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Specifying t-norms Based on the Value of $T(1/2, 1/2)$

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Abstract

We study here the behavior of the t-norms at the point (1/2,1/2). We indicate why this point can be considered as significant in the specification of t-norms. Then, we suggest that the image of this point can be used to classify the t-norms. We consider some usual examples. We also study the case of parameterized t-norms. Finally using the results of this study, we propose a uniform method of computing the parameters. This method allows not only having the same parameter-scale for all the families, but also giving an intuitional sense to the parameters.

Introduction

A goal of Fuzzy Logic is to extend the classical binary logic into an interval valued logic. When we talk about "fuzzy", we think about something between false and truth. If we denote the value truth by one and the value of false by zero, what is more fuzzy, more central than 1/2? An important component of logic is the logical operators like negation, conjunction, disjunction and implication. In extending the binary logic to fuzzy logic an interesting and central question concerns the behavior of these logical operators at this middle point of truth-value. Here we investigate this question.

In this paper we concentrate our attention on one particular logical operator: the and operator. This operator is implemented in the fuzzy logic by the class of operators called t-norms. T-norms have been well-studied and very good overviews and classifications of these operators can be found in the literature, see [1 - 3]. T-norms are usually defined as operators for two variables, associativity allowing the generalization of the definition to n variables. In order to study the t-norms at the "most fuzzy" point we will study the t-norms on (1/2,1/2). We show, taking into account the definitional constraints, how central this point is. We also indicate that defining a t-norm on this point can be a natural step after fulfilling the classical logic constraints.

These results push us to suggest that t-norms can be classified observing their image on the (1/2,1/2) point. We consider some usual t-norms. We pursuit our study by observing what happens in the case of parameterized t-norms. We consider three different families.

Finally using the results of this study and taking into account the classification aspect, we invert our reasoning and we propose a uniform method for computing the parameter of each family. This method allows not only having the same parameter-scale for all the families, but also giving an intuitional sense to the parameters.

1 T-norms

The concept of a triangular norm was introduced by Menger [4] in order to generalize the triangular inequality of a metric. The current notion of a t-norm and its dual operation (t- conorm) is due to Schweizer and Sklar [5].

Both of these operations can also be used as a generalization of the Boolean logic connectives to multi-valued logic. The t-norms generalize the conjunctive 'AND' operator and the t-conorms generalize the disjunctive 'OR' operator. This situation allows them to be used to define the intersection and union operation in fuzzy logic. This possibility was first noted by Höhle [9]. Klement [10], Dubois and Prade [11] and Alsina, Trillas, and Valverde [2] very early appreciated the possibilities of this generalization. Bonissone [12] investigated the properties of these operators with the goal of using them in the development of intelligent systems.

In this paper we will focus on the t-norms, but one should keep in mind that analogous observations could be made for the t-conorms based on the duality between these operators.

1.1. Definition

Formally, a t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$, having the following properties

- $T(a,b) = T(b,a)$ (1) Commutivity
- $T(a,b) \leq (T(c,d)$, if $a \leq c$ and $b \leq d$ (2) Monotonicity
- $T(a,T(b,c)) = T(T(a,b),c)$ (3) Associativity
- $T(a,1) = a$ (4) One as identity

1.2. Properties

A natural consequence of axioms (1-4) is the following property:

- $T(a,0) = 0$ (5)

Proof: We know that $T(a,0) \in [0,1]$, so $T(a,0) \geq 0$. And using axiom (2) with $b = d = 0$ and $a \leq 1$ because $a \in [0,1]$, we obtain the result.

Another property associated with this operator is that $T(a,b) \leq \text{Min}(a,b)$, a special case of this is that $T(a,a) \leq a$. Viewed as a logical connective, "*and*" operator, the t-norm has the general tendency of making truths decrease.

Using the commutivity property (1), we have the limit properties:

- $T(a,1) = a$ (L1)
- $T(1,a) = a$ (L2)
- $T(a,0) = 0$ (L3)
- $T(0,a) = 0$ (L4)

In other words the t-norms are completely defined on the edges of the unit square as shown in Figure #1.

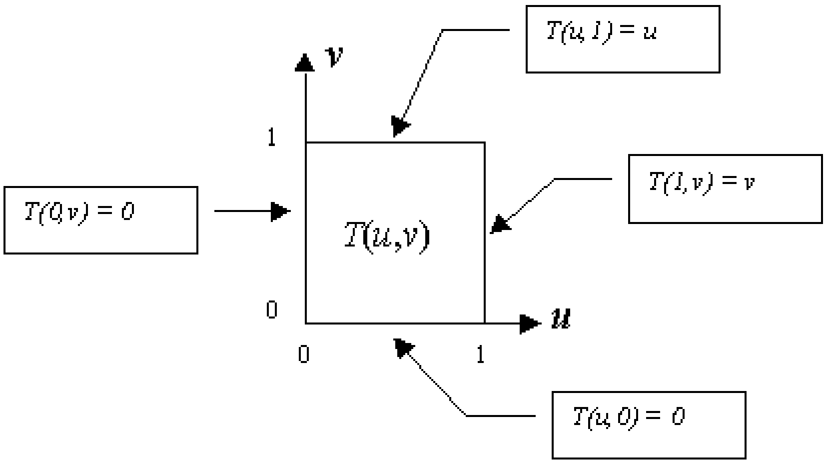

Figure #1: Definition of t-norms on edges of the unit square

We notice that the vertices of the unit square correspond to the arguments of the classical binary logic and here the t-norm emulates the classical AND operator:

- $T(1,1) = 1$ is the value of "True AND True" = "True"
- $T(1,0) = 0$ is the value of "True AND False" = "False"
- $T(0,1) = 0$ is the value of "False AND True" = "False"
- $T(0,0) = 0$ is the value of "False AND False" = "False"

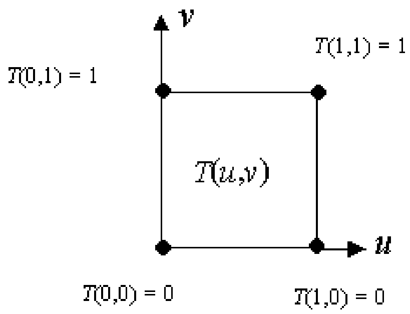

Figure #2: Definition of t-norms on the vertices of the unit square

1.3. The middle point

As noted the t-norms are constrained not only to follow the classical behavior at vertices of the unit square, but also to satisfy the limit properties (L1-L4), on the edges of the unit square. We observe that in the middle area of the unit square we have the freedom of choice. It is in this area we distinguish between different t-norm operators. An interesting point in this middle area, because of its central position is the point (1/2, 1/2). We know that it is the gravitational center of the square, it is also the intersection of the diagonals, the intersection of the middle lines and the barycenter of the edges of the square (the classical points). It also can be shown that it is the point of the unit square that is the maximal distance to the points of the border. In other words it is the "most distant" point of the already defined points and therefore it should be a useful point for characterizing t-norm operators.

We can also attach a sense to this point. The value 1/2 is exactly the point between False (0 point) and True (1 point). So studying the image of the point (1/2, 1/2) we are examining the behavior of the t-norm on the more fuzzy point. More particularly if a is a truth value then $Fuz(a) = 1 - |a - (1 - a)|$ measures the degree of fuzziness of the value a. The degree of fuzziness for a pair of points (a, b) can be measured as the average of the two, $Fuz(a) = 1/2(Fuz(a) + Fuz(b))$. Thus at the (1/2, 1/2) we are dealing with the most confused situation. Thus it would appear specifying the value of a t-norm would very useful in characterizing a t-norm.

1.4. Usual T-norms

Let us now take a look at what the image of (1/2, 1/2) is in some of the most typical t-norms. First we note that while we have some freedom in selecting T(1/2, 1/2) we have some restrictions. In particular we note that $T(1/2, 1/2) \leq T(1, 1/2) = 1/2$ and $T(1/2, 1/2) \geq T(0, 1/2) = 0$, thus $T(1/2, 1/2) \in [0, 1/2]$. We see that the common t-norms [1-3] have different values for the image of (1/2, 1/2). The most common t-norm is the Zadeh t-norm defined by:

$$\min(u, v) \tag{1}$$

We observe of course that all the conditions (L1-L4) that define a t-norm are fulfilled. The value of this t-norm at the point (1/2, 1/2) is 1/2. This operator takes the most uncertain truth-value. Actually this is the largest possible value we can get at the point (1/2, 1/2).

Consider now the probabilistic t-norm, defined by:

$$u \cdot v \tag{2}$$

The value of this t-norm at the point (1/2, 1/2) is 1/4.

Another interesting case is the Lukasiewicz t-norm, defined by:

$$\max(u + v - 1, 0) \tag{3}$$

Here we have that the middle value is 0.

It would be nice if we could uniquely define a t-norm only by giving its middle value. But things are not so easy. In fact we can have quite different t-norms for the same $T(1/2, 1/2)$ value. A very good example is the Lukasiewicz t-norm (3) and the drastic t-norm defined by:

$$\begin{cases} u & \text{if } v = 1 \\ v & \text{if } u = 1 \\ 0 & \text{anywhere else} \end{cases} \tag{4}$$

In both cases the middle point value equal 0.

As indicated above $T(1/2, 1/2)$ is a value of the [0,1/2] interval. Since we have shown an example for each extreme, this interval cannot be reduced.

While we have shown that in general it is not possible to uniquely specify a t-norm by indicating its value at the middle point for some classes of t-norms the specification of the middle can be used to uniquely identify a t-norm. According to this goal we shall look at some parameterized t-norms.

2 Parameterized t-norms

Considerable interest in the literature on t-norms has focused on the study of parameterized families of t-norms. Klir and Folger [1] provide a comprehensive list of families of parameterized t-norms. We are going to study here the Hamacher, the Weber and the Yager t-norms.

2.1. Hamacher

The Hamacher t-norms [6] are defined for $\gamma > 0$ by:

$$\frac{u \cdot v}{\gamma + (1 - \gamma) \cdot (u + v - u \cdot v)} \tag{5}$$

This equation is reduced in the middle point to:

$$\frac{1}{4 \cdot \gamma + 3 \cdot (1 - \gamma)} \tag{6}$$

If we plot the value of $T(1/2, 1/2)$ we will obtain the following graph:

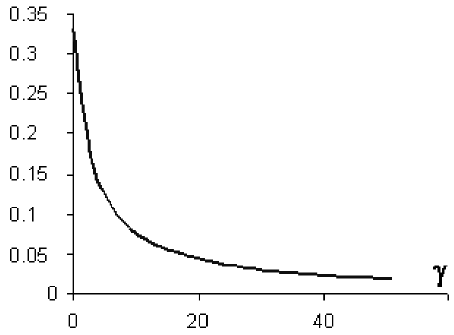

Figure #3: $T(1/2, 1/2)$ for different parameter values of the Hamacher t-norm

We see that $T(1/2, 1/2)$ varies between 0 and 1/3. So we can only obtain t-norms with the middle point in this domain. We see that it is a bijection in this region. It means that for each γ we have a unique value of $T(1/2, 1/2)$ and vice-versa for each value of $T(1/2, 1/2)$ we can find a γ. The formula that allows us to obtain γ from the middle point value is:

$$\gamma = \frac{1}{T(1/2, 1/2)} - 3 \qquad (7)$$

We see that this formula allows us to obtain a γ by giving the value of $T(1/2, 1/2) \in]0, 1/3]$. We can now study interesting particular cases:

- $T(1/2, 1/2) = 1/4$. With (7) we obtain $\gamma = 1$ and replacing this in (5) we obtain the product t-norm (2).
- $T(1/2, 1/2) \to 0^+$. With (7) we obtain $\gamma + \infty$ and replacing this in (5) we obtain the t-norm tends to the drastic t-norm (4).
- $T(1/2, 1/2) = 1/2$. We observed for $\gamma > 0$, $T(1/2, 1/2) > 1/3$ is impossible. So we can immediately conclude that Hamacher's t-norm cannot either generalize or even approach the Zadeh t-norm (1).

2.2. Weber

The Weber t-norms [7] are defined for $\lambda > -1$ by:

$$\max\left(\frac{u + v - 1 + \lambda \cdot u \cdot v}{1 + \lambda}, 0\right) \qquad (8)$$

This equation is reduced in the middle point to:

$$\max\left(\frac{\lambda}{4 \cdot (1 + \lambda)}, 0\right) \qquad (9)$$

If we plot the value of $T(1/2, 1/2)$ we will obtain the following graph:

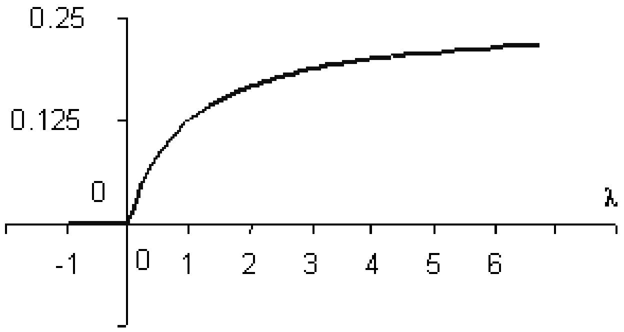

Figure #4: $T(1/2, 1/2)$ for different parameter values of the Weber t-norm

We obtain that $T(1/2, 1/2)$ lies between 0 and 1/4. We observe that for $\lambda \in [-1, 0]$, $T(1/2, 1/2)$ is constant and equals zero. For $\lambda \geq 0$, we have a bijection. It means that for each λ positive we have a different value of $T(1/2, 1/2)$ and vice-versa for each value of $T(1/2, 1/2) \in]0, 1/4]$ we can find a λ. For $\lambda \in [-1, 0]$, we note that $T(1/2, 1/2) = 0$, in other words we have an infinity of Weber t-norms having the middle point value equal to zero. The formula that allows us to obtain $\lambda \geq 0$ from the middle point value $T = T(1/2, 1/2) \in]0, 1/4]$ is:

$$\lambda = \frac{4 \cdot T}{1 - 4 \cdot T} \tag{10}$$

We see that this formula allows us to obtain a λ by giving the value of $T(1/2, 1/2) \in]0, 1/4]$. We can now note interesting particular cases:

- $T(1/2, 1/2) = 1/2$. We observed in the graph that for $\lambda > -1$, $T(1/2, 1/2) < 1/4$. So we can immediately conclude that Weber's t-norm cannot generalize (even approach) the Zadeh t-norm (1).
- $T(1/2, 1/2) \to 1/4$. With (10) we obtain $\lambda \to \infty$, we obtain that (8) tends to the product t-norm (2).
- $T(1/2, 1/2) = 0$. With (10) we obtain $\lambda = 0$ and replacing this in (8) we obtain the Lukasiewicz t-norm (3).

We notice for $T = 0$ we obtain the Lukasiewicz t-norm using the formula (10) to compute λ. But since we do not have a bijection for $-1 \leq \lambda \leq 0$ we cannot obtain, using this formula, the t-norms with $\lambda \in [-1, 0]$. In particular we cannot obtain the drastic t-norm, that is the particular case of (8) for $\lambda \to -1$.

2.3. Yager

The Yager t-norms [8] are defined for $p > 0$ by:

$$\max\left(1 - [(1-u)^p + (1-v)^p]^{1/p}, 0\right) \tag{11}$$

This equation is reduced in the middle point to:

$$\max\left(1 - 2^{\frac{1-p}{p}}, 0\right) \tag{12}$$

If we plot the value of $T(1/2, 1/2)$ we will obtain the following graph:

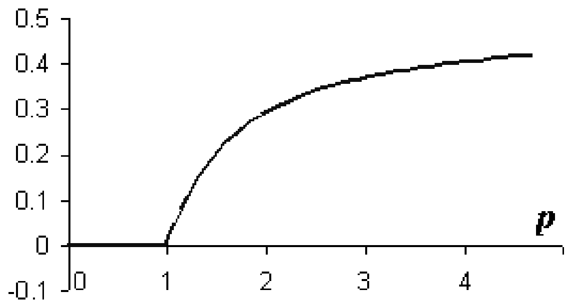

Figure #5: $T(1/2, 1/2)$ for different parameter values of the Yager t-norm

We see that $T(1/2, 1/2)$ lies between 0 and 1/2. We observe that for $p \in [0, 1], T(1/2, 1/2)$ is constant and equals zero, in other words we have an infinity of Yager t-norms having the middle point value equal to zero. For $p \geq 1$, we have a bijection. This means that for each $p \geq 1$ we have a different value of $T(1/2, 1/2)$ and vice-versa for each value of $T(1/2, 1/2) \in]0, 1/2]$ we can find a distinct value of p. The formula that allows us to obtain $p \geq 1$ from the middle point value $T = T(1/2, 1/2) \in]0, 1/2]$ is:

$$p = \frac{1}{1 + \ln_2(1-T)} \tag{13}$$

We see that this formula allows us to obtain a p by giving the value of $T(1/2, 1/2) \in [0, 1/2[$. We now note interesting particular cases:

- $T(1/2, 1/2) \to 1/2$. With (13) we obtain $p \to +\infty$ and replacing this in (11) we obtain that the t-norm tends to Zadeh's t-norm (1).
- $T(1/2, 1/2) = 1/4$. With (13) we obtain $p = 1.709$ and replacing this in (11) we obtain a t-norm that is not the product t-norm (2), but is the closest one in the Yager family.
- $T(1/2, 1/2) = 0$. With (13) we obtain $p = 1$ and replacing this in (11) we obtain the Lukasiewicz t-norm (3).

We notice for $T = 0$ we obtain the Lukasiewicz t-norm using formula (13) to compute p. But since we don't have a bijection for $0 \leq p \leq 1$ we can't obtain using this formula the t-norms with $p \in [0, 1[$. In particular we can't obtain the drastic t-norm, that is the particular case of (11) for $p \to 0$.

3 The inverting functions

In the precedent paragraph we introduced for each of the considered parameterized t-norms an inverting function: for Hamacher's t-norm it was formula (7), for Weber's t-norm it was formula (10) and for Yager's t-norm it was (13). These inverting functions allow us to obtain the parameter required so that the parameterized t-norm has a particular $T(1/2, 1/2)$ value. In other words if we impose the value of the $T(1/2, 1/2)$, using these formulas we can obtain the parameter that allows the t-norm to have this particular value.

From another perspective we can consider that the value of $T(1/2, 1/2)$ is the parameter, through the inverting formulas. This perspective has the advantage of having a uniform parameter for all the parameterized t-norms:

- The parameter T varies in all cases inside the [0,1/2] interval. So, we work always on the same interval.
- We observe that using this parameterization we will always evolve from the most drastic t-norm for $T = 0$ to the closest to the 'min' operator for $T = 1/2$. So, we have always the same kind of variation when the parameter T increases or decreases.
- We also notice that in order to obtain for example the product t-norm the parameter T will equal $1/4$, for any parameterized family. So, we have necessarily to use the same parameter for all parameterized t-norm families in order to obtain a certain particular case. For Zadeh's t-norm $T = 1/2$ and for Lukasiewicz and Drastic t-norms $T = 0$.

Notes: The fact that $T(1/2, 1/2)$ has a particular value does not guarantee that we generalize a behavior. For example we can with this method obtain a Yager t-norm satisfying $T(1/2, 1/2) = 1/4$. We will observe that it is not the product t-norm, but another t-norm of the class $T = 1/4$. But we can say that we have "the closest" Yager t-norm to the product, in the sense that they are equal on all the edges of the unit square and also in the middle point.

The value of $T(1/2, 1/2)$ says only in which class of t-norm we are. But it does not say anything about which t-norm in this class we are generalizing. For example for $T = 0$ we can have the drastic t-norm or the Lukasiewicz t-norm.

Another important thing that we would like to point out here is that $T(1/2, 1/2)$ in function of the parameters is usually only a bijection on a part of the [0,1/2] interval. The consequence of this is that the inverting function does not cover all the values of the parameter. In other words there are particular t-norms that are generalized by the usual definition of the parameterized t-norm family, that cannot be obtained using the inverting function. An example of that is that Yager's t-norm generalizes the drastic t-norm for $p = 0$, but we cannot obtain this value with the inverting formula (13) for $T \in [0, 1/2]$.

Conclusions

In this paper we studied the behavior of t-norms at the (1/2,1/2) point. We showed that this point is central in the definition of a t-norm. These observations encouraged us to suggest that t-norms can be classified with respect to their value at the (1/2,1/2) point. We considered some typical t-norms. We continued by observing what happens in the case of some parameterized t-norms. We considered three different families. We found that by studying the (1/2,1/2) we can learn about the generalization abilities of the parameterized family.

Finally using the results of this study and taking into account the classification aspect, we inverted our reasoning and we proposed a uniform method for computing the parameter of each family. This method allows not only to have the same parameter-scale for all the families, but also to give an intuitional sense to the parameters.

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