

On the Stability of T-S Fuzzy Control for Non-Linear Systems

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Abstract

This work concerns the stability analysis of a non-linear system controlled by a fuzzy T-S control law. It is shown that the closed loop system is in general expressed by a T-S fuzzy system composed of rules with affine linear systems in their consequent parts. The stability of affine T-S systems is then investigated for a special case using as an example the regulation problem of a single link robot arm. Stability conditions are derived using the indirect and direct Lyapunov method and simulation results are presented.

Keywords: Fuzzy control, Non-linear systems, Stability

1 Introduction

The main feature of a Takagi-Sugeno (T-S) fuzzy model is to express the local dynamics of each fuzzy implication (rule) by a linear system model. The overall fuzzy model of the system is achieved by fuzzy “blending” of the linear system models. Parallel or feedback connections of T-S fuzzy systems which preserve the properties of each system are possible [1]. Thus a simple and straightforward approach for the control of non-linear systems has emerged [2]. By representing the non-linear system by a T-S type fuzzy model, linear feedback control techniques can be utilized to design a linear controller for each local linear model. The overall controller is a fuzzy blending of each individual linear controller, and therefore, non-linear but very simple to design. The closed loop fuzzy control system derived in this way is, in general, a system composed of rules with affine linear systems in their consequent parts. This design approach for the control of non-linear systems does not have the local character of a linear control design for the linearized, around

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an operating point, non-linear system, nor the complication and involvement of feedback linearization controllers. Its appeal is, however, dependent on the stability issues involved. While stability conditions have been exploited for T-S fuzzy systems composed of rules with linear consequent parts, the stability of T-S fuzzy systems composed of rules with affine linear system in their consequent parts needs further investigation.

2 Background material

A continuous T-S fuzzy plant model is composed of n plant rules that can be represented as

$$\text{Plant rule } i : \text{IF } x_1 \text{ is } M_{1i} \text{ AND } x_2 \text{ is } M_{2i} \dots x_r \text{ is } M_{ri} \text{ THEN } \dot{x} = A_i x + B_i u \quad (1)$$

where M_{pi} , $p = 1, \dots, r$ are fuzzy sets whose membership functions denoted by the same symbols are continuous piecewise polynomial functions and $x = [x_1, \dots, x_r]^T$ is the state vector. Then, the final output of the T-S fuzzy system is inferred as follows:

$$\dot{x} = \sum_{i=1}^n w_i(x)(A_i x + B_i u) \quad (2)$$

where the membership functions

$$w_i(x) = h_i(x) / \sum_{j=1}^r h_j(x), \quad h_i(x) = \prod_{j=1}^r M_{ji}(x_j) \quad (3)$$

are non-negative and normalized. That is,

$$\sum_{i=1}^n w_i(x) = 1 \quad (4)$$

We may also consider a T-S fuzzy control model composed of n rules having the same premises as those of the above plant, i.e.:

$$\text{Controller rule } i : \text{IF } x_1 \text{ is } M_{1i} \text{ AND } x_2 \text{ is } M_{2i} \dots x_r \text{ is } M_{ri} \text{ THEN } h = K_i x \quad (5)$$

Then, the final output of the T-S fuzzy controller is inferred as follows:

$$h = \sum_{i=1}^n w_i(x) K_i x \quad (6)$$

A closed loop control system can be constructed with a feedback connection of the two fuzzy blocks so that the control input of the plant is $u = r - h$ where r is a reference input (Figure 1).

Figure 1: Closed loop control system

The resulting closed loop control system is expressed by the fuzzy system:

$$\text{System rule } ij: \text{ IF } x \text{ is } (M_{pi} \text{ AND } M_{pj}) \text{ THEN } \dot{x} = (A_i - B_i K_j)x + B_i r \quad (7)$$

where x is $M_{pi} \Leftrightarrow x_1$ is M_{1i} AND ... AND x_r is M_{ri} . The membership function of the fuzzy set $(M_{pi} \text{ AND } M_{pj})$ is defined as $M_{pi} \times M_{pj}$ which is a continuous piecewise polynomial function (not necessarily convex) [1].

Thus, the final output of the closed loop T-S model is

$$\dot{x} = \sum_{i=1}^n \sum_{j=1}^n w_i(x)w_j(x)[A_{ij}x + B_i r] \quad (8)$$

where $A_{ij} = A_i - B_i K_j$.

The stability of the free fuzzy system (2) ($u = 0$) has been investigated in [1] using the Lyapunov direct method. The following stability theorem for the continuous time system holds.

Theorem 1 : *The equilibrium of the free fuzzy system*

$$\dot{x} = \sum_{i=1}^n w_i(x)A_i x \quad (9)$$

is asymptotically stable in the large if there exist a common positive definite matrix $P = P^T > 0$ such that the Lyapunov inequality holds:

$$A_i^T P + P A_i < 0, i = 1, \dots, n \quad (10)$$

Theorem 1, can be used to derive the stability condition for system (8) when the reference input is zero ($r = 0$). That is, the stability condition for the closed loop system with zero reference

$$\dot{x} = \sum_{i=1}^n \sum_{j=1}^n w_i(x)w_j(x)A_{ij}x \quad (11)$$

is to find a common matrix $P = P^T > 0$ such that the following Lyapunov inequality holds:

$$A_{ij}^T P + P A_{ij} < 0 \quad i, j = 1, 2, \dots, n \quad (12)$$

We can write 11 as [1]:

$$\dot{x} = \sum_{i=1}^n w_i(x) w_i(x) \left[A_{ii} x + 2 \sum_{i < j} w_i(x) w_j(x) G_{ij} x \right] \quad (13)$$

where $G_{ij} = \frac{A_{ij} + A_{ji}}{2}$, $i < j$.

Thus, we can state the stability for the closed loop system in the following theorem [2].

Theorem 2 : *The equilibrium of the fuzzy system (11) or (13) is asymptotically stable in the large if there exist a common positive definite matrix $P = P^T > 0$ such that the following inequalities hold:*

$$A_{ii}^T P + P A_{ii} < 0, \quad G_{ij}^T P + P G_{ij} < 0, \quad i, j = 1, 2, \dots, n \quad (14)$$

Stability theorems 1 and 2 involve the solution of matrix inequalities and are solved numerically by using LMI convex programming techniques.

An analytical stability solution exists in the following special case:

- If
- (i) (A_i, B_i) are controllable pairs
 - (ii) $B_i = B$ and
 - (iii) we can find K_i such that:

$$A = A_i - B K_i, i = 1, \dots, n \quad (15)$$

with A Hurwitz, then the system is stable.

Note than in this case, $G_{ij} = A$ and we can therefore choose P such that $A^T P + P A < 0$. However, a choice of K_i such that $A = A_i - B K_i$ may not be possible even if (A_i, B_i) is controllable.

The conditions for quadratic stability described by the above theorems are often found to be conservative. One of the reasons for conservatism is the requirement that the Lyapunov function be globally quadratic. Many systems do not allow the finding of such a function. A way to relax these conditions is to take into account the structural information given by the membership functions and thus require only that:

$$x^T (A_i^T P + P A_i) x < 0 \quad \forall x: w_i(x) > 0$$

A procedure to introduce these conditional requirements in the stability analysis is refered in [3]. An even more powerful relaxation comes from the consideration of Lyapunov functions that are piecewise quadratic.

3 Stability of affine T-S systems

So far, stability conditions in theorem 1 and 2 have been derived for T-S free systems like (9), (11), i.e. systems composed of rules with linear systems in their consequent parts. In general, however, the bias component in (7) may be either a non-zero constant (for example, in a regulation problem $r = const$) or time varying (in trajectory following $r(t)$). Furthermore, fuzzy modeling of a non-linear systems may result in fuzzy systems composed of rules with an additional constant term in their consequent parts. As shown in [4] the function approximation capabilities of the T-S system is substantially improved when constant terms are allowed. We will subsequently call these systems affine T-S systems.

An affine T-S fuzzy model of a system is composed of n rules that can be represented as:

$$\text{System rule } i : \text{ IF } x_1 \text{ is } M_{1i} \text{ AND } x_2 \text{ is } M_{2i} \dots x_r \text{ is } M_{ri} \text{ THEN } \dot{x} = A_i x + d_i \quad (16)$$

The final output of the T-S fuzzy system is inferred as:

$$\dot{x} = \sum_{i=1}^n w_i(x) [A_i x + d_i] \quad (17)$$

Let us assume that the equilibrium point of this system is $x = 0$. That is, we assume that:

$$\sum_{i=1}^n w_i(0) d_i = 0 \quad (18)$$

For convenience of notation, we can also express the affine T-S fuzzy system model as follows:

$$\text{System rule } i : \text{ IF } x_1 \text{ is } M_{1i} \text{ AND } x_2 \text{ is } M_{2i} \dots x_r \text{ is } M_{ri} \text{ THEN } \dot{\bar{x}} = \bar{A}_i \bar{x} \quad (19)$$

where $\bar{A}_i = \begin{bmatrix} A_i & d_i \\ 0 & 0 \end{bmatrix}$ and $\bar{x} = [x \quad 1]^T$.

Thus, the final output of the T-S fuzzy system is inferred as:

$$\dot{\bar{x}} = \sum_{i=1}^n w_i(x) \bar{A}_i \bar{x} \quad (20)$$

There is an increasing interest in studying the stability of affine T-S fuzzy systems. Using piecewise quadratic Lyapunov functions, stability conditions expressed as a set of LMIs have been suggested in [3] for a class of affine T-S systems which assume a zero constant term ($d_i = 0$) for all local models whose membership function contains the equilibrium. Note however, that (18) does not necessary imply zero d_i for the local models interpolated around the origin. Local stability of affine T-S systems through system linearization around the equilibrium point is examined in [5].

Let the Lyapunov candidate function for this system be $V(x) = x^T P x$, $P > 0$. Then, the equilibrium $x = 0$ of the affine fuzzy system is asymptotically stable in the large if the derivative of the Lyapunov function along the system's trajectories is negative definite, i.e. if

$$x^T (A_i^T P + P A_i) x + 2d_i^T P x < 0 \quad i = 1, 2, \dots, n \quad (21)$$

which can be written compactly as

$$\bar{x}^T (\bar{A}_i^T \bar{P} + \bar{P} \bar{A}_i) \bar{x} < 0, \quad i = 1, 2, \dots, n, \quad \text{with} \quad \bar{P} = \begin{bmatrix} P & 0 \\ 0 & p_e \end{bmatrix}, \quad p_e > 0 \quad (22)$$

It is interesting to note from the above relation that the stability of the fuzzy system with zero bias is not sufficient to guarantee the stability of the affine fuzzy system. That is, the existence of a common positive definite matrix $P = P^T > 0$ such that $A_i^T P + P A_i < 0$, $i = 1, 2, \dots, n$ does not imply the satisfaction of (21). It would be useful however to relate the stability of the affine fuzzy system with the stability of the corresponding bias free system and thus, investigate the extra conditions which are required for ensuring the stability of the equilibrium point and the parameters which influence stability. We will do so with the help of a simple example.

4 Example: Single-link robot arm

We demonstrate the fuzzy control design for the regulation of a single link robot arm (driven pendulum). The arm is governed by the equation:

$$\ddot{\theta} + a\dot{\theta} + b \sin \theta = u \quad a, b > 0 \quad (23)$$

where corresponds to the lower vertical position, u is the normalised torque input, the a term corresponds to viscous damping while b depends on the gravity and the distribution of mass.

As a first step in the design procedure, we must represent the non-linear system by a T-S fuzzy model. The above system may be approximated by a fuzzy blend of linear systems in the form of (2). For the regulation problem the model scheduling is governed by one variable, specifically, the link's angular position. To minimize the design effort and complexity we try to use as few rules as possible. Thus, we approximate the system by a three rule fuzzy model in a circle $[-\pi, \pi]$ which is the arm's full workspace, as follows:

$$x = [\theta \quad \dot{\theta}]^T, \quad A_i = \begin{bmatrix} 0 & 1 \\ -b \sin \theta_{oi} / \theta_{oi} & -a \end{bmatrix}, \quad \theta_{oi} = 0, \pm\pi/2, \pi, \quad B_i = B = [0 \quad 1]^T \quad (24)$$

Specifically,

$$A_1 = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -\frac{2b}{\pi} & -a \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix}.$$

Clearly, the approximation concerns the $\sin(\cdot)$ nonlinearity existing in the system (23). The membership functions describing the fuzzy sets are triangular and are shown in figure 2.

Figure 2. Membership functions

By introducing the following partitioning into the space of the link's angular position

$$D_1 = [-\frac{\pi}{2}, \frac{\pi}{2}], \quad D_2 = [\frac{\pi}{2}, \pi] \cup [-\pi, -\frac{\pi}{2}]$$

we define the weights associated with each rule as follows:

$$w_1(\theta) = \begin{cases} 1 - |\theta| \frac{2}{\pi}, & \theta \in D_1 \\ 0, & \theta \in D_2 \end{cases}, \quad w_3(\theta) = \begin{cases} 0, & \theta \in D_1 \\ |\theta| \frac{2}{\pi} - 1, & \theta \in D_2 \end{cases}, \quad dw_2(\theta) = 1 - w_1 - w_3.$$

Figure 3. Approximation of $\sin(\cdot)$ by the three rule fuzzy model

The T-S fuzzy model which approximates the non-linear system is given by the inferred system:

$$\dot{x} = \sum_{i=1}^n w_i(\theta)(A_i x + B u), \quad n = 3 \quad (25)$$

The goodness of approximation of the $\sin(\cdot)$ nonlinearity by this fuzzy model is shown in figure 3.

The next step is the design of a linear controller for each local linear model. The control objective is to reach a desired angle θ_d , i.e. $x_d = [\theta_d \ 0]^T$.

- (i) If the desired angle θ_d is zero, the fuzzy control input is defined as $u = 0 - \sum_{i=1}^n w_i(\theta) K_i x$ and the closed loop system is a free fuzzy system which further belongs to the special case discussed in section 2. Thus, it can be reduced to a LTI system. Specifically, since $B = B_i$ we can define a stability matrix A expressing the desired closed loop dynamics of the arm behaviour and try to find feedback gains K_i satisfying matrix equation (15). Let, $A = \begin{bmatrix} 0 & 1 \\ -c & -d \end{bmatrix}$, $c, d > 0$ and K_i are calculated to satisfy $A = A_i - BK_i$. Then, the T-S fuzzy model of the closed loop system becomes:

$$\dot{x} = \sum_{i=1}^n \sum_{j=1}^n w_i(\theta) w_j(\theta) (A_i - BK_i) x \Leftrightarrow \dot{x} = Ax \quad (26)$$

- (ii) When the desired angle is non zero, we can transform the equilibrium point of the non-linear system to zero, by providing the torque required at steady state, $u_s = b \sin \theta_d$. Thus, if we set $u = u_s + u'$, and define the state variables as $e = x - x_d = [\theta - \theta_d \ \dot{\theta}]^T$, we can write the T-S fuzzy plant model (25) as follows:

$$\dot{e} = \sum_{i=1}^n w_i(\theta) (A_i e + B u' + [A_i x_d + B u_s]) \quad (27)$$

The quantity into brackets is a constant bias and therefore the system is now modelled as a fuzzy blend of affine T-S systems. Note that although a constant term does not appear in the plant's fuzzy model, the existence of a non-zero reference input to the control system produced an affine T-S system.

It is easy to show that the free system equilibrium is zero, i.e. ($e = 0$ or $x = x_d$ for $u' = 0$):

$$0 = \sum_{i=1}^n w_i(\theta_d) (A_i x_d + B u_s) \quad (28)$$

We design the fuzzy controller as before and set:

$$u = - \sum_{i=1}^n w_i(\theta) K_i e \quad (29)$$

The T-S fuzzy model of the closed loop system is then given by:

$$\begin{aligned} \dot{e} &= \sum_{i=1}^n \sum_{j=1}^n w_i(\theta) w_j(\theta) (A_i - BK_i) + \sum_{i=1}^n w_i(\theta) (A_i x_d + B u_s) \\ \text{or } \dot{e} &= Ae + \sum_{i=1}^n w_i(\theta) (A_i x_d + B u_s) \end{aligned} \quad (30)$$

For asymptotic stability of this system at x_d it is necessary that the linearization of (30) yields a stable system. Locally, (30) can be described using (28) by the following linear system:

$$\dot{e} = A_{x_d} e \quad \text{where} \quad A_{x_d} = A + \sum_i A_i x_d \frac{\partial w_i}{\partial x}(x_d) \quad (31)$$

In this example, $A_i x_d = \begin{bmatrix} 0 \\ -\frac{b \sin \theta_{oi}}{\theta_{oi}} \theta_d \end{bmatrix}$ and after some algebra we obtain that

$$A_{x_d} = \begin{cases} A + \begin{bmatrix} 0 & 0 \\ b \frac{2}{\pi} (1 - \frac{2}{\pi}) |\theta_d| & 0 \end{bmatrix}, & \theta_d \in D_1 \\ A + \begin{bmatrix} 0 & 0 \\ b (\frac{2}{\pi})^2 |\theta_d| & 0 \end{bmatrix}, & \theta_d \in D_2 \end{cases} \quad (32)$$

Note that the derivative in (31) is defined everywhere except $\pm \frac{\pi}{2}$. The system is locally stable if

$$-c + b \frac{2}{\pi} \left(1 - \frac{2}{\pi}\right) |\theta_d| < 0, \quad \theta_d \in D_1 \quad (33)$$

$$\text{and } -c + b \left(\frac{2}{\pi}\right)^2 |\theta_d| < 0, \quad \theta_d \in D_2 \quad (34)$$

In order to ensure local stability in this example, c had to be large enough to dominate in the above terms. This can be achieved with faster closed loop dynamics. Therefore, a stable A , ($c, d > 0$), is not sufficient for the system stability. For global stability further analysis is required.

5 Lyapunov stability conditions for the simplified system

Consider the closed loop equation of a simplified plant like (30):

$$\dot{e} = Ae + \sum_{i=1}^n w_i(x) (A_i x_d + B u_s) \quad (35)$$

We may rewrite this system using the equation of the equilibrium condition (28) as:

$$\dot{e} = Ae + \sum_{i=1}^n A_i x_d [(w_i(x) - w_i(x_d))] \quad \text{or} \quad \dot{e} = Ae + C_d \Delta_d(x) \quad (36)$$

where C_d is a matrix and Δ_d a vector defined as

$$C_d = [A_1 x_d \dots A_n x_d] \quad \text{and} \quad \Delta_d = [(w_1(x) - w_1(x_d)), \dots, (w_n(x) - w_n(x_d))]^T.$$

Let the Lyapunov candidate function be $V(e) = e^T P e$, $P > 0$. Differentiate over time along system trajectories to obtain:

$$\dot{V}(e) = -e^T Q e + 2e^T P C_d \Delta_d(x) \quad (37)$$

where Q is the positive definite matrix given by:

$$A^T P + P A + Q = 0 \quad (38)$$

Now assuming $w_i(x)$ are Lipschitz, and since $\Delta_d(x_d) = 0$, there exists a positive γ such that

$$\|C_d \Delta_d(e + x_d)\| \leq \gamma \|e\| \quad (39)$$

for $e \in D$, the domain of Δ_d , provided that D is a compact set. Then, (39) holds for the whole of the state-space and from (37) it follows that:

$$\dot{V}(e) \leq -e^T Q e + 2\gamma \|P\| \|e\|^2 = -e^T (Q - 2\gamma \|P\| I) e \quad (40)$$

Theorem 3 : *The equilibrium $x = x_d$ of the affine fuzzy system (35) is asymptotically stable in the large if there exist a positive such that (39) is satisfied and*

$$\lambda_{\min}(Q) - 2\gamma \|P\| > 0 \quad (41)$$

where Q and P are defined in (38).

The above condition is conservative as long as it corresponds to the negative definiteness of $\dot{V}(e)$ in the whole of the state space. We note that this condition depends on γ which depends on $w_i(x)$ and x_d . Thus, we can relax (41) if we can find γ_i such that $\|C_d \Delta_d(e + x_d)\| \leq \gamma_i \|e\| \quad \forall x \in D_i$ where D_i corresponds to a partitioning of state induced by the membership functions and require that

$$-e^T (Q - 2\gamma_i \|P\| I) e < 0 \quad \forall x \in D_i.$$

6 Simulation

We have simulated a single link robot arm with dynamics expressed by (23) under the suggested control law consisting of the input torque at steady state and the fuzzy control law whose gains are designed to achieve a desired closed loop performance expressed by matrix A .

We consider two cases in the control design. In the first case matrix A is chosen so that it corresponds to a second order system with critical damping and a settling time of 0.5 sec. Figure 4 shows the response of the link's angular position to a step input of 1 rad for various initial positions belonging to different regions while figure 5 shows the system response in steps of increasing size covering almost the total region.

Figure 4. Angular position step responses in different regions

Figure 5. Angular position responses in steps of various sizes

Simulation results show that the system's response is asymptotically stable in all cases. Stability conditions are in this case satisfied for every desired position.

In the second case, matrix A is designed so as to correspond to critical damping but to a much slower system (with a settling time of 2 sec). Figure 6 shows the response of the link's angular position when the desired position is 2 rads. Despite considering initial positions close to the desired final position, the system response diverges from the desired value exhibiting an unstable behaviour. Stability conditions are not satisfied in this case.

Figure 6. Angular position responses in an unstable system

7 Conclusions

Stability issues regarding the T-S fuzzy control of non-linear systems have been considered and illustrated with the use of the single link robot regulation problem. Past work on stability of fuzzy systems can be easily extended when the aim is to stabilize the system at zero reference. However, when a non-zero set-point is present, stability conditions of the free system can not be carried through. Extra conditions are needed to guarantee stability. These conditions depend on the choice of the particular fuzzy set memberships. Future work includes the stability study for the general case with constant and time varying bias. Also of concern are robustness issues under modeling errors and parameter uncertainties.

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