Mathware & Soft Computing 8 (2001) 5-20

On the Generalizations of Siegel's Fixed Point Theorem

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Submitted by S. Sessa

Abstract

In this paper, we establish a new version of Siegel's fixed point theorem in generating spaces of quasi-metric family. As consequences, we obtain general versions of the Downing-Kirk's fixed point and Caristi's fixed point theorem in the same spaces. Some applications of these results to fuzzy metric spaces and probabilistic metric spaces are presented. [1]
1991 AMS Subject Classification: 47H10, 54A40, 54E70
Key words and phrases: Generating spaces of quasi-metric family, fixed points, fuzzy metric spaces, probabilistic metric spaces.

1 Introduction

In the last few years, Caristi's fixed point theorem [2] is a very useful tool in the theory of nonlinear analysis. Because the theorem does not require the continuity of mapping, it has applications in many fields. Among those applications are the inward mapping theory, normal solvability theory, results concerning metric convexity, dissipative dynamical systems, and many others. For the literature, see [1, 2, 5, 12, 15].

Also several generalizations of the theorem were given by a number of authors; for instance, generalizations for single-valued mapping were given by Downing and

[1]This study is supported by the academic research fund of Ministry of Education, Korea, 1997, Project No. BSRI-97-1405, and NON-DIRECTED RESEARCH FUND, Korea Research Foundation, 1995-1996.

Kirk [6], Park [15] and Siegel [17], the multi-valued version of the theorem was obtained by Chang and Luo [3] and Mizoguchi and Takahashi [13], independently, the uniform space version of the theorem was provided by Mizoguchi [14], and extensions to fuzzy metric spaces and probabilistic metric spaces were by Chang, Chen and Guo [4], Chang et al. [5], He [8], Jung, Cho and Kim [9], and Jung et al. [10]. In particular, Siegel [17] gave a generalization with simple constructive proof. A version of Siegel's theorem which includes the result of Downing and Kirk [6] was obtained by Park [15].

In this paper, we establish a new version of Siegel's fixed point theorem [17] in generating spaces of quasi-metric family. As consequences we obtain general versions of Downing-Kirk's fixed point theorem and Caristi's fixed point theorem in the same spaces. Simultaneously, applying these results to fuzzy metric spaces and probabilistic metric spaces, we present the corresponding results. Our results generalize and improve upon the corresponding results of [2, 6, 8, 9, 15, 17].

2 Siegel's Fixed Point Theorem in Generating Spaces of Quasi-Metric Family

In this section, we give a general version of Siegel's fixed point theorem in generating spaces of quasi-metric family. Using the result, we also present general versions of Downing-Kirk's fixed point theorem and Caristi's fixed point theorem.

First, we give the definition, some properties and examples of generating spaces of quasi-metric family.

Definiton 2.1 *[5] Let X be a nonempty set and $\{d_\alpha : \alpha \in (0,1]\}$ be a family of mappings d_α of $X \times X$ into $\mathbb{R}^+$. $(X, d_\alpha : \alpha \in (0,1])$ is called a generating space of quasi-metric family if it satisfies the following conditions:*

(QM-1) $d_\alpha(x,y) = 0$ for all $\alpha \in (0,1]$ if and only if $x = y$,

(QM-2) $d_\alpha(x,y) = d_\alpha(y,x)$ for all $x,\ y \in X$ and $\alpha \in (0,1]$,

(QM-3) For any $\alpha \in (0,1]$, there exists a number $\mu \in (0,\alpha]$ such that

$$d_\alpha(x,y) \le d_\mu(x,z) + d_\mu(z,y), \quad x,\ y,\ z \in X,$$

(QM-4) For any $x,\ y \in X$, $d_\alpha(x,y)$ is nonincreasing and left continuous in α.

In what follows $\{d_\alpha : \alpha \in (0,1]\}$ will be called a family of quasi-metrics.

Example 2.1 *Let (X,d) be a metric space. Letting $d_\alpha(x,y) = d(x,y)$ for all $\alpha \in (0,1]$ and $x,\ y \in X$, (X,d) is a generating space of quasi-metric family. Furthermore, every fuzzy metric space (see Definition 3.1) and every probabilistic metric space (see Definition 4.1) are both the examples of generating spaces of quasi-metric family (the proof will be given in the sections 3 and 4 below).*

In [7], Fan proved that if $(X, d_\alpha : \alpha \in (0,1])$ is a generating space of quasi-metric family, then there exists a topology $\mathcal{T}_{\{d_\alpha\}}$ on X such that $(X, \mathcal{T}_{\{d_\alpha\}})$ is a

Hausdorff topological space and $\mathcal{U}(x) = \{U_x(\epsilon, \alpha) : \epsilon > 0,\ \alpha \in (0,1]\}$, $x \in X$, is a basis of neighborhoods of the point x for the topology $\mathcal{T}_{\{d_\alpha\}}$, where

$$U_x(\epsilon, \alpha) = \{y \in X : d_\alpha(x, y) < \epsilon\}.$$

Throughout this paper, we assume that $k : (0,1] \to (0,\infty)$ is a nondecreasing function satisfying the following condition:

$$M = \sup_{\alpha \in (0,1]} k(\alpha) < \infty. \tag{2.1}$$

Now, we begin with following lemmas.

Lemma 2.1 *Let $(X, d_\alpha : \alpha \in (0,1])$ and $(Y, \delta_\alpha : \alpha \in (0,1])$ be two complete generating spaces of quasi-metric family, $f : X \to Y$ a closed mapping, and $\varphi : f(X) \to \mathbb{R}$ a lower semicontinuous function, bounded from below. Let $\{x_i\}$ be a sequence in X such that*

$$\max\{d_\alpha(x_i, x_{i+1}), c\,\delta_\alpha(f(x_i), f(x_{i+1}))\} \leq k(\alpha)\{\varphi(f(x_i)) - \varphi(f(x_{i+1}))\} \tag{2.2}$$

for any $\alpha \in (0,1]$ and i, where $c > 0$ is a given constant. Then $\lim_{i\to\infty} x_i = \bar{x}$ exists and

$$\max\{d_\alpha(x_i, \bar{x}), c\,\delta_\alpha(f(x_i), f(\bar{x}))\} \leq k(\alpha)\{\varphi(f(x_i)) - \varphi(f(\bar{x}))\}$$

for any $\alpha \in (0,1]$ and i.

Proof. First for $j > i+1$, we prove inductively that

$$\max\{d_\alpha(x_i, x_j), c\,\delta_\alpha(f(x_i), f(x_j))\} \leq k(\alpha)\{\varphi(f(x_i)) - \varphi(f(x_j))\} \tag{2.3}$$

for any $\alpha \in (0,1]$ and i. To this end, let $j = i+2$. Since $\{d_\alpha : \alpha \in (0,1]\}$ and $\{\delta_\alpha : \alpha \in (0,1]\}$ are both families of quasi-metrics, they are nonincreasing in α. Hence for any given $\alpha \in (0,1]$, there exists a common number $\mu \in (0,1]$ such that

$$d_\alpha(x_i, x_{i+2}) \leq d_\mu(x_i, x_{i+1}) + d_\mu(x_{i+1}, x_{i+2})$$

and

$$\delta_\alpha(f(x_i), f(x_{i+2})) \leq \delta_\mu(f(x_i), f(x_{i+1})) + \delta_\mu(f(x_{i+1}), f(x_{i+2})).$$

By (2.2), we have

$$\begin{aligned} d_\alpha(x_i, x_{i+2}) &\leq k(\mu)\{\varphi(f(x_i)) - \varphi(f(x_{i+1})) + \varphi(f(x_{i+1})) - \varphi(f(x_{i+2}))\} \\ &= k(\mu)\{\varphi(f(x_i)) - \varphi(f(x_{i+2}))\} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} c\ \delta_\alpha(f(x_i), f(x_{i+2})) &\leq k(\mu)\{\varphi(f(x_i)) - \varphi(f(x_{i+1})) + \varphi(f(x_{i+1})) - \varphi(f(x_{i+1}))\} \\ &= k(\mu)\{\varphi(f(x_i)) - \varphi(f(x_{i+2}))\}. \end{aligned} \tag{2.5}$$

Noting that the function k is nondecreasing, from (2.4) and (2.5) we have

$$\begin{aligned}\max\{d_\alpha(x_i, x_{i+2}), c\,\delta_\alpha(f(x_i), f(x_{i+2}))\} &\leq k(\mu)\{\varphi(f(x_i)) - \varphi(f(x_{i+2}))\}\\ &\leq k(\alpha)\{\varphi(f(x_i)) - \varphi(f(x_{i+2}))\}.\end{aligned}$$

Suppose that

$$\max\{d_\alpha(x_i, x_{j-1}), c\,\delta_\alpha(f(x_i), f(x_{j-1}))\} \leq k(\alpha)\{\varphi(f(x_i)) - \varphi(f(x_{j-1}))\} \tag{2.6}$$

for any $\alpha \in (0, 1]$. Then by the just above argument, for any $\alpha \in (0, 1]$ we get

$$d_\alpha(x_i, x_j) \leq d_\mu(x_i, x_{j-1}) + d_\mu(x_{j-1}, x_j)$$

and

$$\delta_\alpha(f(x_i), f(x_j)) \leq \delta_\mu(f(x_i), f(x_{j-1})) + \delta_\mu(f(x_{j-1}), f(x_j))$$

for some $\mu \in (0, \alpha]$. By (2.2) and (2.6), we have

$$\begin{aligned}d_\alpha(x_i, x_j) &\leq k(\mu)\{\varphi(f(x_i)) - \varphi(f(x_{j-1})) + \varphi(f(x_{j-1})) - \varphi(f(x_j))\}\\ &= k(\mu)\{\varphi(f(x_i)) - \varphi(f(x_j))\}\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}c\,\delta_\alpha(f(x_i), f(x_j)) &\leq k(\mu)\{\varphi(f(x_i)) - \varphi(f(x_{j-1})) + \varphi(f(x_{j-1})) - \varphi(f(x_j))\}\\ &= k(\mu)\{\varphi(f(x_i)) - \varphi(f(x_j))\}.\end{aligned} \tag{2.8}$$

Again, noting the function k is nondecreasing, from (2.7) and (2.8) we get

$$\max\{d_\alpha(x_i, x_j), c\,\delta_\alpha(f(x_i), f(x_j))\} \leq k(\alpha)\{\varphi(f(x_i)) - \varphi(f(x_j))\}$$

for any $\alpha \in (0, 1]$. This implies that (2.3) is true.

On the other hand, since $\{\varphi(f(x_i))\}$ is monotonically decreasing, by the boundedness from below of φ, there exists a finite number r such that

$$\lim_{i\to\infty} \varphi(f(x_i)) = r.$$

Hence for any given $\lambda > 0$ and $\varepsilon > M\lambda$ (where M is a constant defined by (2.1)), there exists i_0 such that for $i \geq i_0$,

$$r \leq \varphi(f(x_i)) < r + \lambda.$$

Hence for any $j > i \geq i_0$, we have

$$0 \leq \varphi(f(x_i)) - \phi(f(x_j)) < r + \lambda - r = \lambda.$$

Therefore, for any $\alpha \in (0, 1]$ and $j > i \geq i_0$, we have from (2.3)

$$\begin{aligned}\max\{d_\alpha(x_i, x_j), c\,\delta_\alpha(f(x_i), f(x_j))\} &\leq k(\alpha)\{\varphi(f(x_i)) - \varphi(f(x_j))\}\\ &\leq M\lambda < \varepsilon.\end{aligned}$$

This implies that $\{x_i\}$ and $\{f(x_i)\}$ are both Cauchy sequences in X and Y, respectively. By the completeness of X and Y, we assume that $x_i \to \bar{x} \in X$ and $f(x_i) \to \bar{y} \in Y$. Since f is a closed mapping, $\bar{y} = f(\bar{x})$. By the lower semicontinuity of φ, we have

$$\varphi(f(\bar{x})) \le \liminf_{i\to\infty} \varphi(f(x_i)) = \lim_{i\to\infty} \varphi(f(x_i)) = r \le \varphi(f(x_i)) \tag{2.9}$$

for each i. Hence, from (2.9) we have

$$\begin{aligned} \max\{d_\alpha(x_i,\bar{x}), c\,\delta_\alpha(f(x_i), f(\bar{x}))\} &= \max\{\lim_{j\to\infty} d_\alpha(x_i,x_j), c\lim_{j\to\infty} \delta_\alpha(f(x_i), f(x_j))\} \\ &\le k(\alpha)\{\varphi(f(x_i)) - \lim_{j\to\infty}\varphi(f(x_j))\} \\ &= k(\alpha)\{\varphi(f(x_i)) - r\} \\ &\le k(\alpha)\{\varphi(f(x_i)) - \varphi(f(\bar{x}))\} \end{aligned}$$

for any $\alpha \in (0,1]$ and i, which completes the proof.

Let $h_i : X \to X$, $1 \le i < \infty$. The countable composition oh h_i is defined by

$$\prod_{i=1}^{\infty} h_i(x) = \lim_{i\to\infty} h_i h_{i-1} \cdots h_1(x)$$

if the limit exists for each $x \in X$.

Let Φ^* denote the set of all $h : X \to X$ satisfying the condition;

$$\max\{d_\alpha(x,h(x)), c\delta_\alpha(f(x), f(h(x)))\} \le k(\alpha)\{\varphi(f(x)) - \varphi(f(h(x)))\} \tag{2.10}$$

for any $x \in X$ and $\alpha \in (0,1]$, where $c > 0$ is a given constant.

Lemma 2.2 Φ^* *is closed under countable composition.*

Proof. Let h_1, $h_2 \in \Phi^*$. Since $\{d_\alpha : \alpha \in (0,1]\}$ and $\{\delta_\alpha : \alpha \in (0,1]\}$ are both families of quasi-metrics, they are nonincreasing in α. Hence for any given $\alpha \in (0,1]$, there exists a common number $\mu \in (0,\alpha]$ such that

$$d_\alpha(x, h_2(h_1(x))) \le d_\mu(x, h_1(x)) + d_\mu(h_1(x), h_2(h_1(x)))$$

and

$$\delta_\alpha(f(x), f(h_2(h_1(x)))) \le \delta_\mu(f(x), f(h_1(x))) + \delta_\mu(f(h_1(x)), f(h_2(h_1(x)))).$$

By (2.10), we have

$$\begin{aligned} & d_\alpha(x, h_2(h_1(x))) \\ \le\; & k(\mu)\{\varphi(f(x)) - \varphi(f(h_1(x))) + \varphi(f(h_1(x))) - \varphi(f(h_2(h_1(x))))\} \quad (2.11) \\ =\; & k(\mu)\{\varphi(f(x)) - \varphi(f(h_2(h_1(x))))\} \end{aligned}$$

and

$$\begin{aligned} & c\,\delta_\alpha(f(x), f(h_2(h_1(x)))) \\ \le\ & k(\mu)\{\varphi(f(x)) - \varphi(f(h_1(x))) + \varphi(f(h_1(x))) - \varphi(f(h_2(h_1(x))))\} \\ =\ & k(\mu)\{\varphi(f(x)) - \varphi(f(h_2(h_1(x))))\} \end{aligned} \tag{2.12}$$

Noting that the function k is nondecreasing, from (2.11) and (2.12) we have

$$\begin{aligned} & \max\{d_\alpha(x, h_2(h_1(x))), c\delta_\alpha(f(x), f(h_2(h_1(x))))\} \\ \le\ & k(\alpha)\{\varphi(f(x)) - \varphi(f(h_2(h_1(x))))\} \end{aligned}$$

for any $\alpha \in (0,1]$. This shows that Φ^* is closed under composition. By putting $x_i = h_i h_{i-1} \cdots h_1(x)$ for each $x \in X$, from Lemma 2.1, we have the conclusion.

For any $A \subset X$, let $r(A) = \inf_{x\in A}\{\varphi(f(x))\}$. Then $B \subset A$ implies $r(B) \ge r(A)$. For any $\Phi \subset \Phi^*$, let $\Phi(x) = \{h(x) : h \in \Phi\}$. For $A \subset X$, define the diameter of A by

$$\mathrm{diam} A = \sup_{x_i, x_j \in A} d_\alpha(x_i, x_j) \quad \text{for any } \alpha \in (0,1].$$

Lemma 2.3 $\mathrm{diam}\Phi(x) \le 2M\{\varphi(f(x)) - r(\Phi(x))\}$, *where M is a constant defined by* (2.1).

Proof. For any $h_1,\ h_2 \in \Phi$, it follows from the same argument that

$$d_\alpha(h_1(x), h_2(x)) \le d_\mu(x, h_1(x)) + d_\mu(x, h_2(x))$$

for some $\mu \in (0, \alpha]$. By (2.10), we have

$$d_\alpha(h_1(x), h_2(x)) \le k(\mu)\{\varphi(f(x)) - \varphi(f(h_1(x))) + \varphi(f(x)) - \varphi(f(h_2(x)))\}.$$

Since k is nondecreasing, for any $\alpha \in (0,1]$ we have

$$\begin{aligned} d_\alpha(h_1(x), h_2(x)) &\le 2k(\alpha)\{\varphi(f(x)) - r(\Phi(x))\} \\ &\le 2M\{\varphi(f(x)) - r(\Phi(x))\}, \end{aligned}$$

which completes the proof.

Now we are in a position to give a main result.

Theorem 2.1 *Let $(X, d_\alpha : \alpha \in (0,1])$ and $(Y, \delta_\alpha : \alpha \in (0,1])$ be two complete generating spaces of quasi-metric family, $f : X \to Y$ a closed mapping, and $\varphi : f(X) \to \mathbb{R}$ a lower semicontinuous function, bounded from below. Let $c > 0$ be a given constant and let Φ^* be the family of all $h : X \to X$ satisfying*

$$\max\{d_\alpha(x, h(x)), c\delta_\alpha(f(x), f(h(x)))\} \le k(\alpha)\{\varphi(f(x)) - \varphi(f(h(x)))\} \tag{2.13}$$

for any $\alpha \in (0,1]$ and $x \in X$. Let $\Phi \subset \Phi^*$ be closed under composition and $x_0 \in X$.

(a) *If Φ is closed under countable composition, then there exists an $\bar{h} \in \Phi$ such that $\bar{x} = \bar{h}(x_0)$ and $h(\bar{x}) = \bar{x}$ for all $h \in \Phi$.*

(b) *If each mapping in Φ is continuous, then there exist a sequence $\{h_i\}$ in Φ and*

$$\bar{x} = \lim_{i \to \infty} h_i h_{i-1} \cdots h_1(x_0)$$

in X such that $h(\bar{x}) = \bar{x}$ for each $h \in \Phi$.

Proof. Let $\{\varepsilon_i\}$ be a positive sequence converging to 0. Choose an $h_1 \in \Phi$ such that

$$\varphi(f(h_1(x_0))) - r(\Phi(x_0)) < \frac{\varepsilon_1}{2M},$$

where M is a constant defined by (2.1). Set $x_1 = h_1(x_0)$ and for any $\alpha \in (0, 1]$,

$$\begin{aligned} \operatorname{diam}\Phi(x_1) &\leq 2M\{\varphi(f(x_1)) - r(\Phi(x_1))\} \\ &\leq 2M\{\varphi(f(h_1(x_0))) - r(\Phi(x_0))\} \\ &< 2M \cdot \frac{\varepsilon_1}{2M} = \varepsilon_1. \end{aligned}$$

Repeating this process, we obtain a sequence $\{h_i\}$ such that

$$x_{i+1} = h_i(x_i), \quad \Phi(x_{i+1}) \subseteq \Phi(x_i) \text{ and } \operatorname{diam}\Phi(x_i) < \varepsilon_i.$$

(a) Let $\bar{h} = \prod_{i=1}^{\infty} h_i$ and $\bar{x} = \bar{h}(x_0)$. Since $\bar{x} = \prod_{j=i+1}^{\infty} h_j(x_i)$, we have $\bar{x} \in \Phi(x_i)$ for each i. Furthermore, since $\lim_{i \to \infty} \operatorname{diam}\Phi(x_i) = 0$, we have $\bar{x} \in \bigcap_{i=1}^{\infty} \Phi(x_i)$.

Now we show that $h(\bar{x}) = \bar{x}$ for each $h \in \Phi$. In fact, since $h(\bar{x}) = h(\prod_{j=i+1}^{\infty} h_j(x_i))$, we have $h(\bar{x}) \in \Phi(x_i)$ for each i, and hence $h(\bar{x}) = \bar{x}$.

(b) Let $\bar{x} = \lim_{i \to \infty} h_i h_{i-1} \cdots h_1(x_0) = \lim_{i \to \infty} x_i$. Since $\{x_j\} \subset \Phi(x_i)$ for each i, we have $\bar{x} \in \operatorname{cl}\Phi(x_i)$, the closure of $\Phi(x_i)$. Since $\operatorname{diam}(\operatorname{cl}\Phi(x_i)) = \operatorname{diam}\Phi(x_i)$, we get $\bar{x} = \bigcap_{i=0}^{\infty} \operatorname{cl}\Phi(x_i)$.

To show that $h(\bar{x}) = \bar{x}$ for all $h \in \Phi$, observe $h(x_i) \in \Phi(x_i)$ for each i. Since h is continuous, for any $\varepsilon > 0$, there exists i_0 such that for any $\gamma \in (0, 1]$ and $i > i_0$,

$$d_\gamma(h(\bar{x}), h(x_i)) < \varepsilon \text{ and } d_\gamma(h(x_i), \bar{x}) \leq \operatorname{diam}\Phi(x_i) < \varepsilon_i. \tag{2.14}$$

Since $\{d_\alpha : \alpha \in (0, 1]\}$ is a family of quasi-metrics, for any given $\alpha \in (0, 1]$ there exists $\gamma \in (0, \alpha]$ such that

$$d_\alpha(h(\bar{x}), \bar{x}) \leq d_\gamma(h(\bar{x}), h(x_i)) + d_\gamma(h(x_i), \bar{x}).$$

¿From (2.14) it follows that

$$d_\alpha(h(\bar{x}), \bar{x}) < \varepsilon + \varepsilon_i$$

for all $i > i_0$, and so we get $d_\alpha(h(\bar{x}), \bar{x}) < \varepsilon$ for any $\alpha \in (0, 1]$ because $\varepsilon_i \to 0$. Since ε is arbitrary, we have $h(\bar{x}) = \bar{x}$, which completes the proof.

As a consequence of Lemma 2.2 and Theorem 2.1, we can obtain the following:

Corollary 2.2 Let $(X, d_\alpha : \alpha \in (0,1])$, $(Y, \delta_\alpha : \alpha \in (0,1])$, f and Φ be the same as in Theorem 2.1. Then the family Φ^* has a common fixed point. Futhermore, if $h \in \Phi^*$ is continuous, then for any $x_0 \in X$, $\bar{x} = \lim_{i\to\infty} h^i(x_0)$ is a fixed point of h.

By putting $X = Y$, $f = I_X$, the identity mapping, and $c = 1$, we establish a general version of Siegel's fixed point theorem in generating spaces of quasi-metric family.

Theorem 2.3 *Let $(X, d_\alpha : \alpha \in (0,1])$ be a complete generating space of quasi-metric family, $\varphi : X \to \mathbb{R}^+$ a lower semicontinuous function, and Φ^* the family of all $h : X \to X$ satisfying*

$$d_\alpha(x, h(x)) \le k(\alpha)\{\varphi(x) - \varphi(h(x))\}$$

for any $x \in X$ and $\alpha \in (0,1]$. Let $\Phi \subset \Phi^*$ be closed under composition and $x_0 \in X$. Then the conclusions of Theorem 2.1 hold.

As a consequence of Corollary 2.2 for a single mapping, we also have the following result, which is a general version of Downing-Kirk's fixed point theorem in generating spaces of quasi-metric family.

Theorem 2.4 *Let $(X, d_\alpha : \alpha \in (0,1])$, $(Y, \delta_\alpha : \alpha \in (0,1])$, f and φ be the same as in Theorem 2.1. If $c > 0$ is a given constant and a mapping $h : X \to X$ satisfies*

$$\max\{d_\alpha(x, h(x)), c\delta_\alpha(f(x), f(h(x)))\} \le k(\alpha)\{\varphi(f(x)) - \varphi(f(h(x)))\}$$

for any $\alpha \in (0,1]$ and $x \in X$, then h has a fixed point.

If we take $X = Y$, $f = I_X$, and $c = 1$ in Theorem 2.4, we obtain the following:

Theorem 2.5 *Let $(X, d_\alpha : \alpha \in (0.1])$ be a complete generating space of quasi-metric family and let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function. If $h : X \to X$ is a mapping satisfying*

$$d_\alpha(x, h(x)) \le k(\alpha)\{\varphi(x) - \varphi(h(x))\}$$

for any $\alpha \in (0,1]$ and $x \in X$, then h has a fixed point.

Remark 2.1 (1) Theorem 1 in [15] is a special case of Theorem 2.1 with X being a metric space and $k(\alpha) \equiv 1$.

(2) Siegel's fixed point theorem in [17] is a special case of Theorem 2.3 with $k(\alpha) \equiv 1$ and X being a metric space.

(3) When X is a metric space and $k(\alpha) \equiv 1$, from Theorem 2.4 and Theorem 2.5, we obtain Downing-Kirk's fixed point theorem [6] and Caristi's fixed point theorem [2], respectively.

(4) Corollary 2.2 is also a generalization of Corollary of Theorem 1 in [15].

3 General Versions in Fuzzy Metric Spaces

A mapping $x : \mathbb{R} \to [0,1]$ is called a *fuzzy number*. For $\alpha \in (0,1]$ and a fuzzy number x, the set

$$[x]_\alpha = \{u \in \mathbb{R} : x(u) \geq \alpha\}$$

is called a *α-level set* of x. A fuzzy number x is called to be *convex* if $r,\ s,\ t \in \mathbb{R},\ r \leq s \leq t$, implies

$$\min\{x(r), x(t)\} \leq x(s).$$

A fuzzy number x is said to be *normal* if there exists a point $u \in \mathbb{R}$ such that $x(u) = 1$. If a fuzzy number x is upper semicontinuous, convex and normal, then the α-level set of x is a closed interval $[a^\alpha, b^\alpha]$, that is,

$$[x]_\alpha = [a^\alpha, b^\alpha], \quad \alpha \in (0,1],$$

where the values $a^\alpha = -\infty$ and $b^\alpha = \infty$ are admissible. A fuzzy number x is called *nonnegative* if $x(u) = 0$ for all $u < 0$. The fuzzy number $\bar{\theta}$ is defined by $\bar{\theta}(u) = 1$ for $u = 0$ and $\bar{\theta}(u) = 0$ for $u \neq 0$.

Throughout this section, we denote by G the set of all nonnegative upper semicontinuous normal convex fuzzy numbers and we always assume that $L,\ R : [0,1] \times [0,1] \to [0,1]$ are two functions such that they are nondecreasing in both arguments, symmetric and $L(0,0) = 0,\ R(1,1) = 1$.

Let X be a nonempty set and $d : X \times X \to G$ be a mapping. Denote

$$[d(x,y)]_\alpha = [\lambda_\alpha(x,y),\ \rho_\alpha(x,y)],\ x,\ y \in X, \alpha \in (0,1], \tag{3.1}$$

where $[d(x,y)]_\alpha$ is the α-level set of fuzzy number $d(x,y) \in G$, which actually is a closed interval of $\mathbb{R}$ and $\lambda_\alpha(x,y)$, $\rho_\alpha(x,y)$ are the left and right end points of the closed interval $[d(x,y)]_\alpha$, respectively.

Definition 3.1 *[11] The quadruple (X, d, L, R) is called a fuzzy metric space if the mapping $d : X \times X \to G$ satisfies the following conditions:*
(FM-1) $d(x,y) = \bar{\theta}$ if and only $x = y$,
(FM-2) $d(x,y) = d(y,x)$ for all $x,\ y \in X$,
(FM-3) For any $x,\ y,\ z\ \in X$,

(i) $d(x,y)(s+t) \geq L(d(x,z)(s), d(z,y)(t))$ whenever $s \leq \lambda_1(x,z),\ t \leq \lambda_1(z,y)$ and $s + t \leq \lambda_1(x,y)$,

(ii) $d(x,y)(s+t) \leq R(d(x,z)(s), d(z,y)(t))$ whenever $s \geq \lambda_1(x,z),\ t \geq \lambda_1(z,y)$ and $s + t \geq \lambda_1(x,y)$.

Remark 3.1 By Theorem 3.2 in Kaleva and Seikkala [11], we know that if (X, d, L, R) is a fuzzy metric space with $\lim_{a \to 0^+} R(a,a) = 0$, then there exists a topology $\mathcal{T}_d$ on X such that $(X, \mathcal{T}_d)$ is a Hausdorff topological space and

$$\mathcal{U}(x) = \{U_x(\epsilon, \alpha) : \epsilon > 0,\ \alpha \in (0,1]\}, \quad x \in X,$$

is a basis of neighborhoods of the point x for the topology $\mathcal{T}_d$, where

$$U_x(\epsilon, \alpha) = \{y \in X : \rho_\alpha(x, y) < \epsilon\},$$

and $\rho_\alpha(x, y)$ is the right end point of $[d(x, y)]_\alpha$ defined by (3.1).

Proposition 3.1 [5] *Let (X, d, L, R) be a fuzzy metric space with*

$$\lim_{a \to 0^+} R(a, a) = 0, \quad \lim_{t \to \infty} d(x, y)(t) = 0, \quad x,\ y \in X \tag{3.2}$$

Letting $[d(x, y)]_\alpha = [\lambda_\alpha(x, y), \rho_\alpha(x, y)]$, then $(X, \rho_\alpha : \alpha \in (0, 1])$ is a generating space of quasi-metric family and the topology $\mathcal{T}_{\{\rho_\alpha\}}$ induced by the family $\{\rho_\alpha\}$ of quasi-metrics coincides with the fuzzy topology on $\mathcal{T}_d$ on (X, d, L, R).

Proof. Since $\lim_{t\to\infty} d(x, y)(t) = 0$, it follows that $\rho_\alpha(x, y) < \infty$ for all $\alpha \in (0, 1]$. Next we prove that $(X, \rho_\alpha : \alpha \in (0, 1])$ is a generating space of quasi-metric family. It is obvious that $(X, \rho_\alpha : \alpha \in (0, 1])$ satisfies the conditions (QM-1), (QM-2) and (QM-4) in Definition 2.1. Now we prove that it also satisfies the condition (QM-3).

By the assumption that $\lim_{a\to 0^+} R(a, a) = 0$, for any $\alpha \in (0, 1]$, there exists an $\mu \in (0, \alpha]$ such that $R(\mu, \mu) < \alpha$. For any given $x,\ y,\ z\ \in X$, let

$$\rho_\mu(x, z) = s, \quad \rho_\mu(z, y) = t.$$

By the definition of ρ_μ, it is easy to show $s \geq \lambda_1(x, z)$ and $t \geq \lambda_1(z, y)$.

(i) If $s + t \geq \lambda_1(x, y)$, then for any $\epsilon > 0$ it follows from (FM-3)(ii) that

$$\begin{aligned} (x, y)(s + t + 2\epsilon) &\leq R(d(x, z)(s + \epsilon), d(z, y)(t + \epsilon)) \\ &\leq R(\mu, \mu) < \alpha. \end{aligned}$$

Hence we have $\rho_\alpha(x, y) < 2\epsilon + s + t$. By the arbitrariness of ϵ, we obtain

$$\rho_\alpha(x, y) \leq s + t = \rho_\mu(x, z) + \rho_\mu(z, y). \tag{3.3}$$

(ii) If $s + t < \lambda_1(x, y)$ and $u = \lambda_1(x, y) - (s + t)$, then we have

$$\begin{aligned} 1 &= d(x, y)(\lambda_1(x, y) = d(x, y)(u + s + t) \\ &\leq R(d(x, z)(s + \frac{1}{2}u), d(z, y)(t + \frac{1}{2}u)) \\ &\leq R(\mu, \mu) < \alpha, \end{aligned}$$

which is a contradiction. Therefore, the case (ii) can not happen. This proves that $(X, \rho_\alpha : \alpha \in (0, 1])$ satisfies the condition (QM-3).

On the other hand, by Remark 3.1, the topology $\mathcal{T}_{\{\rho_\alpha\}}$ on the generating space of quasi-metric family $(X, \rho_\alpha : \alpha \in (0, 1])$ coincides with the fuzzy topology $\mathcal{T}_d$ on fuzzy metric space (X, d, L, R). This completes the proof.

From Theorem 2.1 and Proposition 3.1, we can obtain the following:

Theorem 3.2 *Let* $(X_i, d_i, L, R),\ i = 1,\ 2,$ *be two complete fuzzy metric spaces with*

$$\lim_{t\to\infty} d_i(x, y)(t) = 0,\ x,\ y\ \in X_i,\ i = 1,\ 2,\quad \textit{and}\quad \lim_{a\to 0^+} R(a, a) = 0.$$

Let $f : X_1 \to X_2$ *be a closed mapping,* $\varphi : f(X_1) \to \mathbb{R}$ *a lower semicontinuous function, bounded from below, and* Φ^* *the family of all* $h : X \to X$ *satisfying*

$$\max\{\rho_{1\alpha}(x, h(x)), c\,\rho_{2\alpha}(f(x), f(h(x)))\} \le k(\alpha)\{\varphi(f(x)) - \varphi(f(h(x)))\}$$

for all $\alpha \in (0, 1]$ *and* $x \in X_1$*, where* $c > 0$ *is a given constant and* $\{\rho_{i\alpha} : \alpha \in (0, 1]\}$ *is the family of quasi-metrics on* X_i *defined by* (3.1)*,* $i = 1,\ 2.$ *Let* $\Phi \subset \Phi^*$ *be closed under composition and* $x_0 \in X_1$*.*

(a) *If* Φ *is closed under countable composition, then exists* $\bar{h}$ *such that* $\bar{x} = \bar{h}(x_0)$ *and* $h(\bar{x}) = \bar{x}$ *for all* $h \in \Phi$*.*

(b) *If each mapping in* Φ *is continuous, then there exist a sequence* $\{h_i\}$ *in* Φ *and*

$$\bar{x} = \lim_{i\to\infty} h_i h_{i-1} \cdots h_1(x_0)$$

in X_1 *such that* $h(\bar{x}) = \bar{x}$ *for each* $h \in \Phi$*.*

Corollary 3.3 Let (X, d, L, R) be a complete fuzzy metric space with

$$\lim_{t\to\infty} d(x, y)(t) = 0,\ x,\ y \in X,\quad \text{and}\quad \lim_{a\to 0^+} R(a, a) = 0.$$

Let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function and let Φ^* be the family of all $h : X \to X$ satisfying

$$\rho_\alpha(x, h(x)) \le k(\alpha)\{\varphi(x) - \varphi(h(x))\}$$

for any $\alpha \in (0, 1]$ and $x \in X$, where $\{\rho_\alpha : \alpha \in (0, 1]\}$ is the family of quasi-metrics on X defined by (3.1). Let $\Phi \subset \Phi^*$ be closed under composition and $x_0 \in X$. Then the conclusions of Theorem 3.2 hold.

Proof. The results can be obtained from Theorem 3.2 if $X_1 = X_2,\ f = I_{X_1}$ and $c = 1$.

Corollary 3.4 Let $(X_i, d_i, L, R),\ i = 1,\ 2,\ f,$ and φ be the same as in Theorem 3.2, c a positive constant, and $h : X_1 \to X_1$ a mapping satisfying

$$\max\{\rho_{1\alpha}(x, h(x)), c\,\rho_{2\alpha}(f(x), f(h(x)))\} \le k(\alpha)\{\varphi(f(x)) - \varphi(f(h(x)))\}$$

for any $\alpha \in (0, 1]$ and $x \in X_1$, where $\{\rho_\alpha : \alpha \in (0, 1]\}$ is the family of quasi-metrics on X defined by (3.1). Then h has a fixed point .

Proof. The result follows from Theorem 2.4 and Proposition 3.1.

Corollary 3.5 Let (X, d, L, R) be a complete fuzzy metric space with

$$\lim_{t\to\infty} d(x, y)(t) = 0,\ x,\ y \in X,\quad \text{and}\quad \lim_{a\to 0^+} R(a, a) = 0.$$

Let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function and let $h : X \to X$ be a mapping satisfying

$$\rho_\alpha(x, h(x)) \leq k(\alpha)\{\varphi(x) - \varphi(h(x))\}$$

for any $\alpha \in (0, 1]$ and $x \in X$, where $\{\rho_\alpha : \alpha \in (0, 1]\}$ is the family of quasi-metrics on X defined by (3.1). Then h has a fixed point.

Remark 3.2 (1) Theorem 3.2 is a generalization of Theorem 1 in [15] to the case of fuzzy metric spaces.

(2) Corollary 3.3 extends Siegel's fixed point theorem [17] to the case of fuzzy metric spaces.

(3) Corollary 3.3 and Corollary 3.6 in [10] are special cases of Corollary 3.4 and Corollary 3.5 with $R = \max$. Theorem 4.1 in [8] is also a special case of Corollary 3.5 with $R = \max$ and $k(\alpha) \equiv 1$.

4 Generalizations in Probabilistic Metric Spaces

In this section, by using Theorem 2.1, we give generalizations of Siegel's fixed point theorem [17] in probabilistic metric spaces.

Throughout this section, we denote by $\mathcal{D}$ the set of all left continuous distribution functions.

A function $\triangle : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if the following conditions are satisfied:

(TN-1) $\triangle(a, b) = \triangle(b, a)$,
(TN-2) $\triangle(a, 1) = a$,
(TN-3) $\triangle(a, \triangle(b, c)) = \triangle(\triangle(a, b), c)$,
(TN-4) $\triangle(a, b) \leq \triangle(c, d)$ for $a \leq c$ and $b \leq d$.

Definition 4.1 *[16] A triple $(X, F, \triangle)$ is called a Menger probabilistic metric space (briefly, a Menger PM-space) if X is a nonempty set, $\triangle$ is a t-norm and $F : X \times X \to \mathcal{D}$ is a mapping satisfying the following conditions (we shall denote $F(x, y)$ by $F_{x,y}$):*

(PM-1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,
(PM-2) $F_{x,y}(0) = 0$,
(PM-3) $F_{x,y} = F_{y,x}$,
(PM-4) $F_{x,y}(s + t) \geq \triangle(F_{x,z}(s), F_{z,y}(t))$ for all $x,\ y,\ z\ \in X,\ s,\ t \geq 0$.

Remark 4.1 It is pointed out in Schweizer and Sklar [16] that if $\triangle$ satisfies the condition $\sup_{t<1} \triangle(t, t) = 1$, then there exists a topology $\mathcal{T}$ on X such that $(X, \mathcal{T})$ is a Hausdorff topological space and the family of sets

$$\mathcal{U}(p) = \{U_p(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0, 1]\}, \quad p \in X,$$

is a basis of neighborhoods of the point p for $\mathcal{T}$, where

$$U_p(\epsilon, \lambda) = \{x \in X : F_{x,p}(\epsilon) > 1 - \lambda\}.$$

Usually, the topology $\mathcal{T}$ is called (ϵ, λ)-topology on $(X, F, \triangle)$.

Proposition 4.1[5] *Let $(X, F, \triangle)$ be a Menger probabilistic metric space with the t-norm $\triangle$ satisfying the condition*:

$$\sup_{t<1} \triangle(t, t) = 1. \tag{4.1}$$

For any $\alpha \in (0, 1]$, we define $d_\alpha : X \times X \to \mathbb{R}^+$ as follows:

$$d_\alpha(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \alpha\}. \tag{4.2}$$

Then (1) *$(X, d_\alpha : \alpha \in (0, 1])$ is a generating space of quasi-metric family and*

(2) *the topology $\mathcal{T}_{\{d_\alpha\}}$ on $(X, d_\alpha : \alpha \in (0, 1])$ coincides with the (ϵ, λ)-topology $\mathcal{T}$ on $(X, F, \triangle)$.*

Proof. (1) From the definition of $\{d_\alpha : \alpha \in (0, 1]\}$, it is easy to see that $\{d_\alpha : \alpha \in (0, 1]\}$ satisfies the conditions (QM-1) and (QM-2) in Definition 2.1. Besides, it follows clearly that d_α is nonincreasing in α.

Next we prove that d_α is left continuous in α. In fact, for any given $\alpha_1 \in (0, 1]$ and $\epsilon > 0$, from the definition of d_α, there exists a $t_1 > 0$ such that $t_1 < d_{\alpha_1}(x, y) + \epsilon$ and $F_{x,y}(t_1) > 1 - \alpha_1$. Letting $\delta = F_{x,y}(t_1) - (1 - \alpha_1) > 0$ and $\lambda \in (\alpha_1 - \delta, \alpha_1]$, we have

$$1 - \alpha_1 < 1 - \lambda < 1 - (\alpha_1 - \delta) = F_{x,y}(t_1),$$

which implies that $t_1 \in \{t > 0 : F_{x,y}(t) > 1 - \lambda\}$. Hence we have

$$\begin{aligned} d_{\alpha_1}(x, y) &\leq d_\lambda(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\} \\ &\leq t_1 < d_{\alpha_1}(x, y) + \epsilon, \end{aligned}$$

which shows that d_α is left continuous in α.

Finally, we prove that $(X, d_\alpha, : \alpha \in (0, 1])$ also satisfies the condition (QM-3).

By the condition (4.1), for any given $\alpha \in (0, 1]$, there exists an $\mu \in (0, \alpha]$ such that

$$\triangle(1 - \mu, 1 - \mu) > 1 - \alpha.$$

Letting $d_\mu(x, z) = \sigma$ and $d_\mu(z, y) = \beta$, from (4.2), for any given $\epsilon > 0$, we have

$$F_{x,z}(\sigma + \epsilon) > 1 - \mu, \quad F_{z,y}(\beta + \epsilon) > 1 - \mu$$

and so

$$\begin{aligned} F_{x,y}(\sigma + \beta + 2\epsilon) &\geq \triangle(F_{x,z}(\sigma + \epsilon), F_{z,y}(\beta + \epsilon)) \\ &\geq \triangle(1 - \mu, 1 - \mu) > 1 - \alpha. \end{aligned}$$

Hence we have

$$d_\alpha(x, y) \leq \sigma + \beta + 2\epsilon = d_\mu(x, z) + d_\mu(z, y) + 2\epsilon.$$

By the arbitrariness of $\epsilon > 0$, we have

$$d_\alpha(x, y) \leq d_\mu(x, z) + d_\mu(z, y).$$

(2) To prove the conclusion (2), it is enough to prove that for any $\epsilon > 0$ and $\alpha \in (0, 1]$,

$$d_\alpha(x, y) < \epsilon \quad \text{if and only if} \quad F_{x,y}(\epsilon) > 1 - \alpha.$$

In fact, if $d_\alpha(x, y) < \epsilon$, from (4.2), we have $F_{x,y}(\epsilon - \mu) > 1 - \alpha$.

Conversely, if $F_{x,y}(\epsilon) > 1 - \alpha$, since $F_{x,y}$ is a left continuous distribution function, there exists an $\mu > 0$ such that $F_{x,y}(\epsilon - \mu) > 1 - \alpha$, and so $d_\alpha(x, y) \le \epsilon - \mu < \epsilon$. This completes the proof.

From Theorem 2.1 and Proposition 4.1, we can obtain the following:

Theorem 4.2 *Let $(X_i, F_i, \triangle_i)$, $i = 1, 2$, be two complete Menger probabilistic metric spaces with the t-norms $\triangle_i$ satisfying the condition (4.1), $f : X_1 \to X_2$ a closed mapping, $\varphi : f(X_1) \to \mathbb{R}$ a lower semicontinuous function, bounded from below, c a positive constant, and Φ^* the family of all $h : X \to X$ satisfying*

$$\max\{\inf\{t > 0 : \ F_{1x,h(x)}(t) > 1 - \alpha\}, c \inf\{t > 0 : F_{2f(x),f(h(x))}(t) > 1 - \alpha\}\} \le k(\alpha)\{\varphi(f(g(x)) - \varphi(f(y))\}$$

for any $\alpha \in (0, 1]$ and $x \in X_1$. Let $\Phi \subset \Phi^*$ be closed under composition and $x_0 \in X_1$.

(a) *If Φ is closed under countable composition, then there exists $\bar{h} \in \Phi$ such that $\bar{x} = \bar{h}(x_0)$ and $h(\bar{x}) = \bar{x}$ for all $h \in \Phi$.*

(b) *If each mapping in Φ is continuous, then there exist a sequence $\{h_i\}$ in Φ and*

$$\bar{x} = \lim_{i \to \infty} h_i h_{i-1} \cdots h_1(x_0)$$

in X such that $h(\bar{x}) = \bar{x}$ for each $h \in \Phi$.

In Theorem 4.2, if $(X, F, \triangle) \equiv (X_i, F_i, \triangle_i)$, $i = 1,\ 2$, $f = I_{X_1}$, and $c = 1$, then we can obtain the following:

Corollary 4.3 Let $(X, F, \triangle)$ be a complete Menger probabilistic metric space with the t-norm $\triangle$ satisfying the condition (4.1), $\varphi : X \to \mathbb{R}^+$ a lower semicontinuous function, bounded from below, and Φ^* the family of all $h : X \to X$ satisfying

$$\inf\{t > 0 : F_{1x,h(x)}(t) > 1 - \alpha\} \le k(\alpha)\{\varphi(x) - \varphi(h(x))\}$$

for any $\alpha \in (0, 1]$ and $x \in X$. Let $\Phi \subset \Phi^*$ be closed under composition and $x_0 \in X$. Then the conclusions of Theorem 4.2 hold.

For a single mapping $h : X_1 \to X_1$, we also have the following:

Corollary 4.4 Let $(X_i, F_i, \triangle_i)$, $i = 1,\ 2,\ f,\ \varphi$ be the same as in Theorem 4.2. If c is a positive constant and $h : X_1 \to X_1$ is a mapping satisfying

$$\max\{\inf\{t > 0 : F_{1x,h(x)}(t) > 1 - \alpha\}, c \inf\{t > 0 : F_{2f(x),f(h(x))}(t) > 1 - \alpha\}\} \le k(\alpha)\{\varphi(f(g(x)) - \varphi(f(y))\}$$

for any $\alpha \in (0,1]$ and $x \in X_1$, then h has a fixed point.

Corollary 4.5 Let $(X, F, \triangle)$ be a complete Menger probabilistic metric space with the t-norm $\triangle$ satisfying the condition (4.1), $\varphi : X \to \mathbb{R}^+$ a lower semicontinuous function, bounded from below, and $h : X \to X$ a mapping satisfying

$$\inf\{t > 0 : F_{1x,h(x)}(t) > 1 - \alpha\} \leq k(\alpha)\{\varphi(x) - \varphi(h(x))\}$$

for any $\alpha \in (0,1]$ and $x \in X$. Then h has a fixed point.

Proof. The result follows from Corollary 4.4 with $X_1 = X_2$, $F_1 = F_2$, $\triangle_1 = \triangle_2$, $f = I_{X_1}$, and $c = 1$.

Remark 4.2 (1) Corollary 4.3 extends Siegel's fixed point theorem [17] to the case of probabilistic metric spaces.

(2) Corollary 4.4 and Corollary 4.5 generalize Theorem 8 in [9] and Theorem 5.1 in [8], respectively.

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