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On Contrast Intensification Operators and Fuzzy Equality Relations

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Abstract

The class of contrast intensification operators is formally defined and it's lattice structure studied. The effect of these operators in the referential classifications derived from special kinds of fuzzy relations is also determined. Results and examples are presented providing contrast intensification operators which keep quasi-uniformity structures generated by fuzzy relations while diminishing the fuzziness or the entropy of the relations.

Keywords. Hedges, entropy, fuzzy similarity, weak properties.

1 Introduction

In ordinary language and in approximate reasoning, linguistic instead of numerical variables or values are commonly used. With the help of such linguistic values, the range of each variable can be separated into a finite number of approximate descriptors –low, medium, high etc.– thus allowing to reasoning or computing on the variables in a summarizing simplified way, as a result of the use of the discretized finite range for the variables, instead of the infinite numerical one, diminishing the number of the rules in the descriptions of the systems and making them more manageable.

Linguistic variables are obtained from primary ones, which are fuzzy subsets. These are modified by the so-called linguistic hedges.

Hedges appeared early in Zadeh's works (cf. [18]). They can be see as linguistic terms by means of which other linguistic terms are modified. They consist on unary operations in the unit interval, which are applied to the membership values of the fuzzy subsets.

Examples of this modifiers are concentration, dilatation, fuzzyfication and others.

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The first issue we are dealing with here is an special kind of hedges, the contrast intensification operators –hereafter *sharpeners*–, which act upon the fuzzy subsets making then less fuzzy, crisper or having less entropy.

Zadeh (see [19]) and Ragade-Gupta (see [13]) proposed examples of such operators which are given as combinations of concentrations and dilatations in some segments of the unit interval.

These operators are of interest in some decision problems, where at the end one has to consider fuzzy subsets or systems which appear in the evaluation of situations, for which the measure of entropy is relevant in the choosing of good methods for the construction of the decision devices (see [2]). Examples of such decision situations appear in the desing of fuzzy systems described by means of fuzzy relational equations (see [2], [4]).

Another important subject we are concerned here, if that of similarities and fuzzy classifications, which play an important role in reasoning with incomplete information, and in the control techniques based upon it, where the degree in which properties of one object can be inferred from those of another similar objects depending on their degree of equality. This situation occurs on the basis of interpolative and analogical reasoning, and so can be found in the methodologies for constructing fuzzy *if-then* rules by means of interpolation and approximation from paradigmatic examples. (see [15], [12]).

Our aim in this paper is to study the structure of the sets of sharpeners ,and to use it to study the effect these operators produce in the classifications derived from fuzzy relations of preorder or similarity, the latter being considered in a wide sense.

The paper is structured as follows: in the second section the concept of sharpened order is remembered and the concept of contrast intensification operator – hereinafter *sharpener* – is formalized. Also the structure of the set of sharpeners is analyzed.

In the third section the definitions of sharpened order and of sharpeners are extended to the classes of fuzzy sets and of fuzzy relations by setting their action in the corresponding power sets. In the fourth section the notions and results on weak properties for fuzzy relations are briefly summarized, and the main results, concerning the actions of the sharpeners on the pseudometrical classificatory structures derived from the relations, are presented.

2 Sharpeners on the unit interval

By $F(X)$ we denote $[0,1]^X$, the set of fuzzy subsets on the universe X. Their elements by A, B, etc., which are determined by the characteristic functions $A(x)$. In the Zadeh's sense, subset operations are defined

$$(A \cup B)(x) = A(x) \vee B(x) \qquad (A \cap B)(x) = A(x) \wedge B(x)$$

$$cA(x) = 1 - A(x)$$

where $\vee, \wedge$ denote the supremum and infimum operations in $[0,1]$. When the operations are defined via different triangular conorms S or norms T, they are denoted by $A \cup_S B$ and $A \cap_T B$.

In the interval $[0,1]$, we define the *sharpened* order relation $\preceq$

$$\alpha \preceq \beta \Longleftrightarrow (\frac{1}{2} \leq \alpha \leq \beta \qquad \text{or} \qquad \beta \leq \alpha < \frac{1}{2})$$

which describes the fact of being nearest to crisp values. This is a partial order which is relevant in the study of measures of fuzziness or entropy of fuzzy sets.(see [3], [14])

By using $\preceq$ it is possible define the concept of sharpener operator in [0,1].

Definition 1 *A sharpener on* $[0,1]$ *is a* $\Phi \in F([0,1])$ *verifying*

$$(S1) \qquad \alpha \preceq \Phi(\alpha) \qquad \forall \alpha \in [0,1]$$

$$(S2) \qquad \alpha \preceq \beta \Longrightarrow \Phi(\alpha) \preceq \Phi(\beta) \qquad \forall \alpha, \beta \in [0,1]$$

Example 2 *The following functions are sharpeners:*

1) The identity function $I(\alpha) = \alpha \qquad \forall \alpha \in [0,1]$

2) The closest border Ξ defined by $\quad \Xi(\alpha) = \begin{cases} 1 & if \quad \alpha \geq \frac{1}{2} \\ 0 & if \quad \alpha < \frac{1}{2} \end{cases}$

3) Convex combinations T_ε of the former ones

$$T_\varepsilon(\alpha) = (1-\varepsilon) \cdot \alpha + \varepsilon \cdot \Xi(\alpha)$$

(see [6]).

4) Given T and S, some triangular norm and conorm respectively, the members of the family $\{\Omega_{\nu\varepsilon}\}_{\nu,\epsilon \in [0,1]}$ defined

$$\Omega_{\nu\varepsilon}(\alpha) = \begin{cases} S(\alpha, \varepsilon) & if \quad \alpha \geq \frac{1}{2} \\ T(\alpha, \nu) & if \quad \alpha < \frac{1}{2} \end{cases}$$

are sharpeners because they verify (S1):

$$\begin{aligned} \alpha &\geq \frac{1}{2} \Longrightarrow \Omega_{\nu\varepsilon}(\alpha) = S(\alpha, \varepsilon) \geq \alpha \vee \varepsilon \geq \alpha \\ \alpha &< \frac{1}{2} \Longrightarrow \Omega_{\nu\varepsilon}(\alpha) = T(\alpha, \nu) \leq \alpha \wedge \nu < \alpha \end{aligned}$$

and (S2):

$$\frac{1}{2} \leq \alpha < \beta \Longrightarrow \Omega_{\nu\varepsilon}(\beta) = S(\beta, \varepsilon) \geq S(\alpha, \varepsilon) = \Omega_{\nu\varepsilon}(\alpha)$$

$$\alpha \leq \beta < \frac{1}{2} \Longrightarrow \Omega_{\nu\varepsilon}(\alpha) = T(\alpha, \nu) \leq T(\beta, \nu) = \Omega_{\nu\varepsilon}(\beta)$$

5) In a similar way, one can prove that the double complement Φ^* of any sharpener Φ:

$$\Phi^*(\alpha) = \begin{cases} 1 - \Phi(1-\alpha) & \text{if} \quad \alpha \neq \frac{1}{2} \\ 1 - \bigvee\limits_{0 \leq \gamma < \frac{1}{2}} \Phi(\gamma) & \text{if} \quad \alpha = \frac{1}{2} \end{cases}$$

is a sharpener.

We shall denote the set of sharpeners on $[0,1]$ by $\mathcal{S}[0,1]$.

Proposition 3 *(Structure of the set of sharpeners).*

(a) $(\mathcal{S}[0,1], \preceq)$ is a complete, partially ordered set, which has I and Ξ as universal bounds. The corresponding lattice is completely distributive.

(b) $(\mathcal{S}[0,1], \circ, \preceq)$ is a monoid. For every pair of sharpeners Φ and Ψ, one has $\Phi \curlyvee \Psi \preceq \Phi \circ \Psi$, where "$\circ$" denotes the ordinary composition and "$\curlyvee$" the join operation corresponding to the partial order $\preceq$.

Proof. (a) It is easy to see that $\preceq$ is a partial order on $\mathcal{S}[0,1]$. For every sharpener Φ, according to (S1), one has

$$\frac{1}{2} \leq \alpha \Longrightarrow \frac{1}{2} \leq \alpha = I(\alpha) \leq \Phi(\alpha) \leq 1 = \Xi(\alpha)$$

$$\alpha < \frac{1}{2} \Longrightarrow \Xi(\alpha) = 0 \leq \Phi(\alpha) \leq \alpha = I(\alpha) < \frac{1}{2}$$

so $I \preceq \Phi \preceq \Xi$ for every Φ.

For three given sharpeners Θ, Φ and Ψ, with $\Theta \preceq \Phi$ and $\Theta \preceq \Psi$, due to (S2), one has

$$\frac{1}{2} \leq \alpha \Longrightarrow \frac{1}{2} \leq \Theta(\alpha) \leq \Phi(\alpha) \quad \text{and} \quad \frac{1}{2} \leq \Theta(\alpha) \leq \Psi(\alpha)$$

$$\alpha < \frac{1}{2} \Longrightarrow \Phi(\alpha) \leq \Theta(\alpha) < \frac{1}{2} \quad \text{and} \quad \Psi(\alpha) \leq \Theta(\alpha) < \frac{1}{2}$$

so

$$\frac{1}{2} \leq \alpha \Longrightarrow \frac{1}{2} \leq \Theta(\alpha) \leq \Phi(\alpha) \leq \Phi(\alpha) \wedge \Psi(\alpha)$$

$$\alpha < \frac{1}{2} \Longrightarrow \Phi(\alpha) \vee \Psi(\alpha) \leq \Theta(\alpha) < \frac{1}{2}$$

and defining $\Phi \curlywedge \Psi$ and $\Phi \curlyvee \Psi$ by

$$(\Phi \curlywedge \Psi)(\alpha) = \begin{cases} \Phi(\alpha) \wedge \Psi(\alpha) & if \quad \alpha \geq \frac{1}{2} \\ \Phi(\alpha) \vee \Psi(\alpha) & if \quad \alpha < \frac{1}{2} \end{cases}$$

$$(\Phi \curlyvee \Psi)(\alpha) = \begin{cases} \Phi(\alpha) \vee \Psi(\alpha) & if \quad \alpha \geq \frac{1}{2} \\ \Phi(\alpha) \wedge \Psi(\alpha) & if \quad \alpha < \frac{1}{2} \end{cases}$$

it can be proved that $\curlywedge$ and $\curlyvee$ are the meet and join operations corresponding to the partial order $\preceq$. It is plain that, for every family $\{\Phi_i\}_{i \in L}$ of sharpeners, the application

$$\underset{i \in L}{\curlywedge} \Phi_i(\alpha) = \begin{cases} \bigwedge_{i \in L} \Phi_i(\alpha) & if \quad \alpha \geq \frac{1}{2} \\ \bigvee_{i \in L} \Phi_i(\alpha) & if \quad \alpha < \frac{1}{2} \end{cases}$$

is a sharpener. Moreover, it is the infimum of $\{\Phi_i\}_{i \in L}$. In a similar way the supremum is obtained. Hence the lattice is complete.

For every non void family of index sets $\{A_z\}_{z \in C}$

$$\underset{z \in C}{\curlywedge} (\underset{t \in A_z}{\curlyvee} \Phi_{z,t}(\alpha)) = \begin{cases} \bigwedge_{z \in C} (\bigvee_{t \in A_z} \Phi_{z,t}(\alpha)) & if \quad \alpha \geq \frac{1}{2} \\ \bigvee_{z \in C} (\bigwedge_{t \in A_z} \Phi_{z,t}(\alpha)) & if \quad \alpha < \frac{1}{2} \end{cases}$$

and, as $([0,1], \vee, \wedge)$ is completely distributive (see [10]), when F is considered being the set of all functions f with C as domain and such that $f(z) \in A_z \quad \forall z$, one can assure that the previous value $\underset{z \in C}{\curlywedge} (\underset{t \in A_z}{\curlyvee} \Phi_{z,t}(\alpha))$ equals

$$= \left\{ \begin{array}{lll} \bigvee_{f \in F} (\bigwedge_{z \in C} \Phi_{z,f(z)}(\alpha)) & if & \alpha \geq \frac{1}{2} \\ \bigwedge_{f \in F} (\bigvee_{z \in C_z} \Phi_{z,f(z)}(\alpha)) & if & \alpha < \frac{1}{2} \end{array} \right\} = \underset{f \in F}{\curlyvee} (\underset{z \in C}{\curlywedge} \Phi_{z,f(z)}(\alpha))$$

therefore $(\mathcal{S}[0,1], \curlywedge, \curlyvee)$ is completely distributive.

(b) Let Φ, Ψ be sharpeners. For every α we have $\alpha \preceq \Psi(\alpha) \preceq \Phi(\Psi(\alpha))$, and due to the transitivity of $\preceq$, $\Phi \circ \Psi$ fulfils (S1), and for $\alpha \preceq \beta$ we have $\Psi(\alpha) \preceq \Psi(\beta)$ and $\Phi(\Psi(\alpha)) \preceq \Phi(\Psi(\beta))$, thus verifying (S2). The composition is associative over all the class $F([0,1])$ so it is over the part $\mathcal{S}[0,1]$. To end, according to (S1), for every Φ and Ψone has

$$\frac{1}{2} \leq \alpha \Longrightarrow \frac{1}{2} \leq \alpha \leq \Psi(\alpha) \Longrightarrow \frac{1}{2} \leq \Psi(\alpha) \leq \Phi(\Psi(\alpha))$$

and due to (S2)

$$\frac{1}{2} \leq \alpha \Longrightarrow \frac{1}{2} \leq \alpha \leq \Psi(\alpha) \Longrightarrow \alpha \preceq \Psi(\alpha) \Longrightarrow \Phi(\alpha) \preceq \Phi(\Psi(\alpha))$$

and in consequence $\frac{1}{2} \leq \alpha \Longrightarrow \Phi(\alpha) \curlyvee \Psi(\alpha) \leq \Phi(\Psi(\alpha))$ and analogously for $\alpha < \frac{1}{2} \quad \Phi(\Psi(\alpha)) \leq \Phi(\alpha) \wedge \Psi(\alpha)$, which guarantees $\Phi \curlyvee \Psi \preceq \Phi \circ \Psi$. ■

As $\mathcal{S}[0,1]$ is a subset of the set $F([0,1])$, it is natural to ask about the stability of the first set under the usual operations of the second one:

Proposition 4 $(\mathcal{S}[0,1], \cup, \cap)$ *is a complete sublattice of* $F([0,1])$.

Proof. For every $\{\Phi_i\}_{i\in L} \subseteq \mathcal{S}[0,1]$, the union and intersection operations in $F([0,1])$ are given by

$$(\underset{i\in L}{\cup} \Phi_i)(\alpha) = \underset{i\in L}{\vee} (\Phi_i(\alpha)) \qquad (\underset{i\in L}{\cap} \Phi_i)(\alpha) = \underset{i\in L}{\wedge} (\Phi_i(\alpha))$$

which belong to $\mathcal{S}[0,1]$ because

$$\frac{1}{2} \leq \alpha \Longrightarrow \alpha \leq \Phi_i(\alpha) \quad \forall i \qquad \Longrightarrow \alpha \leq \underset{i\in L}{\wedge} \Phi_i(\alpha) \leq \underset{i\in L}{\vee} \Phi_i(\alpha)$$

$$\alpha < \frac{1}{2} \Longrightarrow \Phi_i(\alpha) \leq \alpha < \frac{1}{2} \quad \forall i \qquad \Longrightarrow \underset{i\in L}{\wedge} \Phi_i(\alpha) \leq \underset{i\in L}{\vee} \Phi_i(\alpha) \leq \alpha < \frac{1}{2}$$

thus fulfilling (S1) and, for $\alpha \preceq \beta$

$$\tfrac{1}{2} \leq \alpha \leq \beta \Longrightarrow \tfrac{1}{2} \leq \Phi_i(\alpha) \leq \Phi_i(\beta) \quad \forall i \qquad \Longrightarrow \left\{ \begin{array}{l} \frac{1}{2} \leq \underset{i\in L}{\wedge} \Phi_i(\alpha) \leq \underset{i\in L}{\wedge} \Phi_i(\beta) \\ \frac{1}{2} \leq \underset{i\in L}{\vee} \Phi_i(\alpha) \leq \underset{i\in L}{\vee} \Phi_i(\beta) \end{array} \right.$$

or

$$\beta \leq \alpha < \tfrac{1}{2} \Longrightarrow \Phi_i(\beta) \leq \Phi_i(\alpha) \leq \alpha < \tfrac{1}{2} \quad \forall i \Longrightarrow \left\{ \begin{array}{l} \underset{i\in L}{\wedge} \Phi_i(\beta) \leq \underset{i\in L}{\wedge} \Phi_i(\alpha) < \frac{1}{2} \\ \underset{i\in L}{\vee} \Phi_i(\beta) \leq \underset{i\in L}{\vee} \Phi_i(\alpha) \leq \alpha < \frac{1}{2} \end{array} \right.$$

so that $(\underset{i\in L}{\cup} \Phi_i)(\alpha) \preceq (\underset{i\in L}{\cup} \Phi_i)(\beta)$ and $(\underset{i\in L}{\cap} \Phi_i)(\alpha) \preceq (\underset{i\in L}{\cap} \Phi_i)(\beta)$ that is, unions and intersections of families of sharpeners also verify (S2) thus ending the proof of completeness of the sublattice. ■

One can observe that the universal bounds of this sublattice are the fuzzy subsets N and U defined

$$N(\alpha) = \left\{ \begin{array}{lll} \alpha & if & \alpha \geq \frac{1}{2} \\ 0 & if & \alpha < \frac{1}{2} \end{array} \right. \qquad U(\alpha) = \left\{ \begin{array}{lll} 1 & if & \alpha \geq \frac{1}{2} \\ \alpha & if & \alpha < \frac{1}{2} \end{array} \right.$$

as much as they are sharpeners and

$$\begin{array}{rcl} \frac{1}{2} & \leq & \alpha \Longrightarrow \alpha \leq \Phi(\alpha) \leq 1 \\ \alpha & < & \frac{1}{2} \Longrightarrow 0 \leq \Phi(\alpha) \leq \alpha \end{array}$$

and consequently $N \subseteq \Phi \subseteq U \qquad \forall \Phi \in \mathcal{S}[0,1]$.

In the Fuzzy Set Theory, the use of triangular conorms S and norms T , different from $\vee$ and $\wedge$,it is established as alternative operators to compose the membership values of elements in fuzzy subsets, thus providing an adequate degree of flexibility for defining the operations of union and intersection

$$A \cup_S B(x) = S(A(x), B(x)) \qquad A \cap_T B(x) = T(A(x), B(x))$$

with respect to this operations one has

Proposition 5 *(a) Considered as subset of* $F([0,1])$*,* $\mathcal{S}[0,1]$ *is closed with respect to* $\cup_S$*, if and only if*

$$S(\alpha,\alpha)=\alpha \qquad \forall\alpha<\frac{1}{2} \tag{1}$$

(b) $\mathcal{S}[0,1]$ *is closed for the intersection* $\cap_T$ *if and only if* $T(\alpha,\alpha)=\alpha \qquad \forall\ \alpha\geq\frac{1}{2}$.

Proof. First at all, one can observe that, $\vee$ being the smallest of the conorms, and as triangular conorms are monotonous, when (1) occurs, then

$$\alpha\vee\beta\leq S(\alpha,\beta)\leq S(\alpha\vee\beta,\alpha\vee\beta)=\alpha\vee\beta \qquad \forall\alpha,\beta\in[0,\frac{1}{2}[\tag{2}$$

Condition (2) is sufficient to guarantee the thesis:

Let Φ and Ψ be sharpeners, $\Phi\cup_S\Psi$ verifies condition (S1) in definition 1, because for $\alpha\geq\frac{1}{2}$, $\Phi(\alpha)\geq\alpha\geq\frac{1}{2}$ and $\Psi(\alpha)\geq\alpha\geq\frac{1}{2}$, and consequently $S(\Phi,\Psi)(\alpha)\geq\Phi(\alpha)\vee\Psi(\alpha)\geq\alpha\vee\alpha=\alpha\geq\frac{1}{2}$. For $\alpha\leq\frac{1}{2}$ one has $\Phi(\alpha)\leq\alpha<\frac{1}{2}$ and $\Psi(\alpha)\leq\alpha<\frac{1}{2}$, and therefore according to (2), $S(\Phi,\Psi)(\alpha)=\Phi(\alpha)\vee\Psi(\alpha)\leq\alpha\vee\alpha=\alpha<\frac{1}{2}$.

$\Phi\cup_S\Psi$ satisfies (S2) since, if $\alpha\preceq\beta$ and $\frac{1}{2}\leq\alpha\leq\beta$, as Φ and Ψ are sharpeners, $\frac{1}{2}\leq\Phi(\alpha)\leq\Phi(\beta)$ and $\frac{1}{2}\leq\Psi(\alpha)\leq\Psi(\beta)$, thus being $\frac{1}{2}\leq\Phi(\alpha)\vee\Psi(\alpha)\leq(\Phi\cup_S\Psi)(\alpha)\leq(\Phi\cup_S\Psi)(\beta)$. For $\beta\leq\alpha<\frac{1}{2}$, $\Phi(\alpha)\leq\Phi(\beta)<\frac{1}{2}$ and $\Psi(\alpha)\leq\Psi(\beta)<\frac{1}{2}$ which together with (2) guaranties $(\Phi\cup_S\Psi)(\beta)=\Phi(\beta)\vee\Psi(\beta)\leq(\Phi\cup_S\Psi)(\alpha)=\Phi(\alpha)\vee\Psi(\alpha)<\frac{1}{2}$.

Condition (1) is necessary:

If there is some $\alpha_0<\frac{1}{2}$ such that $\alpha_0<S(\alpha_0,\alpha_0)$, then for the sharpener I, $\alpha_0<S(\alpha_0,\alpha_0)=(I\cup_S I)(\alpha_0)$, thus showing that $I\cup_S I$ fails at (S1).

The proof for (b) is similar. ■

Example 6 *Let* S *be a triangular conorm. The following functions are conorms and verify* $S(\alpha,\alpha)=\alpha \quad \forall\alpha\leq k$ *with* $k\in[0,1]$.

$$\text{(E1)} \qquad S_1(\alpha,\beta)=\begin{cases} k+(1-k)\cdot S(\frac{\alpha-k}{1-k},\frac{\beta-k}{1-k}) & \text{if} \quad \alpha\wedge\beta>k \\ \alpha\vee\beta & \text{if} \quad \alpha\wedge\beta\leq k\end{cases}$$

$$\text{(E2)} \qquad S_2(\alpha,\beta)=\begin{cases} 1 & \text{if} \quad \alpha\vee\beta>k \text{ and } \alpha\wedge\beta>0 \\ \alpha\vee\beta & \text{otherwise}\end{cases}$$

Proposition 7 *A triangular conorm is continuous and satisfies* $S(\alpha,\alpha)=\alpha$ *for every* $\alpha\leq k$ *if and only if it is of the type seen in example (E1).*

Proof. For every triangular conorm satisfying the condition, the associativeness implies that for $\alpha\leq k\leq\gamma$

$$S(\alpha,S(k,\gamma))=S(S(\alpha,k),\gamma)=S(\alpha\vee k,\gamma)=S(k,\gamma)$$

and then, for $k\leq\gamma$ and $\delta=S(k,\gamma)$ one has

$$S(\alpha,\delta)=\delta=\alpha\vee\delta \quad \forall\alpha\leq k \tag{3}$$

Now, as $S(k,k) = k$ and $S(k,1) = 1$, the continuity of S implies the continuity of function $S(k,\nu)$ in the variable ν and consequently for every $\delta \geq k$ there is some $\nu \geq k$ such that $\delta = S(k,\nu)$, which added to (3) assures $S(\alpha,\delta) = \alpha \vee \delta \quad \forall \alpha \leq k \leq \delta$, thus showing that S is of type (E1). Let be remarked that the boundary conditions of triangular conorms makes $\wedge$ and $S(\frac{\alpha-k}{1-k}, \frac{\beta-k}{1-k})$ overlap properly in order to continuity because

$$k+(1-k)\cdot S(\frac{\gamma-k}{1-k}, \frac{k-k}{1-k}) = k+(1-k)\cdot S(\frac{\gamma-k}{1-k}, 0) = k+(1-k)\cdot \frac{\gamma-k}{1-k} = \gamma = \gamma \vee k$$

for every $\gamma \in [k,1]$, and similarly in the other argument. ■

Easily one can see that the dual norm of the one in example (E1), with $k = \frac{1}{2}$ satisfies the condition (b) of proposition 3; the uniqueness stated in proposition 5 and the duality conditions guarantee that this is the only continuous triangular norm which does so.

3 Sharpeners and fuzzy sets

Every sharpener Φ provides a map $A \longmapsto \overline{\Phi}(A)$ from the class of fuzzy sets $F(X)$ on itself defined by

$$\overline{\Phi}(A)(x) = \Phi(A(x)) \qquad \forall x \in X \tag{4}$$

which can be interpreted as an operator that reduces the entropy of fuzzy subsets, that is, for the sharpener order $\preceq$ inducted in $F(X)$ as direct product, i.e. point-by-point extension of the sharpened order in $[0,1]$, it is verified that $A \preceq \overline{\Phi}(A) \quad \forall A \in F(X)$ and that $A \preceq B \Longrightarrow \overline{\Phi}(A) \preceq \overline{\Phi}(B)$.

The set of inducted sharpeners, given by (4), will be denoted $\mathcal{S}F(X)$.

Proposition 8 *For every sharpener $\Phi \in \mathcal{S}[0,1]$, the associated $\overline{\Phi} \in \mathcal{S}F(X)$ is isotonous with respect to the inclusion order $\subseteq$ in $F(X)$.*

Proof. Suppose $\alpha \leq \beta$. If $\alpha \leq \frac{1}{2} \leq \beta$ then $\Phi(\alpha) \leq \alpha \leq \frac{1}{2} \leq \beta \leq \Phi(\beta)$ due to (S1); (S2) gives $\frac{1}{2} \leq \Phi(\alpha) \leq \Phi(\beta)$ when $\frac{1}{2} \leq \alpha \leq \beta$ and $\Phi(\alpha) \leq \Phi(\beta) \leq \frac{1}{2}$ for $\alpha \leq \beta < \frac{1}{2}$. Hence $A \subseteq B$ implies $A(x) \leq B(x)$ and so $\Phi(A(x)) \leq \Phi(B(x)) \quad \forall x \in X$, that is $\overline{\Phi}(A) \subseteq \overline{\Phi}(B)$. ■

Corollary 9 *For every $\Phi \in \mathcal{S}[0,1]$, $\overline{\Phi}$ is a morphism from the De Morgan algebra $(F(X), \cup, \cap)$ onto itself, that is, for A and B in $F(X)$ one has*

$$\overline{\Phi}(A \cup B) = \overline{\Phi}(A) \cup \overline{\Phi}(B) \quad \text{and} \quad \overline{\Phi}(A \cap B) = \overline{\Phi}(A) \cap \overline{\Phi}(B).$$

4 Fuzzy relations and uniformities

The connections between indistinguishability and distance are well known (see [1], [16], [11], [15]). The processes of classification through fuzzy similarities and of covering referentials by means of the open balls corresponding to distances are dual. Stated in simple terms, similarities are the complements of S-pseudometrics.

A fuzzy T-*preorder* is a fuzzy relation R *reflexive*, i.e. $R(x,x) = 1 \quad \forall x \in X$, and T-*transitive*, i.e. $R(x,z) \geq T(R(x,y), R(y,z)) \quad \forall x,y,z \in X$. When it is *symmetric*, i.e. $R(x,y) = R(y,x) \quad \forall x,y \in X$, it is called T-*similarity.*

The concepts of fuzzy preorder and similarity can be related to the concept of distance by means of the concepts of weak transitivity and weak symmetry in the following way (for details and proofs see [1], [9]).

Definition 10 *A fuzzy relation R is weakly transitive if there is some function $f \in [0,1[^{[0,1[}$, not decreasing, with $f(\alpha) \geq \alpha$ and such that*

$$R(x,y) \wedge R(y,z) > f(\alpha) \Longrightarrow R(x,z) > \alpha \qquad \forall x,y,z \in X, \quad \forall \alpha \in [0,1[.$$

Definition 11 *The relation R is weakly symmetric if there is some $g \in [0,1[^{[0,1[}$, not decreasing, with $g(\alpha) \geq \alpha$ and such that*

$$R(x,y) > g(\alpha) \Longrightarrow R(y,x) > \alpha \qquad \forall x,y \in X, \quad \forall \alpha \in [0,1[.$$

The upper level sets $\{(x,y) \in X^2 \text{ st. } R(x,y) > \alpha\}$ associated to the a fuzzy relation R are denoted by R_α. One has

Proposition 12 *(See [9]) Every fuzzy relation R, transitive with respect to a triangular norm continuous at $(1,1) \in [0,1]^2$, is weakly transitive.*

Proposition 13 *(See [1]) $\{R_\alpha\}_{\alpha \in [0,1[}$ is a basis for a quasi-uniformity $\mathcal{U}_R$ in X if and only if R is reflexive and weakly transitive. This quasi-uniformity is a uniformity if and only if R weakly symmetric. These quasi-uniformities or uniformities are pseudometrizable.*

Proposition 14 *(See [8]) A transformation $\overline{\Phi}$ transfers the weak transitivity and weak symmetry conditions of every fuzzy relation R to the transformed relation $\overline{\Phi}(R)$ if and only if*

$$\forall \alpha \in [\frac{1}{2}, 1[\quad \exists \beta \textit{ such that } \Phi(\alpha) < \Phi(\beta) \tag{5}$$

or equivalently

$$\exists h \in [0,1[^{[0,1[} \text{ such that } \Phi(\alpha) < \Phi(h(\alpha)) \quad \forall \alpha \in [0,1[.$$

Example 15 *(1) N and $N \cap_\bullet N$, where N is the universal bound of the lattice considered in proposition 2, are sharpeners fulfilling the condition of the proposition.*

(2) U, also considered after proposition 2, and Ξ, from example 1, are sharpeners which does not satisfies the condition.

Let it be remarked that when R is reflexive and weakly transitive, then for every Φ fulfilling the conditions of the former proposition one has

$$R(x,y) > \frac{\alpha+1}{2} \Longrightarrow \Phi(R(x,y)) \geq \Phi(\frac{\alpha+1}{2}) \geq \frac{\alpha+1}{2} > \alpha$$

and

$$\Phi(R(x,y)) \geq \Phi(\alpha) \Longrightarrow R(x,y) > \alpha$$

hence $R_{\frac{\alpha+1}{2}} \subseteq (\overline{\Phi}(R\ ))_\alpha$ and $(\overline{\Phi}(R\ ))_{\Phi(\alpha)} \subseteq R_\alpha$ for every $\alpha < 1$. Consequently, the uniformities $\mathcal{U}_R$ and $\mathcal{U}_{\overline{\Phi}(R)}$ which, according to the previous propositions are generated by the relations, are identical.

Proposition 16 *For any given fuzzy relation R having weak properties, there exists a sharpener $\overline{\Phi}$ which transforms the relation in another $\overline{\Phi}(R)$ arbitrarily closed to some crisp one, while the correspondent quasi-uniformity structures remain identical.*

Proof. For every triangular conorm S, one can consider the ones S_m, $\quad m = 2, 3, ...$ defined by

$$S_m(\alpha,\beta) = \begin{cases} \alpha \vee \beta & \text{if } \ \{\alpha,\beta\} \nsubseteq [\frac{1}{2}, \frac{m-1}{m}] \\ \frac{1}{2} + \frac{m-2}{2\cdot m} \cdot S(\frac{2\cdot m}{m-2}\cdot(\alpha-\frac{1}{2}), \frac{2\cdot m}{m-2}\cdot(\beta-\frac{1}{2})) & \text{otherwise} \end{cases}$$

which are of type (E1), with $k = \frac{1}{2}$, hence the union of sharpeners made with them is a sharpener according to proposition 3(b). Then, for the below universal bound N, $\Phi_m = N \cup_{S_m} N$ is a sharpener, and as

$$(N \cup_{S_m} N)(\alpha) = \begin{cases} \alpha & \text{if } \ \frac{m-1}{m} < \alpha \\ S_m(\alpha,\alpha) & \text{if } \ \frac{1}{2} \leq \alpha \leq \frac{m-1}{m} \\ 0 & \text{if } \ \alpha \leq \frac{1}{2} \end{cases}$$

it is clear that Φ_m satisfies (5) and in consequence, the upper level sets of R are the basis of some quasi-uniformity on X if and only if those of $\overline{\Phi}_m(R\ )$ does it so.

In particular, the previous process can be applied to the triangle conorm

$$S(\alpha,\beta) = \begin{cases} 1 & \text{if } \ (\alpha,\beta) \in]0,1[^2 \\ \alpha \vee \beta & \text{otherwise} \end{cases}$$

from which it results

$$\overline{\Phi}_m(R)(x,y) = \begin{cases} R(x,y) & \text{if } \ (x,y) \in R_{\frac{1}{m}} \\ \frac{m-1}{m} & \text{if } \ (x,y) \in R_{\frac{1}{m}} - R_{\frac{1}{2}} \\ 0 & \text{if } \ (x,y) \notin R_{\frac{1}{2}} \end{cases}$$

and , with respect to uniform distance, one has

$$d(\overline{\Phi}_m(R), \overline{\Xi}(R)) = \bigvee_{(x,y)\in X^2} |\Phi_m(R(x,y)) - \Xi(R(x,y)| = \frac{1}{m}$$

which proves that $\{\overline{\Phi}_m(R)\}_{m\geq 2}$ converges to $\Xi(R)$, the ordinary relation nearest to R. ■

Proposition 17 states that for every measure of fuzziness $\mathcal{F}$ (see [14]) continuous with respect to the uniform metric, one has $\lim\limits_{m\to\infty} \mathcal{F}(\overline{\Phi}_m(R)) = 0$, thus the entropy of the transformed relation becomes as little as wished.

5 Conclusion and future works

For a wide class of fuzzy relations, which include the most frequently used T-similarities, the previous results grant the possibility of simplifying the relations, making them crisper, without essential changes in the nearness structure generated by them.

The result stated in Proposition 17 appears to be of some significance, in relation with viewpoints commonly considered in the development of two issues in Fuzzy Sets Theory, one relative to fuzziness measuring and another one concerning to similarity-based reasoning.

How to measure vagueness or fuzziness of fuzzy subsets has been one of the subjects extensively studied all through the development of Fuzzy Sets Theory (Cf. [3], [14]). There is a set of axioms , due to DeLuca and Termini, which are commonly considered as acceptable requirements in order to capture and characterize the concept of degree of fuzziness.

On the other hand , concepts about similarity, proximity -and ,related with them, distance - play an important role in analogical reasoning, in the handling of vague information and in the construction of fuzzy systems.

A concrete issue is given in the similarity-based approach to fuzzy control systems (Cf. [15], [12]), where the similarity of values is used as a measure of the degree in which some representative magnitudes or attributes of some objects can be inferred from those of another ones, thus allowing the extension of information provided by some prototypical values to larger domains of them In this field, for instance (Cf. [12], [15]),there are methodologies for the construction of fuzzy if-then rules as an interpolation of paradigmatic samples made by means of the similarity relations.

On the light of Proposition 17, a certain inconsistence or contraposition can be appreciated between the concept of fuzziness considered in connection with the issue of fuzziness measuring, and the one arousing from the similarity-based approach.

Possibly, the difficulties found in the attempts to fill the gap between, on one hand the axiomatic definitions proposed in order to modelize the concept of measure of fuzziness, and on the other hand the construction of effective fuzziness measures, with its rather defectuous pass from finite to infinite referential cases, could be related with the above mentioned contraposition. This is a point worthy of attention , which will be studied in further works.

References

[1] P. Burillo, L. González, *Pseudometrics associated with fuzzy relations having weak properties*, in Procs. AAECC '97, L. Huguet, A. Poli eds., Lecture Notes in Computer Science **356**, Springer Verlag (1989), 69-74.

[2] E. Czogala, S. Gotwald and W. Pedrycz, *Aspects for the evaluation of decision situations*, in Fuzzy Information and Decision Processes, M.M. Gupta and E. Sanchez (eds), North-Holland (1982) 41-49.

[3] A. de Luca, S. Termini, *On some algebraic aspects of the measures of fuzziness*, in Fuzzy Information and Decision Processes, M.M. Gupta and E. Sanchez (eds), North-Holland (1982) 17-23.

[4] A. Di Nola, S. Sessa, W. Pedrycz and E. Sanchez. Fuzzy relational equations and their applications to knowledge engineering, Kluwer (Dordrecht 1989).

[5] D. Dubois, H. Prade, Fuzzy Sets and Systems, Ac. Press (1980).

[6] A. Fadini, G. Ventre and C. Cella, *Su certi operatori per insiemi fuzzy*, Atti. dell 'Academia Pontaniana Nuova, vol. **XXVIII** (1978) 1-12.

[7] J. Fodor, M. Roubens, *Fuzzy preference modelling and multicruteria decision support*, Kluwer (Dordrecht 1994).

[8] L. González, A. Marín, *Transformations on valued relations with weak structure*, in Proc. FUZZ-IEEE '96 , N. Orleans (1996) 1255-1260.

[9] L. González, A. Marín, *Weak properties and aggregated extension of fuzzy relations*, Fuzzy Sets and Systems **85** (1997) 311-3.

[10] G. Grätzer, *General lattice theory*, Birkhäuser Verlag (Bassel 1978).

[11] J. Jacas, *Fuzzy topologies induced by S-metrics*, The Journal of Fuzzy Mathematics **I** (1993) 173-191.

[12] F. Klawon, J. Gebhard and R. Kruse, *Fuzzy control on the basis of equality relations with an example from idle speed control*, IEEE Trans. on Fuzzy Systems, vol **3** n. 3 (1995) 336-350.

[13] R.K. Ragade, M.M. Gupta, *Fuzzy set theory: introduction*, in Fuzzy Automata and Decision Precesses, M.M. Gupta, G.N. Saridir and B.R. Gaines (eds.), North-Holland (N.York 1977) 105-131.

[14] T. Riera and E. Trillas, *From measures of fuzzyness to booleanity control*, in Fuzzy Information and Decision Processes, M. Gupta and E. Sanchez eds., North Holland (1982) 3-16.

[15] T. Sudkamp, *Similarity interpolation and fuzzy rule construction*, Fuzzy Sets and Systems **58** (1993) 73-86.

[16] E. Trillas, *Apunte sobre la indistinguibilidad*, Theoría **19** (1993) 23-49.

[17] L. Valverde, *On the structure of f-indistinguishability operators*, Fuzzy Sets and Systems **17** (1985) 313-328.

[18] L.A.Zadeh, *A Fuzzy-Set-Theoretical interpretation of linguistic hedges*, Journal of Cybernetics **2**: 2 (1972) 4 -34

[19] L.A. Zadeh, *The concept of linguistic variable and its applications to approximate reasoning*, Information Sciences, vol **8** (1975) 199-249.