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Integral Closure in MV-algebras

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Abstract

We study the consequences of assuming on an MV-algebra A that $\Sigma_n nx$ exists for each $x \in A$

Given an MV-algebra A, it has a reduct $L(A)$ which is a bounded distributive lattice. A is then complete, or α-complete, provided the reduct $L(A)$ is complete, or α-complete respectively. As the basic MV-algebraic operations are not idempotent, we have a third type of possibility for the supremum of certain sets.

Thus, given an MV-algebra A, and a non-empty subset $X \subseteq A$, we say that X is *integrally-closed*, or ω*-closed*, provided, for each $x \in X$, that $\Sigma_n(nx)$ exists in X.

Here, $\Sigma_n(nx)$ denotes the least upper bound in $L(A)$ of the set $\{nx \mid n = 0,\ 1,\ 2, \ldots\}$

Clearly every $\aleph_0$-complete MV-algebra A is integrally-closed. On the other hand, if $\mathcal{Q}$ is the MV-algebra of rational numbers in the unit interval, and Y a non-empty set, then $\mathcal{Q}^Y$ is integrally-closed but not $\aleph_0$-complete.

Complete MV-algebras are also related to a class of MV-algebras called *strongly-stonean* (sstonean). If A is complete, then A is sstonean; A sstonean implies $B(A)$ complete, where $B(A)$ is the Boolean subalgebra of idempotents of A.

In this work we shall study some properties of being integrally-closed on an MV-algebra A. We shall show, for example, that ω-closed is also intimately related to sstonean, and to the weaker notion of stonean.

For reasons to be seen below, we will be dealing with only semi-simple algebras.

Consider a commutative ordered monoid $< A,\ +,\ \leq,\ 0 >$ on which is defined an order reversing involution *. We denote such a system by its underlying set A.

Define on A a product by: $a \cdot b = (a^* + b^*)^*$. Then the system $< A,\ \cdot,\ \leq, 1 >$ becomes a commutative ordered monoid where $1 = 0^*$. We always assume that $0 \neq 1$.

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Next define on A the operations: $a \wedge b = a \cdot (a^* + b)$ and $a \vee b = a + (a^*) \cdot b$.

Recall, then, that an MV-algebra is a system as above that satisfies the identity: $a \vee b = b \vee a$ or, equivalently, $a \wedge b = b \wedge a$.

The induced system $L(A) =< A,\ \vee,\ \wedge,\ 0,\ 1 >$ is a bounded distributive lattice.

An ideal in an MV-algebra A is a subset $I \subseteq A$ such that $0 \in I$, I is closed under $+$, and if $a \in I,\ b \le a$, then $b \in I$.

MV-algebras have the usual universal properties of algebraic systems, that is homomorphisms, quotient algebras, subalgebras, etc.

We assume familiarity with MV-algebras, but will give the meanings of some special notation.

Thus, for an MV-algebra A, $Spec(A)$, $Max(A)$, $Min(A)$ will denote respectively the space of prime ideals, maximal ideals, and minimal prime ideals. For a subset $X \subseteq A$, $id_A(X)$ will denote the ideal in A generated by X. (If there is no confusion, we may just write $id(X)$. By $X^{\perp}$ is meant the set $\{y \in A \mid x \wedge y = 0, \text{for all } x \in X\}$. If $X = \{x\}$, we just write $x^{\perp}$. The set $U(X) = \{P \in Spec(A) \mid X \not\subseteq P\}$ is the open set in $Spec(A)$ determined by X. We note that $U(X) = U(id(X))$.

Let $A,\ A'$ be MV-algebras, $A \subseteq A'$ a subalgebra of A'.

Definition. We shall say that A' is an *integral extension* or an *ω-extension* of A provided for each $a \in A$, we have that $\Sigma_n(na)$ exists in A'. (We point out that by [1], if $\Sigma_n(na)$ exists, it is always an idempotent.)

Definition. We say that A is *integrally closed* or *ω-closed*, if A is an integral extension of itself.

If $A \subseteq A'$ and A' is integrally closed, we shall refer to A' as an integrally closed or ω-closed extension of A.

Definition. If I is an ideal of A such that for each $x \in I,\ \Sigma_n(nx)$ exists and is in I, we will say that I is ω-closed.

It was established in [1], Theorem 6, that if A is integrally closed, then it is semi-simple. In fact,

Proposition 1 *If A has an integral extension, then A is semi- simple.*

Proof. Suppose that $a \in Rad(A)$. Let A' be an integrally closed extension of A. Then $\Sigma_n(na) = e$ exists in A'. We know that e is idempotent. Hence $e = e^2 = \Sigma_{n,\ m}(na \cdot ma)$. Since $na,\ ma \in Rad(A)$, we also know $na \cdot ma = 0$. Therefore $e = 0$ which implies $a = 0$.

We also have the converse,

Proposition 2 *If A is semi-simple it has an integrally closed extension.*

Proof. A is a subalgebra of $\Pi\{A/M \mid M \in Max(A)\}$. Each A/M is locally finite, thus integrally closed. Clearly the product of integrally closed MV-algebras is integrally closed.

We say that a in an MV-algebra A has *ultrafinite order* if $\Sigma_n(na) = 1$. We have,

Proposition 3 *If every non-zero element in A has ultrafinite order, then A is locally finite.*

Proof. Since A is integrally closed, it is semi-simple. Thus we need only show it is linearly ordered. Suppose, then, that $a \wedge b = 0,\ a,\ b \neq 0$. We have by assumption, $\Sigma_n(na) = \Sigma_m(mb) = 1$. Therefore, $1 = \Sigma_n(na) \wedge \Sigma_m(mb) = \Sigma_{n,\ m}(na \wedge mb)$. But $na \wedge mb \leq mn(a \wedge b) = 0$. Hence $1 = 0$ which is absurd.

Corollary. *If A is integrally closed and linearly ordered, then A is locally finite.*

Sometimes a quotient of an integrally closed MV-algebra is integrally closed.

Proposition 4 *Suppose that A is integrally closed and that I is an $\aleph$-closed ideal. Then A/I is integrally closed.*

Proof. Let $a/I \in A/I$ and let $e = \Sigma_n(na)$. Then for each $n > 0$ we have $n(a/I) \leq e/I$. Suppose that $n(a/I) \leq b/I$ for all $n > 0$. Then $na \cdot \bar{b} \in I$ for all $n > 0$. Hence $\Sigma_n(na \cdot \bar{b}$ exists in I. But ([1]) $\Sigma_n(na \cdot \bar{b}) = \bar{b} \cdot \Sigma_n(na) = \bar{b} \cdot e \in I$. Thus, $e/I \leq b/I$ so $\Sigma_n(n(a/I)) = e/I$, and A/I is integrally closed.

Corollary. *If A is integrally closed and P is an $\aleph$-closed prime ideal, then P is maximal.*

The following result shows that in the corollary above, that P cannot be just ω-closed.

Proposition 5 *Let A be integrally closed and $P \in Min(A)$. Then P is ω-closed. If $P \in Spec(A)$ is ω-closed, then $P \in Min(A)$.*

Proof. Let $P \in Min(A)$. Let $x \in P$. Since P is minimal, $x^\perp \not\subseteq P$. Let $e = \Sigma_n(nx)$. Then $x^\perp = id(\bar{e})$. Therefore $\bar{e} \notin P$. But P is prime, so $e \in P$. Thus P is ω-closed. Conversely, suppose that $P \in Spec(A)$ is ω-closed. Let $m \subseteq P$ be a minimal prime. Let $x \in P$. Then $e = \Sigma_n(nx) \in P$. Hence, $\bar{e} \notin m$. But m is prime, so $e \in m$. As $x \leq e$, we see that $x \in m$. Therefore, $P = m \in Min(A)$.

Now let $A = \mathcal{Q}^X$ for an infinite set X. Then A is integrally closed but not quasi-boolean [1]. Thus there are minimal primes that are not maximal. That is, there are ω-closed primes that are not maximal.

We can improve the situation in Proposition 4 and its corollary in the following manner.

Call an ideal *plus-ω*-closed, in symbols, $^+\omega$-closed, if for each $x,\ y \in A$, if $y(nx) \in I$ for each $n > 0$, then $y \cdot \Sigma_n(nx) \in I$. It easily follows that,

Proposition 6 *If A is integrally closed and I an $^+\omega$-closed ideal, then A/I is integrally closed.*

Corollary. *If A is integrally closed and P is a $^+\omega$-closed prime ideal, then P is maximal.*

We make the observation that if A is integrally closed, and J is any ideal, then $J^{\perp}$ is $^+\omega$-closed. From this observation we have, letting $MinMax(A) = Min(A) \cap Max(A)$.

Proposition 7 *If A is integrally closed, then $Min(A) = \mathcal{E}(A) \cup MinMax(A)$, where $\mathcal{E}(A) = \{m \in Min(A) \mid m^{\perp} = 0\}$.*

Proof. From Proposition 10 of [5], we know that if $m \in Min(A)$ and $m^{\perp} \neq 0$, then $m = a^{\perp}$ for some $a \in A$. Hence, for $m \in Min(A)$, either $m^{\perp} = 0$ or m is $^+\omega$-closed.

An MV-algebra A is *stonean* iff for each $a \in A$ the annihilator ideal $a^{\perp} = \{b \in A \mid a \wedge b = 0\}$ is generated by an idempotent. In symbols, $a^{\perp} = id(e)$ for some $e \in B(A)$, where $id(e) = id(\{e\})$. The following two lemmas are well known and easy to show.

Lemma 1 *Let $x,\ y \in A$ and let $e \in B(A)$. Then,*
i) $e \cdot x = e \wedge x$.
ii $e + x = e \vee x$.
iii) $e \cdot (x + y) = ex + ey$.

Lemma 2 *Lemma Suppose that $\Sigma_n(nx)$ and $\Sigma_n(ny)$ exist in A. Then*
i) $\Sigma_n(n(x \wedge y))$ *exists and* $\Sigma_n(n(x \wedge y)) = \Sigma_n(nx) \wedge \Sigma_n(ny)$.
ii) $\Sigma_n(n(x + y))$ *exists and* $\Sigma_n(n(x + y)) = \Sigma_n(nx) + \Sigma_n(ny)$.

Lemma 3 *Lemma Let $e,\ e' \in B(A)$ with $e + e' = 1$. Let $x \in A$. Then there is an integer n_0 such that $x \leq xe + xe' \leq n_0 x$.*

Proof. We clearly have, $x \leq xe + xe' \leq x + x \leq n_0 x$ for $n_0 \geq 2$. The lemma is evident for $n_0 = 1$.

Converse to Proposition 5 above, we have

Proposition 8 *Suppose that A is such that every minimal prime is ω- closed. Then A is stonean. Moreover, if A is not linearly ordered, then A is integrally closed.*

Proof. It suffices to show that A is regular and that $Min(A)$ is compact [6]. A will be regular if for each maximal ideal M of $B(A)$ we have that $id_A(M)$ is prime in A [6]. Thus, let M be maximal in $B(A)$. By the lying over theorem [4], there is a prime P of A such that $M = P \cap B(A)$. Let $m \subseteq P$ be a minimal prime. Then $m \cap B(A) \subseteq M$. Therefore $m \cap B(A) = M$. Then $\hat{M} = id_A(M) \subseteq m$. Let $x \in m$. Then $e = \Sigma_n(nx) \in m \cap B(A) = M$. We then have that $e \in \hat{M}$. But $x \leq e$, and so $x \in \hat{M}$. Thus $\hat{M} = m$, and it follows that A is regular.

Suppose now that $Min(A)$ is not compact. Then for some ideal I of A, we have $Min(A) \subseteq U(I)$ but $Min(A) \nsubseteq U(F)$ for any subset $F \subseteq I$. In particular, $Min(A) \nsubseteq U(x)$ for any $x \in I$. Therfore, if $x \in I$, there is an $m \in Min(A)$ with $x \in m$. As m is ω-closed by hypothesis, we see that $e_x = \Sigma_n(nx)$ exists and is in m. Let $I' = id_A(\{e_x \mid x \in I\}$. Assume that $1 \in I'$. Then there are $x_1,\ x_2,\ \ldots, x_n \in I$ with $1 = e_{x_1} + \cdots e_{x_n}$. By Lemma 2 ii), we see that there is an $x \in I$ with $1 = e_x$. But then, $1 \in m$ for some $m \in Min(A)$. Hence I' is proper. Thus $I' \cap B(A)$ is a proper ideal of $B(A)$ and so is contained in a maximal ideal M of $B(A)$. Since I' is generated by idempotents, we see that $I' = id_A(I' \cap B(A))$. Therefore $I' \subseteq id_A(M)$. Since A is regular, $id_A(M) = m$ for some $m \in Min(A)$. We now have $I \subseteq I' \subseteq m$ and so $m \notin U(I)$ which is absurd. Hence A is stonean.

Suppose further that A is not linearly ordered. Then A contains two distinct minimal primes, $m,\ m'$. Since A is stonean it is hypernormal, therefore $m + m' = A$. So for some $a \in m,\ b \in m'$ we have $a + b = 1$. But then we have $1 = e_a + e_b$. By the preceding lemma, there is an n_0 such that $x \leq xe_a + xe_b \leq n_0 x$.

Since $xe_a \in m$, we have that $\Sigma_n n(xe_a)$ exists. Similarly, $\Sigma_n n(xe_b)$ exists. By Lemma 2 above we now see that $e = \Sigma_n n(xe + xe')$ exists. Now certainly, $nx \leq e$ for all positive integers n. Suppose that for some $y \in A$ that $nx \leq y$ for all n. Then we obtain, for any n, $n(xe_a + xs_b) \leq nn_0 x \leq y$. We can now conclude that $\Sigma_n nx = e$ and thus A is integrally closed.

From the above propositions we see that in an integrally closed MV-algebra A, if I is the intersection of a set of minimal primes, then I is ω-closed. We claim the converse as well.

Lemma 4 *Lemma Let I be an ideal, and P a prime ideal minimal over I. If I is ω-closed, the so is P.*

Proof. We know that $P = I_P = \{x \mid \text{for some } y \notin P,\ x \wedge y \in I\}$ [6]. Let $x \in P$ and let y be such that $x \wedge y \in I$ but $y \notin P$. Now, $\Sigma_n(n(x \wedge y)) \in I$ since I is ω-closed. But $\Sigma_n(nx) \wedge \Sigma_n(ny) = \Sigma_n(n(x \wedge y))$. But $y \notin P$ implies $\Sigma_y(ny) \notin P$. It follows that $\Sigma_n(nx) \in P$. Hence P is ω-closed.

Proposition 9 *Let A be integrally closed. Then I is an ω-closed ideal iff $I = \bigcap\{m \in Min(A) \mid I \subseteq m\}$.*

Proof. We saw above that if I is the intersection of minimal primes, then I is ω-closed. Suppose then that I is ω-closed. Since I is the intersection of all the primes that contain it, clearly, $I = \bigcap\{P \in Spec(A) \mid P \text{ is minimal over } I\}$. Now let $x \in m$ for every minimal prime $m,\ I \subseteq m$. Suppose $P \in Spec(A)$ and $I \subseteq P$. There is a prime ideal $P' \subseteq P$ with P' minimal over I. By the lemma above, P' is ω-closed. By Proposition 5, P' is a minimal prime. Thus $x \in P' \subseteq P$. We see then that $x \in I$.

Proposition 10 *If A is integrally closed, then it is stonean.*

Proof. Let $a \in A$. Let $e = \Sigma_n(na)$. Clearly we have $a^\perp = (id(e))^\perp = id(\bar{e})$. Since $e \in B(A)$ so is $\bar{e}$. So, A is stonean.

The converse is false in general, (just take A to be the special Chang algebra C, [7]) but we have,

Proposition 11 *If A is stonean and semi-simple, then A is integrally closed.*

Proof. Let $a \in A$. We know that $a^\perp = id(e)$ for some $e \in B(A)$. Hence $na \le \bar{e}$ for all $n > 0$. Suppose that $b \in A$ is such that $na \le b$ for all $n > 0$. Let M be a maximal ideal of A. If $a \in M$, then $a \wedge \bar{b} \in M$. Suppose that $a \notin M$. Then a/M has finite order in A/M. Since $n(a/M) \le b/M$ for all $n > 0$, we see that $b/M = 1$. Therefore $\bar{b} \in M$ and so $a \wedge \bar{b} \in M$. Since A is semi-simple, $a \wedge \bar{b} = 0$; so $\bar{b} \le e$. Therefore $\bar{e} \le b$. It follows that $\Sigma_n(na) = \bar{e}$ and so A is integrally closed.

Corollary. *A is integrally closed iff A is stonean and semi-simple.*

An MV-algebra A is *strongly stonean* if for each ideal $I \subseteq A$, $I^\perp = id(e)$ for some $e \in B(A)$. Integral closure relates to strongly stonean in the following manner.

Proposition 12 *If A is integrally closed and $B(A)$ is complete, then A is sstonean.*

Proof. Let I be an ideal of A. For each $a \in I$, let $e_a = \Sigma_n(na)$. Now let $e = \Sigma\{e_a \mid a \in I\}$. Clearly, if $a \in I$, then $a \wedge \bar{e} = 0$. Thus $id(\bar{e}) \subseteq I^\perp$. Now let $b \in I^\perp$. Then $b \wedge e_a = 0$ for all $a \in I$.Therefore $b \wedge e = \Sigma\{b \wedge e_a \mid a \in I\} = 0$. Hence $b \le \bar{e}$. That is, $I^\perp = id(\bar{e})$ and A is sstonean.

Corollary. *An MV-algebra A is sstonean and semi- simple iff A is integrally closed and $B(A)$ is complete.*

Proof. Suppose A is sstonean and semi-simple. Then A is stonean and semi-simple, thus integrally closed by Proposition 11. From [2] we know that $B(A)$ is complete.

We note that for the MV-algebra $C = \{0,\ c,\ 2c,\ \ldots, 1-2c,\ 1-c,\ 1\}$ of [7], that C^X is sstonean and $B(C^X)$ is complete. C^X is neither semi-simple nor integrally closed.

Given, then, a semi-simple MV-algebra A, we can look for three types of integral extensions. Let A be a subalgebra of A'.

Type 1) A' is sstonean and semi-simple.

Type 2) A' is stonean and semi-simple.

Type 3) For each $a \in A$, $\Sigma_n(na)$ exists in A', where A' is semi- simple.

Definition. Suppose A is a subalgebra of $\hat{A}$. We shall call $\hat{A}$ an *integral closure of Type k* if $\hat{A}$ is a minimal integral extension of A of Type k, k=1, 2, 3.

In [9] it is shown, using sheaf theoretic methods, that each MV-algebra A has a *unique* minimal sstonean extension. (In [9] these are called Baer- extensions.) Semi-simplicity is not examined in [9], nor is integral closure. Since every semi-simple MV-algebra is contained in a complete algebra, it automatically has a semi-simple sstonean extension. We shall show (without sheaf theory) that every semisimple MV-algebra has a minimal such extension. As mentioned above, a result of A. Filipou [9] shows that the minimal extension is unique up to isomorphism.

Proposition 13 *Let A be an arbitrary MV-algebra. Then there exists a minimal sstonean MV-algebra $\hat{A}$ with $A \subseteq \hat{A}$.*

Proof. Let $A_0 = \Pi\{A/P \mid P \in Spec(A)\}$. Then A_0, being a direct product of linearly ordered MV-algebras, is sstonean. Let $\mathcal{S} = \{A" \mid A \subseteq A" \subseteq A_0, A"\text{sstonean}\}$. Let $\mathcal{C} \subseteq \mathcal{S}$ be a chain (under $\subseteq$). Let $A' = \bigcap \mathcal{C}$. Then $A \subseteq A' \subseteq A_0$. Clearly, $B(A') = \bigcap\{B(A_u \mid A_u \in \mathcal{C}\}$. Let I be an ideal of A', and for each $A_u \in \mathcal{C}$, let $I_u = id_u(I)$ be the ideal in A_u generated by I. Let $e_u \in B(A_u)$ be such that ${I_u}^{\perp} = id_u(e_u)$. Clearly, if $A_u \subseteq A_w$ in $\mathcal{C}$ then $I_u \subseteq I_w$. Now $e_u \wedge a = 0$ for all $a \in I$. If $b \in I_w$, then $b \leq a$ for some $a \in I$. Therefore, since $e_u \in A_w$ and $e_u \wedge a = 0$, we have $e_u \wedge b = 0$. That is, $e_u \leq e_w$. Let $A_w \in \mathcal{C}$ and let $e = \Pi\{e_u \mid A_u \subseteq A_w\}$. As $B(A_w)$ is complete, we see that e exists. Moreover, we must have that $e = \Pi\{e_u \mid A_u \subseteq A_v\}$ whenever $A_v \subseteq A_w$. Consequently, $e \in \bigcap\{B(A_u) \mid A_u \in \mathcal{C}\}$. It is clear that $e \in I^{\perp} \subseteq A$. Now let $a \in I^{\perp}$, $a \in A$. Then for each $A_u \in \mathcal{C}$, we have $a \in {I_u}^{\perp}$. Thus, $a \leq e_u$ for all $A_u \in \mathcal{C}$ and so $a \leq e$. Therefore we infer $I^{\perp} = id(e) \subseteq A$. Ergo A' is sstonean. By Zorn's Lemma we conclude that $\mathcal{S}$ has minimal elements.

Comment: In the proof above, the only role played by A_0, aside from avoiding set theoretical difficulties, is to guarantee that $\mathcal{S}$ is non-empty. In fact we have proved,

Proposition 14 *If A is a subalgebra of a sstonean algebra A', then there is a sstonean algebra $\hat{A}$ minimal over A with $\hat{A} \subseteq A'$.*

If A is semi-simple, then the algebra A_0 can be taken to be $\Pi\{A/M \mid M \in Max(A)\}$, which is also semi-simple. Since subalgebras of semi-simple algebras are semi-simple, we see that we can find a minimal sstonean extension $\hat{A}$ of A which is semi-simple. Thus,

Proposition 15 *If A is semi-simple, then A has an integral closure $\hat{A}$ of Type 1.*

We next show that if A is bipartite, then so is $\hat{A}$. Recall that A bipartite means that for some maximal ideal $M \subseteq A$, that $A = M \cup M^*$ where $M^* = \{x \in A \mid x^* \in M\}$.

Lemma 5 *Lemma Suppose that an MV-algebra A is a subalgebra of a sstonean algebra A'. If $B(A') \subseteq A$, then A is also sstonean.*

Proof. Let I be an ideal of A. Let $J = id_{A'}(I)$ be the ideal of A' generated by I. Then $J^{\perp} = id_{A'}(e)$ for some $e \in B(A')$. Since e is also in A, we have $I^{\perp} = id_{A'}(e) \cap A = id_a(e)$. Hence A is sstonean.

Proposition 16 *Let A be an arbitrary MV-algebra, I an ideal of A. Let $S(I) = \{a \in A \mid a \wedge a^* \in I\}$. Then,* ([3], [4])

i) $S(I)$ is a subalgebra of A, and I is an ideal of $S(I)$.

ii) $B(A) \subseteq S(I)$.

iii) If P is a prime ideal of A, then $S(P)$ is bipartite.

From this we now have,

Proposition 17 *If A is a bipartite MV-algebra, and $\hat{A}$ a minimal sstonean extension of A, then $\hat{A}$ is bipartite.*

Proof. As A is bipartite, it has a maximal ideal M such that $A = M \cup M^*$. Let $\hat{M} = id_{\hat{A}}(M)$. Then $\hat{M} \neq \hat{A}$ and $S(\hat{M})$ is a subalgebra of $\hat{A}$ and by Proposition 11, $B(\hat{A}) \subseteq S(\hat{M})$. If $a \in A$, then $a \in M$ or $a^* \in M$. Thus $a \wedge a^* \in M$ and therefore $a \wedge a^* \in \hat{M}$. It follows that $A \subseteq S(\hat{M})$. By the preceeding lemma, $S(\hat{A})$ is sstonean. By the minimality of $\hat{A}$, we may infer that $S(\hat{M}) = \hat{A}$. From Proposition 16 we conclude that $\hat{A}$ is bipartite.

OPEN PROBLEM:

What is the integral closure of $\mathcal{C}([0,\ 1],\ [0,\ 1])$, the algebra of continuous functions from $[0,\ 1]$ to $[0,\ 1]$?

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