$ its automorphism group. We say that $A(X)$ is transitive if for any $x, y \in X$ there is $f \in A(X)$ such that $f(x) = y$. If $A(X)$ is transitive, then X is said to be homogeneous.*

Definition 3.2. *If for any $x_1 < \dots < x_n$ and $y_1 < \dots < y_n$ in X there is $f \in A(X)$ such that $f(x_i) = y_i$ for all $i, 1 \leq i \leq n$, we say that $A(X)$ is n -transitive and X is n -homogeneous.*

Obviously, n -homogeneity (n -transitivity) implies m -homogeneity (m -transitivity) for $m \leq n$. The converse, in general, is not true.

Double homogeneous chains are considered to be “well-behaved” in the theory of chains [5, Chapter 2]. The following theorem gives some equivalent descriptions of these chains.

Theorem 3.1. *Let X be a chain. The following conditions are equivalent.*

- (1) X is doubly homogeneous;
- (2) X is n -homogeneous for all $n \geq 2$;
- (3) any open interval in X is homogeneous;
- (4) any open interval in X is n -homogeneous for all $n \geq 2$.

Proof. (1) \Leftrightarrow (2) by [5, Lemma 1.10.1].

(2) \Rightarrow (3) and (2) \Rightarrow (4). Let (a, b) be an open interval in X . Consider two sequences

$$a < x_1 < \dots < x_n < b$$

and

$$a < y_1 < \dots < y_n < b,$$

where $n \geq 1$. Since X is $(n + 2)$ -homogeneous, there is $f \in A(X)$ such that $f(a) = a, f(b) = b$, and $f(x_i) = y_i$ for all $i, 1 \leq i \leq n$. The restriction f^* of f on (a, b) is an automorphism of (a, b) satisfying $f^*(x_i) = y_i$ for all $i, 1 \leq i \leq n$.

(3) \Rightarrow (1). We do not distinguish automorphisms of open intervals and their trivial extensions to X . Let $x < y$ and $u < v$ be elements of X . Consider an open interval (a, b) containing these elements. Since (a, b) is homogeneous, there is an automorphism f such that $f(y) = u$. Since $u < v$, we have $f^{-1}(u) < f^{-1}(v) = y$. Thus $f^{-1}(u) \in (a, y)$. Since $u < v$, we have $f^{-1}(u) < f^{-1}(v) = y$. Thus $f^{-1}(u) \in (a, y)$. Since (a, y) is homogeneous and $x \in (a, y)$, there is an automorphism g of (a, y) such that $g(x) = f^{-1}(u)$. Consider automorphism $h = fg$. We have $h(x) = f(g(x)) = u$ and $h(y) = f(g(y)) = f(y) = v$.

(4) \Rightarrow (3). Trivial. □

Corollary 3.1. *If X is doubly homogeneous, it does not have gaps.*

Proof. Suppose $(a, b) = \emptyset$ for some $a < b$. Since X has no minimal element, there are elements c and d such that $d < c < a < b$. Then (d, b) is an open interval containing at least two elements. Since (a, b) is empty, a is the maximal element in (d, b) and therefore a fixed point of any automorphism of (d, b) . Thus (d, b) is not homogeneous, which contradicts condition (3) of the previous theorem. □

We conclude that all open intervals with distinct end points in a doubly homogeneous chain are nonempty. In fact, each of these intervals contains infinitely many elements.

Now we consider connectedness and compactness in X .

Lemma 3.1. *Let X be a disconnected doubly homogeneous chain. For any two elements $u < v$ in X , there are disjoint open sets U and V such that $u \in U, v \in V$ and $U \cup V = X$.*

Proof. Since X is disconnected, there are two nonempty disjoint open sets A and B such that $A \cup B = X$. Let a and b be some elements in A and B , respectively. We may assume that $a < b$. Since X is doubly homogeneous, there is $f \in A(X)$ such that $f(a) = u$ and $f(b) = v$. Then $U = f(A)$ and $V = f(B)$ satisfy conditions of the lemma. □

Theorem 3.2. *(cf. [9]) A doubly homogeneous chain X is either a connected or totally disconnected topological space.*

Proof. Suppose X is not connected and let K be a component containing at least two elements $u < v$. Let U and V be sets from 3.1. Then $K = (K \cap U) \cup (K \cap V)$ and $(K \cap U) \cap (K \cap V) = \emptyset$, a contradiction. Thus each component contains exactly one element. □

Any topologically connected chain X is locally compact. The only connected subsets of X are the intervals. Closed bounded intervals in X are compact. (See [2, IV, 2, Exercise 7]).

4 Connectedness in $A(X)$

Automorphism groups $A(\mathbb{R})$ and $A(\mathbb{Q})$ provide typical examples for the results of this section. Namely, $A(\mathbb{R})$ is a connected topological group, whereas $A(\mathbb{Q})$ is a totally disconnected topological group. In the rest of this section, X is a doubly homogeneous chain.

First, we consider the case of totally disconnected X .

Theorem 4.1. *If X is totally disconnected, so is $A(X)$.*

Proof. Let K be a component of $A(X)$ containing at least two elements, say, f and g . Then there is $a \in X$ such that $f(a) \neq g(a)$. By Lemma 3.1, there are two nonempty open sets U and V which form a partition of X and such that $f(a) \in U$ and $g(a) \in V$. Consider open sets $U' = (\{a\}, U)$ and $V' = (\{a\}, V)$. Clearly, they form a partition of $A(X)$. Then $K = (K \cap U') \cup (K \cap V')$ which contradicts the definition of a component. Thus each component has exactly one element. □

According to Theorem 3.2, the remaining possibility is that of connected X . Let $B(X)$ be the set of all automorphisms of X with bounded support. (Recall that the support of $f \in A(X)$ is the set of all $x \in X$ such that $f(x) \neq x$.) We shall need the following lemma.

Lemma 4.1. *$B(X)$ is dense in $A(X)$.*

Proof. Let $U = (x_1, \dots, x_n; U_1, \dots, U_n)$ be a nonempty element of the basis of $A(X)$ and $h \in U$. We may assume that $x_1 < \dots < x_n$. Let $y_i = h(x_i)$, $1 \leq i \leq n$. Let $a, b \in X$ be two elements such that

$$a < x_1 < \dots < x_n < b$$

and

$$a < y_1 < \dots < y_n < b$$

There is $f \in A(X)$ such that $f(a) = a$, $f(b) = b$, and $f(x_i) = y_i$, $1 \leq i \leq n$. We define $f^*(x) = f(x)$ for $a \leq x \leq b$, and $f^*(x) = x$, otherwise. Clearly, $f^* \in U$ and $f^* \in B(X)$. □

Theorem 4.2. *If X is connected, so is $A(X)$.*

Proof. Let K be the component of the identity element in $A(X)$. The K is a closed normal subgroup of $A(X)$ ([2, III, 2.2]). By [5, Theorem 2G], $B(X) \subseteq K$. By Lemma 4.1, $K = A(X)$. □

The following theorem summarizes the above results.

Theorem 4.3. *$A(X)$ is connected (respectively, totally disconnected) if and only if X is connected (respectively, totally disconnected).*

It is known [2, III, 2.2] that a connected topological group is generated by any neighborhood of the identity element. In what follows, we show that this true for any $A(X)$ provided X is a doubly homogeneous chain.

We shall use special open neighborhoods of the identity element in $A(X)$. These are neighborhoods in the form

$$U = (x_1, \dots, x_n; J_1, \dots, J_n) = \{f \in A(X) : f(x_i) \in J_i, 1 \leq i \leq n\}$$

where $x_1 < \dots < x_n$ and $x_i \in J_i = (a_i, b_i)$, $1 \leq i \leq n$. In addition, we assume that intervals J_i do not overlap. Clearly, any neighborhood of the identity element contains an open set in this form.

To prove that $A(X)$ is generated by any neighborhood of the identity element we only need to show that any neighborhood in the form $(x_1, \dots, x_n; J_1, \dots, J_n)$ generates $A(X)$. First, we prove two lemmas.

Lemma 4.2. *Let $U = (x; J)$ where $x \in J = (a, b)$. Then U generates $A(X)$.*

Proof. Let h be any automorphism of X . It suffices to show that $h = fg$ for some $f, g \in U$. Let $y = h^{-1}(x)$. We may assume that $y \neq x$. (Otherwise $h \in U$.) If $y < x$, then there is $z \in J$ such that $y < x < z$. If $y > x$, there is $z \in J$ such that $z < x < y$. In both cases, by double homogeneity of X , there is an automorphism g such that $g(y) = x$ and $g(x) = z$. Thus $g \in U$. Let $f = gh^{-1}$. Then $f(x) = h(g^{-1}(x)) = h(y) = x$. Thus $f \in U$ and $h = fg$. □

For a given $U = (x_1, \dots, x_n; J_1, \dots, J_n)$ and $k \leq n$, we define

$$U_k = (x_1, \dots, x_k; J_1, \dots, J_k).$$

Lemma 4.3. *For any $h \in U_k$, $k < n$, there are $f, g \in U_{k+1}$ such that $h = fg$.*

Proof. Let $y = h^{-1}(x_{k+1})$. We may assume that $y \neq x_{k+1}$. Since $x_k \in J_k$ and $J_k \cap J_{k+1} = \emptyset$, we have $h(x_k) < x_{k+1}$ or, equivalently, $x_k < y$. If $y < x_{k+1}$, there is $z \in J_{k+1}$ such that $y < x_{k+1} < z$. Thus we have

$$x_1 < \dots < x_k < y < x_{k+1} < z.$$

If $y > x_{k+1}$, there is $z \in J_{k+1}$ such that $z < x_{k+1} < y$. Note that $z > x_k$, since $J_k \cap J_{k+1} = \emptyset$. Thus we have

$$x_1 < \cdots < x_k < z < x_{k+1} < y.$$

In both cases, by $(k + 2)$ -homogeneity of X , there exists an automorphism g such that $g(x_i) = x_i$, $(1 \leq i \leq k)$, $g(y) = x_{k+1}$, and $g(x_{k+1}) = z$. Clearly, $g \in U_{k+1}$. Let $f = hg^{-1}$. Then $f(x_i) = h(g^{-1}(x_i)) = h(x_i) \in J_i$ for all $i \leq k$, and $f(x_{k+1}) = h(g^{-1}(x_{k+1})) = h(y) = x_{k+1} \in J_{k+1}$. Thus $f \in U_{k+1}$. Clearly, $h = fg$. □

Let $U = U_n = (x_1, \dots, x_n; J_1, \dots, J_n)$. Consider a nested family of neighborhoods of the identity

$$A(X) \supseteq U_1 \supseteq \cdots \supseteq U_k \supset \cdots \supseteq U_n$$

where $U_k = (x_1, \dots, x_k; J_1, \dots, J_k)$, $1 \leq k \leq n$. Obvious inductive argument using lemmas 3 and 4 shows that any element in $A(X)$ is a composition of at most 2^n elements in U . Therefore $A(X)$ is generated by U . We proved the following theorem.

Theorem 4.4. *If X is a doubly homogeneous chain, then the automorphism group $A(X)$ is generated by any neighborhood of its identity element.*

Consider chain $X = \mathbb{Z}$. The automorphism group of this chain is a discrete group isomorphic to \mathbb{Z} . Clearly, $U = \{0\}$ does not generate $A(\mathbb{Z})$. Hence the double homogeneity property is an essential condition in Theorem 10.

5 Compact sets in $A(X)$

In this section we introduce three compactness criteria for subsets of $A(X)$. The first two criteria are obtained by embedding $A(X)$ into standard function spaces. The third criterion employs the lattice structure in $A(X)$. In what follows, we do not assume that X is a doubly homogeneous chain unless it is otherwise specified.

First we note that $A(X)$ is a Hausdorff subspace of X^X in p -topology. Thus we have the following theorem [6, 7.1].

Theorem 5.1. *A subset F of $A(X)$ is compact if and only if*

- (a) *F is closed in X^X , and*
- (b) *for each $x \in X$ the set $\mathcal{E}(F, \{x\})$ has a compact closure.*

The second criterion is a form of the Arzela–Ascoli Theorem. It is established under assumption that X is a k -space [6]. This includes, in particular, all doubly homogeneous topologically connected chains and all chains that are first countable topological spaces.

Suppose X is a k -space. Let $C(X)$ be a space of all continuous functions from X into X endowed with the compact–open topology. Then $A(X)$ is a subspace of $C(X)$ (Theorem 2.4). Thus we have the following version of Arzela–Ascoli Theorem [6, Theorem 7.21] and [8, Theorem 3.2.6].

Theorem 5.2. *If X is a k -space, then a subset F of $A(X)$ is compact if and only if*

- (a) F is closed in $C(X)$,
- (b) the closure of $\mathcal{E}(F, \{x\})$ is compact for each $x \in X$, and
- (c) F is evenly continuous.

The third criterion does not use any assumption about X or embeddings of $A(X)$ into a larger space.

Let L be a lattice. The *interval topology* on L is defined by taking the intervals $[p, \rightarrow)$ and $(\leftarrow, q]$ for all $p, q \in L$ as a subbasis of closed sets. The following theorem is found in [1] and [4].

Theorem 5.3. *A lattice is compact in its interval topology if and only if it is complete.*

We shall establish a similar result for sublattices of topological lattice $A(X)$ endowed with the pointwise topology.

First note that sets $(\{a\}, [b, \rightarrow))$, and $(\{a\}, (\leftarrow, b])$ form a subbasis for the p -topology in $A(X)$. We have

$$[f, \rightarrow) = \{g : g \geq f\} = \bigcap_{x \in X} (\{x\}, [f(x), \rightarrow))$$

Thus $[f, \rightarrow)$ is a closed set for any $f \in A(X)$. Similarly, all sets $(\leftarrow, f]$ are closed sets. Thus interval topology is smaller than the p -topology in $A(X)$.

Let L be a sublattice of $A(X)$. The interval topology in L is smaller than the induced topology. Thus, if L is a compact subset of $A(X)$, then it is also compact in the interval topology. By Theorem 12, L is complete. We proved the following lemma.

Lemma 5.1. *Any compact sublattice of $A(X)$ is complete.*

The following lemma asserts that the induced topology coincides with the interval topology provided L is a complete sublattice of $A(X)$.

Lemma 5.2. *Let L be a complete sublattice of $A(X)$. Then the induced topology is smaller than the interval topology in L .*

Proof. Let $(\{a\}, [b, \rightarrow))$ be an element of subbasis of closed sets in $A(X)$. We define

$$K_{a,b} = L \cap (\{a\}, [b, \rightarrow)) = \{f \in L : f(a) \geq b\}.$$

Since L is a complete lattice, $f_{a,b} = \inf K_{a,b}$ exists and $f_{a,b} \in K_{a,b}$. For any $f \in L$ such that $f \geq f_{a,b}$, we have $f \in K_{a,b}$. Thus $K_{a,b} = [f_{a,b}, \rightarrow)$ in L . Similarly, an intersection of $(\{a\}, (\leftarrow, b])$ with L is a closed interval in the form $(\leftarrow, f]$. Therefore any subset of L which is closed in the induced topology is also closed in the interval topology. □

By Lemma 5.2 and Theorem 5.2, any complete sublattice of $A(X)$ is a compact subset in $A(X)$. We proved the following theorem.

Theorem 5.4. *A sublattice L of $A(X)$ is compact if and only if L is a complete lattice.*

Corollary 5.1. *Let H be a closed subset of $A(X)$. If the lattice $[H]$ generated by H is complete, then H is compact.*

6 Robustness of aggregation procedures

Generally speaking, an aggregation procedure is a function assigning to n -tuples of objects all belonging to a given set X a single object from the same set. Typical examples include $X = \mathbb{R}$ (aggregation of real numbers) and $X = [0, 1]^m$ (aggregation of fuzzy sets with a finite universe).

We begin with the case $X = \mathbb{R}$. Thus we assume that the objects are represented by real numbers; in other words, we deal with *measurements* of real objects rather than with objects themselves.

In this setting, an aggregation procedure is just a function

$$y = M(x_1, x_2, \dots, x_n)$$

from \mathbb{R}^n to \mathbb{R} . Naturally, some conditions on M should be imposed in order to justify the name “aggregation procedure”. Here, we assume that M is a continuous symmetric function of its arguments satisfying the following condition

$$\min\{x_1, x_2, \dots, x_n\} \leq M(x_1, x_2, \dots, x_n) \leq \max\{x_1, x_2, \dots, x_n\}$$

for all n -tuples $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Following Cauchy [3], we call these functions *means* [9]. Yager’s OWA operators [12] are common examples of aggregation functions satisfying the above conditions as well as the usual arithmetic mean.

Because numbers x_1, x_2, \dots, x_n represent measurements, we should specify a scale in which these measurements were performed. Here, we are concerned with the case of ordinal scales. Moreover, we probably want the aggregation function M to represent a *meaningful* relation with respect to this scale. The concept of meaningfulness is formalized in the representational measurement theory [7] as the *invariance* property as follows. For any $f \in A(\mathbb{R})$ (the automorphism group of \mathbb{R}), the equation

$$f(y) = M(f(x_1), f(x_2), \dots, f(x_n))$$

is equivalent to the equation

$$y = M(x_1, x_2, \dots, x_n).$$

Functions from $A(\mathbb{R})$ are called *admissible* transformations in measurement theory [7].

The invariance condition greatly restricts the set of means on \mathbb{R} . The following theorem follows from a more general result established in [9].

Theorem. *An invariant mean on \mathbb{R} is an order statistics.*

The set $A(\mathbb{R})$ of all admissible transformations of \mathbb{R} is a huge set. It consists of all strictly increasing functions from \mathbb{R} onto itself. What happens if we consider only “small” admissible transformations which are “near” the identity transformation? In mathematics, the concept of “nearness” is formalized by introducing a topological structure on the set under consideration. Here, we consider $A(\mathbb{R})$ as a topological group so we can use the results of the previous sections. The transformations that are “near” the identity element belong to some neighborhood V of the identity transformation. These observations motivate the following definition (cf. [10, 11]).

Definition 6.1. *A mean M on \mathbb{R} is V -robust if*

$$M(f(x_1), f(x_2), \dots, f(x_n)) = f(M(x_1, x_2, \dots, x_n))$$

for all $x_1, x_2, \dots, x_n \in \mathbb{R}$ and all $f \in V$ where V is a neighborhood of the identity in $A(\mathbb{R})$.

Clearly, any invariant mean is V -robust for any neighborhood V of the identity. It turns out that in the case $X = \mathbb{R}$ the converse is also true. Namely we have the following theorem [11].

Theorem 6.1. *Any V -robust mean on \mathbb{R} is invariant.*

Proof. By Theorem 4.4, $A(\mathbb{R})$ is generated by V . In other words, for a given $f \in A(\mathbb{R})$ there are transformations $f_1, f_2, \dots, f_k \in V$ such that

$$f(x) = (f_1 f_2 \cdots f_k)(x)$$

for all $x \in \mathbb{R}$. Since

$$M(f_i(x_1), f_i(x_2), \dots, f_i(x_n)) = f(M(x_1, x_2, \dots, x_n))$$

for all $1 \leq i \leq k$, we have

$$M(f(x_1), f(x_2), \dots, f(x_n)) = f(M(x_1, x_2, \dots, x_n)).$$

□

The situation is quite different in the case $X = [0, 1]^m$ if $m > 1$. In this case we are concerned with the aggregation problem for fuzzy sets defined on a finite universe U with $|U| = m$. We denote $x(u)$ the membership function of a fuzzy set $x \in [0, 1]^U$. Consider the set of all “pointwise” admissible transformations of X of the form

$$F(x)(u) = f_u(x(u)),$$

where, for each $u \in U$, f_u is an increasing bijection of $[0, 1]$ onto itself (an automorphism of $[0, 1]$). Thus defined transformations preserve lattice structure on X and form a group $A^o(X)$ which is a proper subgroup of the automorphism group $A(X)$. We have the following theorem.

Theorem 6.2. *Let V be a neighborhood of the identity in $A(X)$ such that $V \subseteq A^\circ(X)$. Then any V -robust mean is $A^\circ(X)$ -robust.*

Proof. The group $A^\circ(X)$ is the direct product of m copies of automorphism groups $A([0, 1])$, i.e.,

$$A^\circ(X) = (A([0, 1]))^m.$$

Since the group $A([0, 1])$ is isomorphic to the group $A(\mathbb{R})$, it is a connected topological group. Thus $A^\circ(X)$ is also connected and therefore is generated by V . Now we apply the same argument as in the proof of the previous theorem to complete the proof. □

The following theorem follows immediately from Theorem 6.1 in [9].

Theorem 6.3. *Let*

$$y(u) = M(x_1(u), x_2(u), \dots, x_n(u))$$

be a V -robust mean on $X = [0, 1]^U$, where $V \subseteq A^\circ(X)$. There exists a function $p : U \rightarrow \{1, 2, \dots, m\}$ such that, for any given $u \in U$, $y(u)$ is the $p(u)$'s order statistic.

It is proven in [9] that any *invariant* mean on X is a p 'th order statistic for some p which does not depend on $u \in U$. Thus, in the case $X = [0, 1]^m$, the class of V -robust means is much wider than the class of invariant means.

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