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Lattical Token Systems

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Abstract

Stochastic token theory is a new branch of mathematical psychology. In this paper we investigate algebraic properties of token systems defined on finite lattices.

1 Introduction

Studies in the area of token theory were originally undertaken by Falmagne in [3] and generalized in [5]. The reader is referred to [3-6] where token theory and its applications are discussed in detail. The following excerpt from [2] presents some motivations from the point of view of cognitive sciences.

"In many important empirical situations, subjects are repeatedly asked to provide judgments concerning commodities or individuals. It is also typical that these judgments take the form of binary relations such as rank orders, quasi orders or some other types of binary relations. For example, the subjects may be required, at times $t_1, \dots, t_k$, to rank order in terms of personal preference the candidates in a political election. Quantitative modeling of the temporal evolution of such judgments requires a thorough understanding of the combinatorics structure of the particular class of ordering considered. Indeed, it is reasonable to suppose that the judgment at time t will much resemble that at time $t+\delta$ if δ is small, and will tend to wander away from it if δ grows larger. Accordingly, if we want to understand the details of this evolution, we must research the structure of relevant families of orderings from the standpoint of the resemblance between particular orderings."

Such a study is a subject of the token theory. A significant part of this theory deals with temporal evolution of token systems and is based on the theory of stochastic processes. Here, we are concerned with the underlying algebraic model of token systems. First, we introduce some of the very basic algebraic concepts of the token theory.

Let $\mathcal{V}$ be a finite set of states. A token is a function $\tau : S \to S\tau$ mapping $\mathcal{V}$ into itself. We use abbreviations $S\tau = \tau(S)$, and $S\tau_1 \dots \tau_n = \tau_n[\dots[\tau_1(S)]$ for the function composition. We suppose that the identity function on $\mathcal{V}$ is not a token.

Let $\mathcal{T}$ be a distinguished set of tokens on $\mathcal{V}$. The pair $(\mathcal{V}, \mathcal{T})$ is called a token system. A token τ' is a reverse of a token τ if for all distinct $S, V \in \mathcal{V}$

$$S\tau = V \Leftrightarrow V\tau' = S.$$

In general, a token may have one or several reverses, or may not have any reverse. A finite composition m= $\tau_1 \ldots \tau_n$ of not necessarily distinct tokens $\tau_1, \ldots, \tau_n \in \mathcal{V}$ such that Sm= V is called a message producing V from S. A message is called consistent if it does not contain both a token and its reverse.

There are many other interesting and important concepts in the token theory. The reader is referred to [3-5] for an elaborate theory of tokens.

A simple example of a token system can be constructed as follows. Let A be a finite set and $\mathcal{B} = \mathbf{2}^A$ be the Boolean lattice of all subsets of A. For every $a \in A$ we define functions τ_a and τ_a' by

$$S\tau_a = S \cup \{a\} \text{ and } S\tau_a' = S \setminus \{a\}$$

for all $S \in \mathcal{B}$, respectively. Clearly, τ_a and τ_a' are unique reverses of each other. Let $\mathcal{T}$ be the set of all such functions. The token system $(\mathcal{B}, \mathcal{T})$ is called a universal token system.

This example motivates our studies of token systems that have certain lattice $\mathcal{L}$ as a set of states. We show that any such token system is essentially a universal one.

The paper is organized as follows. Our second section recalls some standard facts about atomic lattices. Although these facts are basically well known, there are subtle differences between our statements and others found in the pertinent literature. Thus we give complete proofs whenever it is necessary. In Section 3, we introduce lattical token systems and establish their properties including the main result of the paper. The paper ends with some final remarks.

2 Atomic lattices

In what follows, $\mathcal{L}$ is a finite lattice with operations $\wedge$ and $\vee$. We use 0 to denote the least element of $\mathcal{L}$ and 1 to denote the greatest element; x' denotes a complement of x in $\mathcal{L}$.

An element $a \in \mathcal{L}$ is an atom iff $x < a$ implies $x = 0$. An element $a \in \mathcal{L}$ is a dual atom iff $x > a$ implies $x = 1$. Let $\mathcal{A} = \{a_1, \cdots, a_n\}$ be the set of all atoms in $\mathcal{L}$ and $J = \{1, \cdots, n\}$.

Lemma 1. If a is an atom with a unique complement a', then a' is a dual atom.

Proof. Let $x > a'$. Then $a \vee x = 1$ so that $a \wedge x \neq 0$, since a' is unique. Hence, $a \wedge x = a$ which implies $a \vee x = x$. Thus $x = 1$.

□

Lemma 2. ([1, p.121]) If a, b are distinct atoms with unique complements, then $b' \geq a$.

Corollary. If $p \notin X \subset J$ and all atoms in $\{a_i\}_{i\in X}$ have unique complements, then

$$a'_p \geq \bigvee_{i\in X} a_i \,. \tag{1}$$

Lemma 3. If $X \subset J$, all atoms in $\{a_i\}_{i\in X}$ have unique complements, and

$$a_p \leq \bigvee_{i\in X} a_i \,,$$

then $p \in X$.

Proof. Suppose $p \notin X$. Then, by (1),

$$\bigvee_{i\in X} a_i \leq a'_p$$

implying $a_p \leq a'_p$, a contradiction.

□

A lattice $\mathcal{L}$ is atomic if every element in $\mathcal{L}$ is a join of atoms.

Lemma 4. Let $\mathcal{L}$ be an atomic lattice such that every atom has a unique complement. Then every element $x \in \mathcal{L}$ has a unique representation

$$x = \bigvee_{i\in X} a_i \tag{2}$$

for some $X \subset J$. If $x = \bigvee_{i\in X} a_i$ and $y = \bigvee_{i\in Y} a_i$, then

$$x \vee y = \bigvee_{i\in X\cup Y} a_i \quad \text{and} \quad x \wedge y = \bigvee_{i\in X\cap Y} a_i \;.$$

Proof. Suppose $x = \bigvee_{i\in X} a_i = \bigvee_{i\in X'} a_i$. For any $p \in X$, we have

$$a_p \leq \bigvee_{i\in X} a_i = \bigvee_{i\in X'} a_i \;.$$

By Lemma 3, $p \in X'$. Thus $X \subset X'$. By symmetry, $X = X'$.

Clearly, $x \vee y = \bigvee_{i\in X\cup Y} a_i$.

Let $z = x \wedge y$ and $z = \bigvee_{i\in Z} a_i$. Then

$$\bigvee_{i\in Z} a_i = \bigvee_{i\in X} a_i \;\wedge\; \bigvee_{i\in Y} a_i \;.$$

For $p \in Z$ we have $a_p \leq \bigvee_{i\in Z} a_i$. Hence, $a_p \leq \bigvee_{i\in X} a_i$ and $a_p \leq \bigvee_{i\in Y} a_i$. By Lemma 3, $p \in X \cap Y$. Hence, $Z \subset X \cap Y$. For $p \in X \cap Y$, $a_p \leq \bigvee_{i\in X} a_i$ and $a_p \leq \bigvee_{i\in Y} a_i$. Hence, $a_p \leq \bigvee_{i\in Z} a_i$. By Lemma 3, $p \in Z$. Hence, $X \cap Y \subset Z$.

□

Theorem 1. Let $\mathcal{L}$ be an atomic lattice such that every atom has a unique complement. Then $\mathcal{L}$ is isomorphic to the Boolean lattice $2^{\mathcal{A}}$ where $\mathcal{A}$ is the set of all atoms in $\mathcal{L}$.

Proof. By Lemma 3, $x \mapsto X$ in (2) establishes an isomorphism between $\mathcal{L}$ and $2^{\mathcal{A}}$. □

3 Lattical token systems

Let $\mathcal{L}$ be a lattice. For any given $a \in \mathcal{L}$ we define functions ϕ_a and ψ_a (cf. [1, p.73] by

$$\begin{aligned} \phi_a(x) &= x \wedge a \text{ for all } x \in \mathcal{L} \\ \psi_a(x) &= x \vee a \text{ for all } x \in \mathcal{L}. \end{aligned}$$

Let $(\mathcal{L}, \mathcal{T})$ be a token system such that any $\tau \in \mathcal{T}$ is either ϕ_a or ψ_a for some $a \in \mathcal{L}$. We call such a system a lattical token system. Since the identity function on $\mathcal{L}$ is not a token, $a \neq 1$ in ϕ_a and $a \neq 0$ in ψ_a.

A token τ' is a reverse of a token τ if for all $x \neq y$ in $\mathcal{L}$

$$x\tau = y \quad \Leftrightarrow \quad y\tau' = x.$$

Lemma 5. If $\phi_a \in \mathcal{T}$ has a reverse $\tau \in \mathcal{T}$, then $\tau = \psi_b$ where b is an atom in $\mathcal{L}$ and $a = b'$ is a unique complement of b.

Proof. Suppose $\tau = \phi_b$. Then $1\phi_a = a \neq 1$ implies $a\phi_b = a \wedge b = 1$. Thus $a = 1$, a contradiction. Hence, $\tau = \psi_b$ for some $b \in \mathcal{L}$.

For any $x > a$ we have $x\phi_a = x \wedge a = a \neq x$. Thus $x = a\psi_b = a \vee b$. Since $1 > a$, $x = 1$. Hence, a is a dual atom and $a \vee b = 1$.

For any $x < b$ we have $x\psi_b = x \vee b = b \neq x$. Thus $x = b\phi_a = b \wedge a$. Since $0 < b$, $x = 0$. Hence, b is an atom and $a \wedge b = 0$.

Suppose ϕ_a has two reverses, ψ_b and ψ_c. We have $b\phi_a = b \wedge a = 0 \neq b$. Thus $b = 0\psi_c = c$.

□

Lemma 6. If $\psi_a \in \mathcal{T}$ has a reverse $\tau \in \mathcal{T}$, then $\tau = \phi_b$ where $b = a'$ is a unique complement of a.

Proof. The proof is dual to the previous one. Note that b is a dual atom, by Lemma 1.

□

The next theorem follows immediately from lemmas 5 and 6.

Theorem 2. Suppose any token in $(\mathcal{L}, \mathcal{T})$ has a reverse. Then $\mathcal{T} = \{(\psi_a, \phi_{a'})\}_{a \in A}$ where A is a set of atoms such that any $a \in A$ has a unique complement a'. ψ_a and $\phi_{a'}$ are mutual reverses in $(\mathcal{L}, \mathcal{T})$.

Remark. The theorem does not claim that such a token system exists. Moreover, there are lattices with no token systems on them. Examples include nonmodular lattices N_5 and M_3. On the other hand, there is a token system on the nonmodular lattice $N_5 \times C_2$.

Theorem 3. Let $(\mathcal{L}, \mathcal{T})$ be a lattical token system such that every token has a reverse and for any $x \in \mathcal{L}$ there is a consistent message m$= \tau_1 \cdots \tau_m$ transforming 0 into x. Then $\mathcal{L}$ is a Boolean lattice and $\mathcal{T} = \{(\psi_a, \phi_{a'})\}_{a \in A}$ where $\mathcal{A} = \{a_1, \cdots, a_n\}$ is the set of all atoms in $\mathcal{L}$.

Proof. Suppose $x = 0\mathrm{m} = 0\tau_1 \cdots \tau_m$. We may assume that there is k such that $\tau_i = \psi_{a_i}$ for $1 \leq i \leq k$. If $k = m$, then $x = a_1 \vee \cdots \vee a_m$. Otherwise, $\tau_{k+1} = \phi_{a'_{k+1}}$. Since m is consistent, a_{k+1} is different from all $a_1, \cdots, a_k$. By (1),

$$0\tau_1 \cdots \tau_k \tau_{k+1} = (a_1 \vee \cdots \vee a_k) \wedge a'_{k+1} = a_1 \vee \cdots \vee a_k = 0\tau_1 \cdots \tau_k.$$

Thus $x = 0\mathrm{m}'$ where m' is a consistent message obtained from m by removing token τ_{k+1}. We keep applying this process until we obtain $x = 0\mathrm{n}$ where n contains no tokens in the form $\phi_{a'}$. Then x is a join of atoms. Hence, $\mathcal{L}$ is an atomic lattice. Clearly, $\mathcal{T} = \{(\psi_a, \phi_{a'})\}_{a \in A}$. By Theorem 1, $\mathcal{L}$ is a Boolean lattice.

□

This theorem states that any lattical token system satisfying the two conditions of the theorem is isomorphic to the universal token system introduced in Section 1.

4 Concluding remarks

We have shown that under some simple conditions any lattical token system is a token system based on the Boolean lattice of all subsets of a finite set. Thus each token acts on a given state S as an addition (subtraction) of a single element to (from) S. This fact is in compliance with the stochastic token theory where tokens are considered as quantum items of information transforming individual states.

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