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A Theorem for Basis Operators Over Intuitionistic Fuzzy Sets

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Abstract

The concept of s-basis operators over intuitionistic fuzzy sets is introduced and all 2-, 3-, 4- basis operators are listed.

Keywords: Intuitionistic fuzzy set, operator

Let a set E be fixed. An Intuitionistic Fuzzy Set (IFS) A^* on E is an object of the following form ([1], see [4,5]):

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

For simplicity below we shall write A instead of A^*.

Several relations, operations and operators are defined for every two IFSs A and B (see [1-3]). Here we shall introduce only these of them, which are related to the present research:

$$\begin{array}{lcl} A = B & iff & (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x)); \\ \overline{A} & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \\ A \cap B & = & \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\ A \cup B & = & \{\langle x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \end{array}$$

Initially, following [1] we shall introduce two operators over IFSs which will transform every IFS into a fuzzy set (i.e. a particular case of an IFS). These two operators are similar to the operators "necessity" and "possibility" defined in some modal logics. Let for every IFS A:

$$\begin{array}{ll} \Box A & = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}; \\ \Diamond A & = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}. \end{array}$$

Obviously, if A is an ordinary fuzzy set, then:

$$\Box A = A = \Diamond A.$$

These equalities show that these operators have no sense in the case of the ordinary fuzzy set.

Let $\alpha, \beta \in [0, 1]$ be a fixed numbers. The following operators are defined and their properties are studied in [1-3]:

$$\begin{array}{ll} D_\alpha(A) & = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in E\}, \\ F_{\alpha,\beta}(A) & = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \\ G_{\alpha,\beta}(A) & = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\} \\ H_{\alpha,\beta}(A) & = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \\ H^*_{\alpha,\beta}(A) & = \{\langle x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\}, \\ J_{\alpha,\beta}(A) & = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x)\rangle | x \in E\}, \\ J^*_{\alpha,\beta}(A) & = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x)\rangle | x \in E\}, \end{array}$$

where for the operator F $\alpha + \beta \leq 1$.

We must note that the operator D represents simultaneously the operators $\Box$ and $\Diamond$q because $\Box = D_0$ and $\Diamond = D_1$. It has no analogues in the ordinary modal logics, but the author hopes that the search of its analogues in the modal logics will be interesting.

On the other hand, for every $\alpha \in [0, 1]: \ D_\alpha = F_{\alpha, 1-\alpha}$.

Let $s \geq 2$ be a fixed natural number. The s-tuple $(X_1, \ldots, X_s)$, where $X_1, \ldots, X_s \in S = \{D_\alpha, F_{\alpha,\beta}, G_{\alpha,\beta}, H_{\alpha,\beta}, H^*_{\alpha,\beta}, J_{\alpha,\beta}, J^*_{\alpha,\beta}\}$ we shall call a basic s-tuple of operators from S if every one of the operators of S can be represented by the operators of the s-tuple, using the above operations and the "composition" operation over operators.

On Fig. 1 - 7 the geometrical interpretations of the elements of S are shown.

Theorem. *$(D, G), (F, G), (H, J), (H, J^*), (H^*, J)$ and (H^*, J^*) are the only basic 2-tuples of operators.*

Proof: The Theorem will be proved in two steps.

First we shall show that every one of the operators can be represented by every one of the above 2-tuples. To this end we shall construct equalities, connecting every one of the operators with the operators of the particular 2-tuples.

Let us assume that A is a fixed IFS over the universe E and let the real numbers $\alpha, \ \beta \in [0, \ 1]$ be fixed.

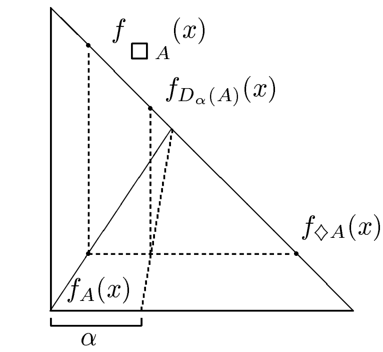

Fig. 1.

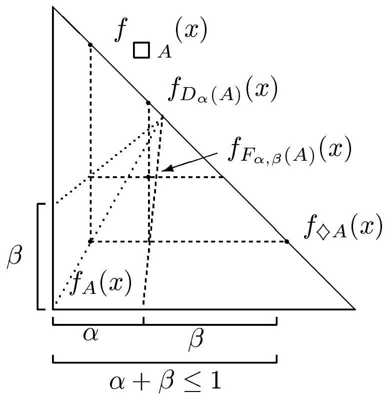

Fig. 2.

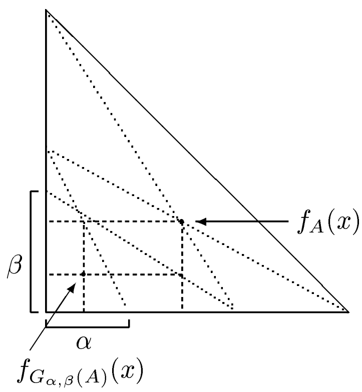

Fig. 3.

For the tuple $(D,\ G)$ we have (let for the first equality below it is valid that $\alpha + \beta \leq 1$):

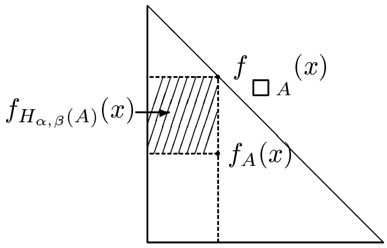

Fig. 4.

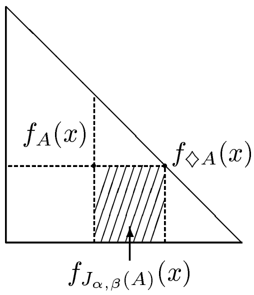

Fig. 5.

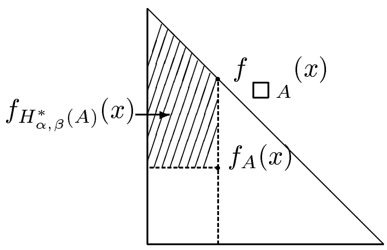

Fig. 6.

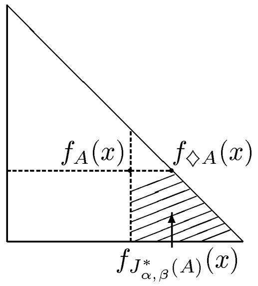

Fig. 7.

$$
\begin{aligned}
& F_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \mu_A(x) + \alpha.\pi_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\
& (\text{ because } \mu_A(x) + \alpha.\pi_A(x) \leq \mu_A(x) + (1-\beta).\pi_A(x)) \\
= \ & \{\langle x,\ \mu_A(x) + \alpha.\pi_A(x),\ 0\rangle | x \in E\} \\
& \cap \{\langle x,\ \mu_A(x) + (1-\beta).\pi_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\
= \ & G_{1,\ 0}(\{\langle x,\ \mu_A(x) + \alpha.\pi_A(x),\ \nu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in E\}) \\
& \cap \{\langle x,\ \mu_A(x) + (1-\beta).\pi_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\
= \ & G_{1,\ 0}(D_\alpha(A)) \cap D_{1-\beta}(A); \\
& H_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},
\end{aligned}
$$

$$
\begin{aligned}
= \ & \{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\} \\
& \cap \{\langle x,\ \mu_A(x) + (1-\beta).\pi_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\
= \ & G_{\alpha,\ \beta}(A) \cap D_{1-\beta}(A);
\end{aligned}
$$

$$
\begin{aligned}
& H^*_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\}) \\
= \ & \{\langle x,\ \min(\alpha.\mu_A(x) + (1-\beta).(1 - \alpha.\mu_A(x) - \nu_A(x)),\ \alpha.\mu_A(x)), \\
& \max(\nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x)),\ \nu_A(x))\rangle | x \in E\}) \\
= \ & \{\langle x,\ \alpha.\mu_A(x) + (1-\beta).(1 - \alpha.\mu_A(x) - \nu_A(x)), \\
& \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\}) \\
& \cap \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)\rangle | x \in E\} \\
= \ & D_{1-\beta}(\{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)\rangle | x \in E\}) \cap \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)\rangle | x \in E\} \\
= \ & D_{1-\beta}(G_{\alpha,\ 1}(A)) \cap G_{\alpha,\ 1}(A);
\end{aligned}
$$

$$
\begin{aligned}
& J_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\
= \ & \{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\} \\
& \cap \{\langle x,\ \mu_A(x) + (1-\beta).\pi_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\
= \ & G_{\alpha,\ \beta}(A) \cap D_{1-\beta}(A);
\end{aligned}
$$

$$
\begin{array}{rl}
 & J^*_{\alpha,\ \beta}(A) \\
= & \{\langle x,\ \mu_A(x)+\alpha.(1-\mu_A(x)-\beta.\nu_A(x)),\ \beta.\nu_A(x)\rangle | x\in E\} \\
= & \{\langle x,\ \mu_A(x)+\alpha.(1-\mu_A(x)-\beta.\nu_A(x)), \\
 & \beta.\nu_A(x)+(1-\alpha).(1-\mu_A(x)-\beta.\nu_A(x))\rangle | x\in E\} \\
 & \cup\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\} \\
= & D_\alpha(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\})\cup\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\} \\
= & D_\alpha(G_{1,\ \beta}(A))\cup G_{1,\ \beta}(A).
\end{array}
$$

For the tuple $(F,\ G)$ we have that (let everywhere below $\alpha+\beta\leq 1$):

$D_\alpha(A)=G_{1,\ 1}(F_{\alpha,\ 1-\alpha})$;

$$
\begin{array}{rl}
 & H_{\alpha,\ \beta}(A) \\
= & \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)+\beta.\pi_A(x)\rangle | x\in E\} \\
= & \{\langle x,\ \min(\mu_A(x),\ \alpha.\mu_A(x)), \max(\nu_A(x)+\beta.\pi(x),\ \nu_A(x))\rangle | x\in E\} \\
= & \{\langle x,\ \mu_A(x),\ \nu_A(x)+\beta.\pi_A(x)\rangle | x\in E\}\cap\{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)\rangle | x\in E\} \\
= & F_{0,\ \beta}(A)\cap G_{\alpha,\ 1}(A);
\end{array}
$$

$$
\begin{array}{rl}
 & J_{\alpha,\ \beta}(A) \\
= & \{\langle x,\ \mu_A(x)+\alpha.\pi_A(x),\ \beta.\nu_A(x)\rangle | x\in E\} \\
= & \{\langle x,\ \mu_A(x)+\alpha.\pi_A(x),\ \nu_A(x)\rangle | x\in E\}\cup\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\} \\
= & F_{\alpha,\ 0}(A)\cup G_{1,\ \beta}(A);
\end{array}
$$

$$
\begin{array}{rl}
 & H^*_{\alpha,\ \beta}(A) \\
= & \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)+\beta.(1-\alpha.\mu_A(x)-\nu_A(x))\rangle | x\in E\} \\
= & F_{0,\ \beta}(\{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)\rangle | x\in E\}) \\
= & F_{0,\ \beta}(G_{\alpha,\ 1}(A));
\end{array}
$$

$$
\begin{array}{rl}
 & J^*_{\alpha,\ \beta}(A) \\
= & \{\langle x,\ \mu_A(x)+\alpha.(1-\mu_A(x)-\beta.\nu_A(x)),\ \beta.\nu_A(x)\rangle | x\in E\} \\
= & F_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\}) \\
= & F_{\alpha,\ 0}(G_{1,\ \beta}(A)).
\end{array}
$$

For the tuple $(H,\ J)$ we have (let for the second equality below $\alpha+\beta\leq 1$):

$$
\begin{array}{rl}
 & D_\alpha(A) \\
= & \{\langle x,\ \mu_A(x)+\alpha.\pi_A(x),\ \nu_A(x)+(1-\alpha).\pi_A(x)\rangle | x\in E\} \\
= & \{\langle x,\ 1-\nu_A(x)-(1-\alpha).\pi_A(x),\quad \nu_A(x)+(1-\alpha).\pi_A(x)\rangle | x\in E\} \\
= & J_{1,\ 1}(\{\langle x,\ 0,\ \nu_A(x)+(1-\alpha).\pi_A(x)\rangle | x\in E\}) \\
= & J_{1,\ 1}(H_{0,\ 1-\alpha}(A));
\end{array}
$$

$$
\begin{array}{rl}
 & F_{\alpha,\ \beta}(A) \\
= & \{\langle x,\ \mu_A(x)+\alpha.\pi_A(x),\ \nu_A(x)+\beta.\pi_A(x)\rangle | x\in E\} \\
= & \{\langle x,\ \min(\mu_A(x)+\alpha.\pi_A(x),\ 1-\nu_A(x)-\beta.\pi_A(x)), \\
 & \max(\nu_A(x)+\beta.\pi_A(x),\ \nu_A(x))\rangle | x\in E\} \\
= & \{\langle x,\ 1-\nu_A(x)-\beta.\pi_A(x),\ \nu_A(x)+\beta.\pi_A(x)\rangle | x\in E\} \\
 & \cap\{\langle x,\ \mu_A(x)+\alpha.\pi_A(x),\ \nu_A(x)\rangle | x\in E\} \\
= & J_{1,\ 1}(\{\langle x,\ 0,\ \nu_A(x)+\beta.\pi_A(x)\rangle | x\in E\})\cap J_{\alpha,\ 1}(A) \\
= & J_{1,\ 1}(H_{0,\ \beta}(A))\cap J_{\alpha,\ 1}(A);
\end{array}
$$

$$\begin{aligned}
& G_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\
= \ & H_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\
= \ & H_{\alpha,\ 0}(J_{0,\ \beta}(A));
\end{aligned}$$

$$\begin{aligned}
& H^*_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\}) \\
= \ & H_{1,\ \beta}(\{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)\rangle | x \in E\}) \\
= \ & H_{1,\ \beta}(H_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle | x \in E\})) \\
= \ & H_{1,\ \beta}(H_{\alpha,\ 0}(J_{0,\ 1}(A)));
\end{aligned}$$

$$\begin{aligned}
& J^*_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\nu_A(x)),\ \beta.\nu_A(x)\rangle | x \in E\} \\
= \ & J_{\alpha,\ 1}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\
= \ & J_{\alpha,\ 1}(H_{1,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\})) \\
= \ & J_{\alpha,\ 1}(H_{1,\ 0}(J_{0,\ \beta}(A))).
\end{aligned}$$

For the tuple $(H,\ J^*)$ we have (let for the second equality below it is valid that $\alpha + \beta \leq 1$):

$$\begin{aligned}
& D_\alpha(A) \\
= \ & \{\langle x,\ \mu_A(x) + \alpha.\pi_A(x),\ \nu_A(x) + (1 - \alpha).\pi_A(x)\rangle | x \in E\} \\
= \ & \{\langle x,\ 1 - \nu_A(x) - (1 - \alpha).\pi_A(x),\quad \nu_A(x) + (1 - \alpha).\pi_A(x)\rangle | x \in E\} \\
= \ & J^*_{1,\ 1}(\{\langle x,\ 0,\ \nu_A(x) + (1 - \alpha).\pi_A(x)\rangle | x \in E\}) \\
= \ & J^*_{1,\ 1}(H_{0,\ 1-\alpha}(A));
\end{aligned}$$

$$\begin{aligned}
& F_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \mu_A(x) + \alpha.\pi_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\
= \ & \{\langle x,\ 1 - \nu_A(x) - \beta.\pi_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\
& \cap \{\langle x,\ \mu_A(x) + \alpha.\pi_A(x),\ \nu_A(x)\rangle | x \in E\} \\
= \ & J^*_{1,\ 1}(\{\langle x,\ 0,\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}) \cap J^*_{\alpha,\ 1}(A) \\
= \ & J^*_{1,\ 1}(H_{0,\ \beta}(A)) \cap J^*_{\alpha,\ 1}(A);
\end{aligned}$$

$$\begin{aligned}
& G_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | xq \in E\}) \\
= \ & H_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\
= \ & H_{\alpha,\ 0}(J^*_{0,\ \beta}(A));
\end{aligned}$$

$$\begin{aligned}
& H^*_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x))\rangle | x \in E\}) \\
= \ & H_{1,\ \beta}(\{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)\rangle | x \in E\}) \\
= \ & H_{1,\ \beta}(H_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle | x \in E\})) \\
= \ & H_{1,\ \beta}(H_{\alpha,\ 0}(J^*_{0,\ 1}(A)));
\end{aligned}$$

$$\begin{aligned}
& J_{\alpha,\ \beta}(A) \\
= \ & \{\langle x,\ \mu_A(x) + \alpha.(1 - \mu_A(x) - \nu_A(x)),\ \beta.\nu_A(x)\rangle | x \in E\} \\
= \ & \{\langle x,\ \mu_A(x) + \alpha.(1 - \mu_A(x) - \nu_A(x)),\ \nu_A(x)\rangle | x \in E\} \\
& \cup \{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\
= \ & J^*_{\alpha,\ 1}(A) \cup H_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\
= \ & J^*_{\alpha,\ 1}(A) \cup H_{\alpha,\ 0}(J^*_{0,\ \beta}(A)).
\end{aligned}$$

For the tuple $(H^*,\ J)$ we have (let for the second equality below $\alpha + \beta \leq 1$):

$$\begin{array}{rl} & D_\alpha(A) \\ = & \{\langle x,\ \mu_A(x)+\alpha.\pi_A(x),\ \nu_A(x)+(1-\alpha).\pi_A(x)\rangle | x\in E\} \\ = & \{\langle x,\ 1-\nu_A(x)-(1-\alpha).\pi_A(x),\ \nu_A(x))+(1-\alpha).\pi_A(x)\rangle | x\in E\} \\ = & J_{1,\ 1}(\{\langle x,\ 0,\ \nu_A(x)+(1-\alpha).\pi_A(x)\rangle | x\in E\}) \\ = & J_{1,\ 1}(\{\langle x,\ \mu_A(x),\ \nu_A(x)+(1-\alpha).\pi_A(x)\rangle | x\in E\} \\ & \cap\{\langle x,\ 0,\ (1-\alpha).\nu_A(x)\rangle | x\in E\}) \\ = & J_{1,\ 1}(H^*_{1,\ 1-\alpha}(A)\cap H^*_{0,\ 0}(\{\langle x,\ \mu_A(x),\ (1-\alpha).\nu_A(x)\rangle | x\in E\})) \\ = & J_{1,\ 1}(H^*_{1,\ 1-\alpha}(A)\cap H^*_{0,\ 0}(J_{0,\ 1-\alpha}(A))) \end{array}$$

$$\begin{array}{rl} & F_{\alpha,\ \beta}(A) \\ = & \{\langle x,\ \mu_A(x)+\alpha.\pi_A(x),\ \nu_A(x)+\beta.\pi_A(x)\rangle | x\in E\} \\ = & \{\langle x,\ 1-\nu_A(x)-\beta.\pi_A(x)),\ \nu_A(x)+\beta.\pi_A(x)\rangle | x\in E\} \\ & \cap\{\langle x,\ \mu_A(x)+\alpha.(1-\mu_A(x)-\nu_A(x)),\ \nu_A(x)\rangle | x\in E\} \\ = & J_{1,\ 1}(\{\langle x,\ 0,\ \nu_A(x)+\beta.(1-\mu_A(x)-\nu_A(x))\rangle | x\in E\}\cap J^*_{\alpha,\ 1}(A) \\ = & J_{1,\ 1}(\{\langle x,\ \mu_A(x),\ \nu_A(x)+\beta.(1-\mu_A(x)-\nu_A(x))\rangle | x\in E\} \\ & \cap\{\langle x,\ 0,\ \beta.\nu_A(x)\rangle | x\in E\})\cap J_{\alpha,\ 1}(A) \\ = & J_{1,\ 1}(H^*_{1,\ \beta}(A)\cap H^*_{0,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\}))\cap J_{\alpha,\ 1}(A) \\ = & J_{1,\ 1}(H^*_{1,\ \beta}(A)\cap H^*_{0,\ 0}(J_{0,\ \beta}(A)))\cap J_{\alpha,\ 1}(A); \end{array}$$

$$\begin{array}{rl} & G_{\alpha,\ \beta}(A) \\ = & \{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\}) \\ = & H^*_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\}) \\ = & H^*_{\alpha,\ 0}(J_{0,\ \beta}(A)); \\ & H_{\alpha,\ \beta}(A) \\ = & \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x)+\beta.\pi_A(x)\rangle | x\in E\} \\ = & \{\langle x,\ \mu_A(x),\ \nu_A(x)+\beta.(1-\mu_A(x)-\nu_A(x))\rangle | x\in E\} \\ & \cap\{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\}) \\ = & H^*_{1,\ \beta}(A)\cap H^*_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\}) \\ = & H^*_{1,\ \beta}(A)\cap H^*_{\alpha,\ 0}(J_{0,\ \beta}(A)); \\ & J^*_{\alpha,\ \beta}(A) \\ = & \{\langle x,\ \mu_A(x)+\alpha.(1-\mu_A(x)-\beta.\nu_A(x)),\ \beta.\nu_A(x)\rangle | x\in E\} \\ = & J_{\alpha,\ 1}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\}) \\ = & J_{\alpha,\ 1}(H^*_{1,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x\in E\})) \\ \\ = & J_{\alpha,\ 1}(H^*_{1,\ 0}(J_{0,\ \beta}(A))). \end{array}$$

Finally, for the tuple $(H^*,\ J^*)$ we have (let for the second equality below $\alpha+\beta\leq 1$):

$$\begin{array}{rl} & D_\alpha(A) \\ = & \{\langle x,\ \mu_A(x)+\alpha.\pi_A(x),\ (1-\alpha).\pi_A(x)\rangle | x\in E\} \\ = & \{\langle x,\ 1-\nu_A(x)-(1-\alpha).\pi_A(x),\ (1-\alpha).\pi_A(x)\rangle | x\in E\} \\ = & J^*_{1,\ 1}(\{\langle x,\ 0,\ \nu_A(x)+(1-\alpha).(1-\mu_A(x)-\nu_A(x))\rangle | x\in E\}) \\ = & J^*_{1,\ 1}(\{\langle x,\ \mu_A(x),\ \nu_A(x)+(1-\alpha).(1-\mu_A(x)-\nu_A(x))\rangle | x\in E\} \\ & \cap\{\langle x,\ 0,\ (1-\alpha).\nu_A(x)\rangle | x\in E\}) \\ = & J^*_{1,\ 1}(H^*_{1,\ 1-\alpha}(A)\cap H^*_{0,\ 0}(\{\langle x,\ \mu_A(x),\ (1-\alpha).\nu_A(x)\rangle | x\in E\})) \\ = & J^*_{1,\ 1}(H^*_{1,\ 1-\alpha}(A)\cap H^*_{0,\ 0}(J^*_{0,\ 1-\alpha}(A))); \end{array}$$

$$\begin{array}{rl} & F_{\alpha,\ \beta}(A) \\ = & \{\langle x,\ \mu_A(x) + \alpha.\pi_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\ = & \{\langle x,\ 1 - \nu_A(x) - \beta.\pi_A(x),\ \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\} \\ & \cap\{\langle x,\ \mu_A(x) + \alpha.(1 - \mu_A(x) - \nu_A(x)),\ \nu_A(x)\rangle | x \in E\} \\ = & J^*_{1,\ 1}(\{\langle x,\ 0,\ \nu_A(x) + \beta.(1 - \mu_A(x) - \nu_A(x))\rangle | x \in E\} \cap J^*_{\alpha,\ 1}(A) \\ = & J^*_{1,\ 1}(\{\langle x,\ \mu_A(x),\ \nu_A(x) + \beta.(1 - \mu_A(x) - \nu_A(x))\rangle | x \in E\} \\ & \cap\{\langle x,\ 0,\ \beta.\nu_A(x)\rangle | x \in E\}) \cap J^*_{\alpha,\ 1}(A) \\ = & J^*_{1,\ 1}(H^*_{1,\ \beta}(A) \cap H^*_{0,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\})) \cap J^*_{\alpha,\ 1}(A) \\ = & J^*_{1,\ 1}(H^*_{1,\ \beta}(A) \cap H^*_{0,\ 0}(J^*_{0,\ \beta}(A))) \cap J^*_{\alpha,\ 1}(A); \end{array}$$

$$\begin{array}{rl} & G_{\alpha,\ \beta}(A) \\ = & \{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\ = & H^*_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\ = & H^*_{\alpha,\ 0}(J^*_{0,\ \beta}(A)); \end{array}$$

$$\begin{array}{rl} & H_{\alpha,\ \beta}(A) \\ = & \{\langle x,\ \alpha.\mu_A(x),\ \nu_A(x) + \beta.(1 - \mu_A(x) - \nu_A(x))\rangle | x \in E\} \\ = & \{\langle x,\ \mu_A(x),\ \nu_A(x) + \beta.(1 - \mu_A(x) - \nu_A(x))\rangle | x \in E\} \\ & \cap\{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\ = & H^*_{1,\ \beta}(A) \cap H^*_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\ = & H^*_{1,\ \beta}(A) \cap H^*_{\alpha,\ 0}(J^*_{0,\ \beta}(A)); \end{array}$$

$$\begin{array}{rl} & J_{\alpha,\ \beta}(A) \\ = & \{\langle x,\ \mu_A(x) + \alpha.\pi_A(x),\ \beta.\nu_A(x)\rangle | x \in E\} \\ = & \{\langle x,\ \mu_A(x) + \alpha.(1 - \mu_A(x) - \nu_A(x)),\ \nu_A(x)\rangle | x \in E\} \\ & \cup\{\langle x,\ \alpha.\mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\ = & J^*_{\alpha,\ 1}(A) \cup H^*_{\alpha,\ 0}(\{\langle x,\ \mu_A(x),\ \beta.\nu_A(x)\rangle | x \in E\}) \\ = & J^*_{\alpha,\ 1}(A) \cup H^*_{\alpha,\ 0}(J^*_{0,\ \beta}(A)). \end{array}$$

Therefore we showed that every one of the above pairs is a basic 2-tuple. Below we will show that these 2-tuples are unique.

Let a universe E be fixed. Let an element $x \in E$ be fixed, too, and let us construct for the fixed set $A \subset E$ the following IFS (below we will use only IFSs A, i.e., no notational collision will arise):

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

for which $\mu_A(x) > 0$ and $\nu_A(x) > 0$. It can be seen directly that the ordered tuple

$$\langle x, 0, 0\rangle \in G_{0,0}(A) = \{\langle x, 0, 0\rangle | x \in E\}$$

cannot be represented by any of the following pairs of operators:

$$(D, F), (D, H), (D, H^*), (D, J), (D, J^*), (F, H), (F, H^*), (F, J), (F, J^*),$$

because none of them nor their combinations applied to the set A yields as a result $\langle x, 0, 0\rangle$ for $x \in E$. This fact can be seen, e.g., from the geometrical interpretations of the individual operators. Analogically, the ordered tuple

$$\langle x, 1, 0\rangle \in J^*_{1,0}(A) = \{\langle x, 1, 0\rangle | x \in E\}$$

cannot be represented by any of the following pairs of operatots: (G, H), (G, H^*), (H, H^*), and the ordered tuple

$$\langle x, 0, 1\rangle \in H^*_{0,1}(A) = \{\langle x, 0, 1\rangle | x \in E\} -$$

by any one of the following pairs of operators: (G, J), (G, J^*), (J, J^*).

Therefore the unique pairs of basic 2-tuple operators are those listed in the theorem. ♢

Corollary 1. The only basic 3-tuples of operators are:

(D, F, G),	(D, G, H),	(D, G, H^*),	(D, G, J),	(D, G, J^*),
(D, H, J),	(D, H, J^*),	(D, H^*, J),	(D, H^*, J^*),	(F, G, H),
(F, G, H^*),	(F, G, J),	(F, G, J^*),	(F, H, J),	(F, H, J^*),
(F, H^*, J),	(F, H^*, J^*),	(G, H, J),	(G, H, J^*),	(G, H^*, J),
(G, H^*, J^*),	(H, H^*, J),	(H, H^*, J^*),	(H, J, J^*),	(H^*, J, J^*).

Proof. These 3-tuples are basic 3-tuples of operators, since each of them contains a basic 2-tuple of operators. For the set A, as in the proof of the theorem 2, we see again that the element $< x, 0, 0 >\in G_{0,0}(A)$ cannot be represented by any of the 3-tuples (D, F, H), (D, F, H^*), (D, F, J), (D, F, J^*), (D, H, H^*), (D, J, J^*), (F, H, H^*), (F, J^*, J^*); the element $< x, 1, 0 >\in J_{1,0}(A)$ cannot be represented by (G, H, H^*); and the element $\langle x, 0, 1\rangle \in H^*_{0,1}(A)$ – by (G, J, J^*). ♢

Corollary 2. $(D, F, H, H^*), (D, F, J, J^*)$ are the only 4-tuples of operators that are not basic 4-tuples of operators.

Indeed, only these two 4-tuples of operators do not contain basic 2-tuples or 3-tuples of operators and it can be proved that they cannot represent the elements $< x, 1, 0 >\in J^*_{1,0}(A)$ and $< x, 0, 1 >\in H^*_{0,1}(A)$, respectively.

Corollary 3. All 5-tuples of operators are basic 5-tuples of operators.

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