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On The Global Stability of Takagi-Sugeno General Model

F. Matía, B.M. Al-Hadithi and A. Jiménez
DISAM
Universidad Politécnica de Madrid
J. Gutiérrez Abascal 2, E-28006 Madrid, Spain
matia@disam.upm.es

Abstract

Global stability of Takagi-Sugeno (T-S) fuzzy model is presented. First, stability conditions for T-S fuzzy model presented by Tanaka and Sugeno are reviewed. Second, new theorems for the stability of the general form of T-S model is derived in the sense of Lyapunov.

The T-S model we studied includes a linear equation with a constant parameter in the consequent part of each rule while other authors have analyzed the model with no constant term, which does not represent a real system. This in turn will impose restrictions on the stability conditions derived in this field. An example is presented to illustrate the new suggested condition.

Keywords: Fuzzy Control, Dynamical Analysis, Stability.

1 Introduction

Recently, fuzzy control has gained wide popularity and has been applied in many industrial applications. On the other hand, fuzzy control is still suffering from the lack of analysis and stability techniques. Stability is considered one of the important issues in the analysis and design of control systems. The difficulties encountered in the analysis of stability of fuzzy systems are due to the nonlinearity of such systems.

Several studies have been made to analyze the stability issue of T-S fuzzy model [1], [2], [3], [4], [5], [6], [7], [9], [10], [11], [12], [13] and [14]. One of the most important works is the one presented by Tanaka and Sugeno [13], in which a sufficient condition which assures the stability of discrete fuzzy systems has been derived. Kosko presented a continuous version of this stability criterion [7]. Tanaka and Sano [12] have extended their study on stability analysis to robust control of fuzzy systems in the presence of premise -parameter uncertainty using the same T-S model.

Tanaka and various others [11], [12], [13] and [14], have considered the linear equation in the consequent part of each one, as being obtained by linearizing the original system with respect to the origin. This means that the constant term is zero which does not represent real systems.

The aim of this work is to derive sufficient stability conditions that take into account the existence of the constant term in T-S fuzzy model. Section 2 presents Takagi and Sugeno fuzzy model and the stability conditions derived for it. Section 3 shows the proposed new stability theorems derived in this paper for the general form of Takagi and Sugeno fuzzy model. Section four gives an illustrative example

2 Takagi-Sugeno model

The model proposed by Takagi and Sugeno consists of a set of IF -THEN rules where the consequent part is a linear function of the inputs. Its principal feature is that it allows to model with high accuracy the original system's dynamics around the linearization points. Each rule $R^{(i_1 \cdots\cdots i_n)}$ has the form:

$$\begin{aligned} &\textit{If } x \textit{ is } M_1^{i_1} \textit{ and } \dot{x} \textit{ is } M_2^{i_2} \ldots \textit{ y } x^{(n-1)} \textit{ is } M_n^{i_n} \\ &\textit{then } \dot{\mathbf{x}} = \mathbf{a}_o^{(i_1 \ldots i_n)} + \mathbf{A}^{(i_1 \ldots i_n)}\mathbf{x} + \mathbf{B}^{(i_1 \ldots i_n)}\mathbf{u} \end{aligned} \tag{1}$$

where $M_1^{i_1}$ $(i_1 = 1, 2, \ldots, r_1)$ are fuzzy sets for x, $M_2^{i_2}$ $(i_2 = 1, 2, \ldots, r_2)$ fuzzy sets for $\dot{x}$ y $M_n^{i_n}$ $(i_n = 1, 2, \ldots, r_n)$ fuzzy sets for x^{n-1} (figure 1).

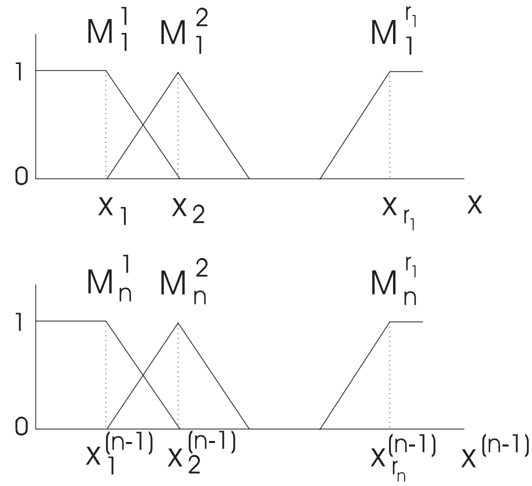

Figure 1: Generic Membership Functions

$\mathbf{x}$ is the state vector and $\mathbf{u}$ is the input vector, so the system is composed of $r_1 r_2 \ldots r_n$ rules. Supposing that the $\mathbf{A}^{(i_1 \ldots i_n)}$ matrix corresponds to the controllable canonical form in which $x = f(\dot{x}, \ldots, x^{(n)})$, this will be given by:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_1^{(i_1 \dots i_n)} & -a_2^{(i_1 \dots i_n)} & \dots & -a_{n-1}^{(i_1 \dots i_n)} & -a_n^{(i_1 \dots i_n)} \end{bmatrix} \tag{2}$$

while

$$\mathbf{a}_0^{(i_1 \dots i_n)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_0^{(i_1 \dots i_n)} \end{bmatrix}, \tag{3}$$

with

$$\mathbf{x}^T = \begin{bmatrix} x & \dot{x} & \dots & x^{n-1} \end{bmatrix} \tag{4}$$

If the input is zero, the final system output is obtained as:

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \dots i_n)} (\mathbf{a}_o^{(i_1 \dots i_n)} + \mathbf{A}^{(i_1 \dots i_n)} \mathbf{x})}{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \dots i_n)}(\mathbf{x})} \\ &= \mathbf{a}_0(\mathbf{x}) + \mathbf{A}(\mathbf{x})\mathbf{x} \end{aligned} \tag{5}$$

being

$$\mathbf{A}(\mathbf{x}) = \frac{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \dots i_n)}(\mathbf{x}) \mathbf{A}^{(i_1 \dots i_n)}(\mathbf{x})}{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \dots i_n)}(\mathbf{x})} \tag{6}$$

$$\mathbf{a}_0(\mathbf{x}) = \frac{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \dots i_n)}(\mathbf{x}) \mathbf{a}_0^{(i_1 \dots i_n)}(\mathbf{x})}{\sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \dots i_n)}(\mathbf{x})} \tag{7}$$

Each consequent equation is a linear subsystem given by the expression $\mathbf{a}_o^{(i_1 \dots i_n)} + \mathbf{A}^{(i_1 \dots i_n)}\mathbf{x}$.

Theorem 1 *If* $\mathbf{a}_0^{(i_1 \dots i_n)} = 0, \quad \forall\, i_1 \dots, i_n,$ *this means,* $\dot{\mathbf{x}} = \mathbf{A}^{(i_1 \dots i_n)}\mathbf{x}$, *then the fuzzy system represented by (5) is asymptotically stable in the large if there exists a positive definite* $\mathbf{P}$ *matrix, such that:*

$$\left(\mathbf{A}^{(i_1 \dots i_n)}\right)^T \mathbf{P} + \mathbf{P}\mathbf{A}^{(i_1 \dots i_n)} < 0 \tag{8}$$

$\forall\, i_1, \dots, i_n$, *this means, for all the subsystems.*

See [7] and [11] for its proof. It is easy to observe that the previous theorem is reduced to Lyapunov stability theorem for linear systems, when $r_1, \dots, r_n = 1$.

Many papers have also analyzed the stability for discrete fuzzy systems [7], [11]. Tanaka and Sugeno have obtained a sufficient condition for stability of discrete fuzzy systems [13]. The analysis of stability for discrete fuzzy systems is of higher complexity than the continuous ones. For example, the discrete fuzzy system may be unstable, although all the linear subsystems are. This does not happen in the continuous case, as it can be deduced from the previous theorem.

3 General stability theorem

In the previous Theorem, it was supposed that $\mathbf{a}_0^{(i_1 \ldots i_n)} = 0$. Nevertheless, a non-linear system linearized in two different points (two rules) gives two linear subsystems that, in general do not pass through the origin $\mathbf{x} = \mathbf{0}$, as it is shown in figure 2.

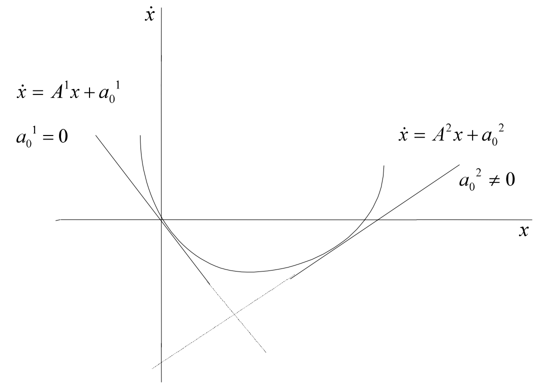

Figure 2: General Takagi-Sugeno Model

If only one rule is applied (just a linear system, non-fuzzy), we can use incremental variables at the linearization point, so the a_0 term would dissapear. Nevertheless, this can not be done in a fuzzy system with several subsystems, because the resultant incremental variables would also be different (because the linearization ponits are different), so it would not be possible later to interpolate them (center of gravity) to obtain the system output.

Now, a theorem for the general case in which $\mathbf{a}_0^{(i_1 \ldots i_n)} \neq 0$ is presented. In other words, the stability for the general form of Takagi and Sugeno fuzzy model is under consideration.

Theorem 2 *The fuzzy system represented by*

$$\mathbf{B}(\mathbf{x})\dot{\mathbf{x}} + \mathbf{x} = \mathbf{b}_0(\mathbf{x}) \tag{9}$$

where

$$\mathbf{B}^{(i_1 \ldots i_n)} = -\left(\mathbf{A}^{(i_1 \ldots i_n)}\right)^{-1} \tag{10}$$

$$\mathbf{b}_0^{(i_1 \ldots i_n)} = \mathbf{B}^{(i_1 \ldots i_n)} \mathbf{a}_0^{(i_1 \ldots i_n)} \tag{11}$$

and in which all the linear subsystems are stable, is asymptotically stable in the large if

$$\left[I - \frac{d\mathbf{b_0}(\mathbf{x})}{d\mathbf{x}}\right] \mathbf{B}^{-1}(\mathbf{x}) > 0, \quad \forall \mathbf{x} \neq 0 \tag{12}$$

Proof. First, the matrix $\mathbf{A}^{i_1 \cdots i_n}$ corresponds to the controllable canonical from $x = f(\dot{x}, \ldots, x^{(n)})$, and is stable, so it is easy to prove that the inverse matrix $\mathbf{B}^{i_1 \cdots i_n}$ exists, and so $\mathbf{B}(\mathbf{x})$ also exists.

Taking as a Lyapunov function:

$$V(\mathbf{x}) = [\mathbf{x} - \mathbf{b_0}(\mathbf{x})]^T [\mathbf{x} - \mathbf{b_0}(\mathbf{x})] \tag{13}$$

this one verifies that

$$V(\mathbf{0}) = 0$$

$$V(\mathbf{x}) > 0, \ \forall \mathbf{x} \neq \mathbf{0}$$

and must satisfy that

$$\dot{V}(\mathbf{0}) = 0$$

$$\dot{V}(\mathbf{x}) < 0, \ \forall \mathbf{x} \neq \mathbf{0}$$

Taking the derivative $V(\mathbf{x})$,

$$\begin{aligned} \dot{V}(\mathbf{x}) &= 2[\mathbf{x} - \mathbf{b_0}(\mathbf{x})]^T \left[\dot{\mathbf{x}} - \frac{d\mathbf{b_0}}{d\mathbf{x}}\dot{\mathbf{x}}\right] \\ &= -2[\mathbf{x} - \mathbf{b_0}(\mathbf{x})]^T \left[I - \frac{d\mathbf{b_0}}{d\mathbf{x}}\right] \\ &\quad \mathbf{B}^{-1}(\mathbf{x}) [\mathbf{x} - \mathbf{b_0}(\mathbf{x})] < 0, \quad \forall \mathbf{x} \neq 0 \end{aligned} \tag{14}$$

And this is true if $\left[I - \frac{d\mathbf{b_0}(\mathbf{x})}{d\mathbf{x}}\right] \mathbf{B}^{-1}(\mathbf{x}) > 0, \ \forall \mathbf{x} \neq 0.$

A more specific theorem follows.

Theorem 3 *The fuzzy system represented by Takagi-Sugeno general model (9), is asymptoticaly stable if* $\forall x_{i_l}^{(l-1)} \leq x^{(l-1)} \leq x_{i_l+1}^{(l-1)}$, $\forall i_l = \{1, \ldots, r_l - 1\}$, $\forall l = \{1, \ldots n\}$, *the following polynomials correspond to stable linear systems:*

$$\begin{aligned} &\left(1 - \frac{b_0^{i_1+1 j_2 \cdots j_n} - b_0^{i_1 j_2 \cdots j_n}}{x_{i_1+1} - x_{i_1}}\right) \lambda^n \\ + \ &\left(b_1^{j_1 \cdots j_n} - \frac{b_0^{j_1 i_2+1 j_3 \cdots j_n} - b_0^{j_1 i_2 j_3 \cdots j_n}}{\dot{x}_{i_2+1} - \dot{x}_{i_2}}\right) \lambda^{n-1} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \ldots + \\
&+ \quad \left(b_{n-1}^{j_1 \ldots j_n} - \frac{b_0^{j_1 \ldots j_{n-1} i_n+1} - b_0^{j_1 \ldots j_{n-1} i_n}}{x_{i_n+1}^{(n-1)} - x_{i_n}^{(n-1)}} \right) \lambda \\
&+ \quad b_n^{j_1 \ldots j_n} = 0 \qquad \forall j_l = \{i_l, i_l+1\}
\end{aligned} \tag{15}
$$

Proof. By simplifying Theorem 2. $\left[I - \frac{d\mathbf{b}_0(\mathbf{x})}{d\mathbf{x}}\right] \mathbf{B}^{-1}(\mathbf{x})$ is the product of matrices:

$$
\begin{bmatrix}
1 - \frac{\partial b_0}{\partial x} & -\frac{\partial b_0}{\partial \dot{x}} & \ldots & \ldots & -\frac{\partial b_0}{\partial x^{(n-1)}} \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & 0 & 1
\end{bmatrix} \tag{16}
$$

and

$$
\begin{bmatrix}
0 & -1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & 0 & -1 \\
\frac{1}{b_n(\mathbf{x})} & \frac{b_1(\mathbf{x})}{b_n(\mathbf{x})} & \ldots & \ldots & \frac{b_{n-1}(\mathbf{x})}{b_n(\mathbf{x})}
\end{bmatrix} \tag{17}
$$

and, so $\left[I - \frac{d\mathbf{b}_0(\mathbf{x})}{d\mathbf{x}}\right] \mathbf{B}^{-1}(\mathbf{x}) = E(\mathbf{x})$ may be expressed as

$$
\begin{bmatrix}
e_1(\mathbf{x}) & e_2(\mathbf{x}) & \ldots & \ldots & \ldots & e_n(\mathbf{x}) \\
0 & 0 & -1 & 0 & \ldots & 0 \\
\vdots & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & -1 \\
\frac{1}{b_n(\mathbf{x})} & \frac{b_1(\mathbf{x})}{b_n(\mathbf{x})} & \ldots & \ldots & \ldots & \frac{b_{n-1}(\mathbf{x})}{b_n(\mathbf{x})}
\end{bmatrix} \tag{18}
$$

with

$$
\left\{
\begin{aligned}
e_1(\mathbf{x}) &= \frac{-1}{b_n(\mathbf{x})} \frac{\partial b_0(\mathbf{x})}{\partial x^{(n-1)}} \\
e_2(\mathbf{x}) &= \left(\frac{\partial b_0(\mathbf{x})}{\partial x} - 1\right) - \frac{b_1(\mathbf{x})}{b_n(\mathbf{x})} \frac{\partial \mathbf{b}_0(\mathbf{x})}{\partial x^{(n-1)}} \\
&\ldots \\
e_n(\mathbf{x}) &= \frac{\partial b_0(\mathbf{x})}{\partial x^{(n-2)}} - \frac{b_{n-1}(\mathbf{x})}{b_n(\mathbf{x})} \frac{\partial b_0(\mathbf{x})}{\partial x^{(n-1)}}
\end{aligned}
\right. \tag{19}
$$

In order to obtain a positive definite $E(\mathbf{x})$ matrix, the real part of the roots of $\left|\lambda I - \left(-E^{-1}(\mathbf{x})\right)\right| = 0$, which are the same as those of $|\lambda E(\mathbf{x}) + \mathbf{I}| = 0$, must be strictly negative $\forall \mathbf{x}$. The determinant $|\lambda E(\mathbf{x}) + \mathbf{I}|$ may be calculated as

$$\left|\begin{array}{cccccc} 1+\lambda e_1(\mathbf{x}) & \lambda e_2(\mathbf{x}) & \dots & \dots & \dots & \lambda e_n(\mathbf{x}) \\ 0 & 1 & -\lambda & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 1 & -\lambda \\ \frac{\lambda}{b_n(\mathbf{x})} & \lambda\frac{b_1(\mathbf{x})}{b_n(\mathbf{x})} & \dots & \dots & \dots & 1+\lambda\frac{b_{n-1}(\mathbf{x})}{b_n(\mathbf{x})} \end{array}\right| \tag{20}$$

which is equal to

$$\begin{array}{rl} & \left(1-\frac{\partial b_0}{\partial x}\right)\lambda^n + \left(b_1(\mathbf{x})-\frac{\partial b_0}{\partial \dot{x}}\right)\lambda^{n-1}+\dots+ \\ + & \left(b_{n-1}(\mathbf{x})-\frac{\partial b_0}{\partial x^{(n-1)}}\right)\lambda+\ b_n(\mathbf{x}) \end{array} \tag{21}$$

Such a polynomial must have roots with negative real parts. For example, for n=3, $E(\mathbf{x})$ may be written as

$$\left[\begin{array}{ccc} -\frac{1}{b_3}\frac{\partial b_0}{\partial \ddot{x}} & \frac{\partial b_0}{\partial x}-1-\frac{b_1}{b_3}\frac{\partial b_0}{\partial \ddot{x}} & \frac{\partial b_0}{\partial \dot{x}}-\frac{b_2}{b_3}\frac{\partial b_0}{\partial \ddot{x}} \\ 0 & 1 & -1 \\ \frac{1}{b_3} & \frac{b_1}{b_3} & \frac{b_2}{b_3} \end{array}\right] \tag{22}$$

so $|\lambda E(\mathbf{x})+\mathbf{I}|$ becomes

$$\left|\begin{array}{ccc} 1-\frac{\lambda}{b_3}\frac{\partial b_0}{\partial \ddot{x}} & \lambda\left(\frac{\partial b_0}{\partial x}-1-\frac{b_1}{b_3}\frac{\partial b_0}{\partial \ddot{x}}\right) & \lambda\left(\frac{\partial b_0}{\partial \dot{x}}-\frac{b_2}{b_3}\frac{\partial b_0}{\partial \ddot{x}}\right) \\ 0 & 1 & -\lambda \\ \frac{\lambda}{b_3} & \lambda\frac{b_1}{b_3} & 1+\lambda\frac{b_2}{b_3} \end{array}\right| \tag{23}$$

which is equal to the following polynomial:

$$\begin{array}{rl} \left(1-\dfrac{\partial b_0}{\partial x}\right)\lambda^3 & + \ \left(b_1(\mathbf{x})-\dfrac{\partial b_0}{\partial \dot{x}}\right)\lambda^2 \\ & + \ \left(b_2(\mathbf{x})-\dfrac{\partial b_0}{\partial \ddot{x}}\right)\lambda+\ b_3(\mathbf{x}) \end{array} \tag{24}$$

whose roots must have negative real part. Anyway, triangular membership functions have been supposed, which verify that $\sum_{i_l=1}^{r_l}\mu_{M_l^{i_l}}(x^{(l-1)})=1$. In [8] the authors have proved that

$$\begin{array}{rl} & \displaystyle\sum_{i_1=1}^{r_1}\cdots\sum_{i_n=1}^{r_n} w^{i_1\dots i_n}(\mathbf{x}) \\ = & \displaystyle\sum_{i_1=1}^{r_1}\cdots\sum_{i_n=1}^{r_n}\prod_{l=1}^{n}\mu_{M_l^{i_l}}(x^{(l-1)})=1 \end{array} \tag{25}$$

so

$$\mathbf{b_0}(\mathbf{x}) = \sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} \prod_{l=1}^{n} \mu_{M_l^{i_l}}(x^{(l-1)}) \mathbf{b_0}^{i_1 \ldots i_n} \tag{26}$$

or even

$$b_0(\mathbf{x}) = \sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} \prod_{l=1}^{n} \mu_{M_l^{i_l}}(x^{(l-1)}) b_0^{i_1 \ldots i_n} \tag{27}$$

And, taking into account that the membership functions overlap by pairs:

$$b_0(\mathbf{x}) = \sum_{j_1=i_1,i_1+1} \cdots \sum_{j_n=i_n,i_n+1} \prod_{l=1}^{n} \mu_{M_l^{j_l}}(x^{(l-1)}) b_0^{j_1 \ldots j_n} \tag{28}$$
$$\forall x_{i_l}^{(l-1)} \leq x^{(l-1)} \leq x_{i_l+1}^{(l-1)}$$
$$\forall i_l = \{1, \ldots, r_1\}, \quad \forall l = \{1, \ldots, n\}$$

Finally, taking the derivative of the previous expression, it may be obtained $\frac{\partial b_0(\mathbf{x})}{\partial x^{(m-1)}}$ as

$$\begin{aligned}
& \sum_{j_1=i_1,i_1+1} \cdots \sum_{j_n=i_n,i_n+1} \\
& \frac{\partial \mu_{M_m^{j_m}}(x^{(m-1)})}{\partial x^{(m-1)}} \prod_{l=1}^{n} \mu_{M_l^{j_l}}(x^{(l-1)}) b_0^{j_1 \ldots j_n} \\
= \; & \sum_{j_1=i_1,i_1+1} \cdots \sum_{j_{m-1}=i_{m-1},i_{m-1}+1} \sum_{j_{m+1}=i_{m+1},i_{m+1}+1} \\
& \cdots \sum_{j_n=i_n,i_n+1} \left[\frac{\partial \mu_{M_m^{i_m}}(x^{(m-1)}) b_0^{j_1 \cdots j_{m-1} i_m j_m \cdots j_n}}{\partial x^{(m-1)}} \right. \\
& \left. + \frac{\partial \mu_{M_m^{i_m+1}}(x^{(m-1)}) b_0^{j_1 \cdots j_{m-1} i_m+1 j_{m+1} \cdots j_n}}{\partial x^{(m-1)}} \right] \\
& \prod_{l=1}^{n} \mu_{M_l^{j_l}}(x^{(l-1)}) \\
= \; & \sum_{j_1=i_1,i_1+1} \cdots \sum_{j_{m-1}=i_{m-1},i_{m-1}+1} \\
& \sum_{j_{m+1}=i_{m+1},i_{m+1}+1} \cdots \sum_{j_n=i_n,i_n+1} \\
& \frac{b_0^{j_1 \cdots j_{m-1} i_m+1 j_{m+1} \cdots j_n} - b_0^{j_1 \cdots j_{m-1} i_m j_{m+1} \cdots j_n}}{x_{i_m+1}^{(m-1)} - x_{i_m}^{(m-1)}} \\
& \prod_{l=1}^{n} \mu_{M_l^{j_l}}(x^{(l-1)})
\end{aligned} \tag{29}$$

Taking into account (28) and (29), equation (21) becomes $\forall x_{i_l}^{(l-1)} \leq x^{(l-1)} \leq x_{i_l+1}^{(l-1)}$, $\forall i_l = \{1, \ldots, r_1\}$, $\forall l = \{1, \ldots, n\}$

$$\begin{aligned}
& \sum_{j_1=i_1}^{i_1+1} \cdots \sum_{j_n=i_n}^{i_n+1} \prod_{l=1}^{n} \mu_{M_l^{j_l}}(x^{(l-1)}) \\
& \left[\left(1 - \frac{b_0^{i_1+1 j_2 \ldots j_n} - b_0^{i_1 j_2 \ldots j_n}}{x_{i_1+1} - x_{i_1}} \right) \lambda^n \right. \\
+ & \left(b_1^{j_1 \ldots j_n} - \frac{b_0^{j_1 i_2+1 j_3 \ldots j_n} - b_0^{j_1 i_2 j_3 \ldots j_n}}{\dot{x}_{i_2+1} - \dot{x}_{i_2}} \right) \lambda^{n-1} + \ldots \\
+ & \left(b_{n-1}^{j_1 \ldots j_n} - \frac{b_0^{j_1 \ldots j_{n-1} i_n+1} - b_0^{j_1 \ldots j_{n-1} i_n}}{x_{i_n+1}^{(n-1)} - x_{i_n}^{(n-1)}} \right) \lambda \\
+ & \left. b_n^{j_1 \ldots j_n} \right] = 0
\end{aligned} \tag{30}$$

$\forall j_l = \{i_l, i_l + 1\}$.
So it must be satisfied that the real part of the roots of

$$\begin{aligned}
& \left(1 - \frac{b_0^{i_1+1 j_2 \ldots j_n} - b_0^{i_1 j_2 \ldots j_n}}{x_{i_1+1} - x_{i_1}} \right) \lambda^n + \\
& \left(b_1^{j_1 \ldots j_n} - \frac{b_0^{j_1 i_2+1 j_3 \ldots j_n} - b_0^{j_1 i_2 j_3 \ldots j_n}}{\dot{x}_{i_2+1} - \dot{x}_{i_2}} \right) \lambda^{n-1} + \ldots \\
& \ldots + \left(b_{n-1}^{j_1 \ldots j_n} - \frac{b_0^{j_1 \ldots j_{n-1} i_n+1} - b_0^{j_1 \ldots j_{n-1} i_n}}{x_{i_n+1}^{(n-1)} - x_{i_n}^{(n-1)}} \right) \lambda \\
& + b_n^{j_1 \ldots j_n} = 0
\end{aligned} \tag{31}$$

is strictly negative. Finally, note that if $x^{(l-1)} \notin \left[x_1^{(l-1)}, x_{r_l}^{(l-1)} \right]$ for some $l = \{1, \ldots, n\}$ (this means that x is outside the universe of discourse), the system is stable, provided that all the subsystems $\mathbf{B}^{j_1 \ldots j_{l-1} 1 j_{l+1} \ldots j_n}$ and $\mathbf{B}^{j_1 \ldots j_{l-1} r_l j_{l+1} \ldots j_n}$ are stable.

4 Example

Let us analyze the stability of a fuzzy system given by

$$\begin{aligned}
R^1 &: \; If \; \left(x \; is \; M_1^1 \right) \; then \; \dot{x} + x = -2 \\
R^2 &: \; If \; \left(x \; is \; M_1^2 \right) \; then \; \dot{x} + x = 2
\end{aligned} \tag{32}$$

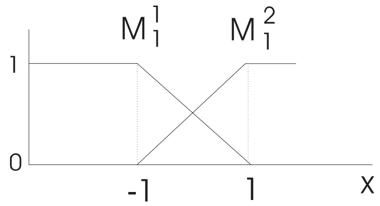

Figure 3: Membership Functions

It is easy to identify that $\mathbf{B}^1 = \mathbf{B}^2 = 1$ and $\mathbf{b_0}^1 = -2$, $\mathbf{b_0}^2 = 2$. Figure 3 shows the membership functions for the rules conditions.

The parameters of the fuzzy system may be obtained as follows:

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \frac{\sum_{i_1=1}^{2} w^{i_1}(\mathbf{x})\mathbf{B}^{i_1}(\mathbf{x})}{\sum_{i_1=1}^{2} w^{i_1}(\mathbf{x})} \\ &= \frac{0.5(1-x)(1)+0.5(1+x)(1)}{0.5(1-x)+0.5(1+x)} \\ &= 1 \\ \mathbf{b_0}(\mathbf{x}) &= \frac{\sum_{i_1=1}^{2} w^{i_1}(\mathbf{x})\mathbf{b_0}^{i_1}(\mathbf{x})}{\sum_{i_1=1}^{2} w^{i_1}(\mathbf{x})} \\ &= \frac{0.5(1-x)(-2)+0.5(1+x)(2)}{0.5(1-x)+0.5(1+x)} \\ &= 2x \end{aligned} \tag{33}$$

It should be observed that $\mathbf{b_0}(\mathbf{0}) = \mathbf{0}$.

Both linear subsystems are stable, because $\mathbf{B_1}^1$ and $\mathbf{B_1}^2$ are positive definite. According to the results obtained in [7] and [11] (Theorem 1), the fuzzy system should be asymptotically stable in the large, because there exists a positive definite matrix $\mathbf{P}$, such that $\forall i_1$, $(\mathbf{A}^{i_1})^T\mathbf{P} + \mathbf{P}\mathbf{A}^{i_1} < 0$. If $\mathbf{P} = \lambda = 1$ is chosen,

$$\begin{aligned} (\mathbf{A}^{i_1})^T\mathbf{P} + \mathbf{P}\mathbf{A}^{i_1} &= \left[(\mathbf{B}^{i_1})^{-1}\right]^T\mathbf{P} + \mathbf{P}(\mathbf{B}^{i_1})^{-1} \\ &= \frac{1}{(-1)}(1) + \frac{1}{(1)}(-1) \\ &= -1 < 0 \end{aligned} \tag{34}$$

Nevertheless, the resultant fuzzy system is unstable. In fact, it can be verified that the asymptotic stability in the large can not be guarranteed, by applying Theorem 2:

$$\left[I - \frac{d\mathbf{b_0}(\mathbf{x})}{d\mathbf{x}}\right]\mathbf{B}^{-1}(\mathbf{x}) = (1-2)\frac{1}{1} = -1 \tag{35}$$

which is not strictly possitive $\forall x \neq 0$.

Nevertheless, to apply this theorem, we need to obtain an explicit expression for $\mathbf{B}(\mathbf{x})$. By doing so, the fuzzy system can be rewritten as:

$$\mathbf{B}(\mathbf{x})\dot{\mathbf{x}} + \mathbf{x} = \mathbf{b_0}(\mathbf{x}) \;\Leftrightarrow\; \dot{x} + x = 2x \;\Leftrightarrow\; \dot{x} - x = 0 \tag{36}$$

which is clearly an unstable system.

It is more useful to apply Theorem 3. It must be verified that $\forall x_{i_1} \leq x \leq x_{i_1+1}$, $\forall i_1 = \{1, \ldots, r_1 - 1\}$, the following polynomials correspond to stable linear systems:

$$\left(1 - \frac{b_0^{i_1+1} - b_0^{i_1}}{x_{i_1+1} - x_{i_1}}\right)\lambda + \; b_1^{j_1} = 0 \qquad \forall j_1 = \{i_1, i_1+1\} \tag{37}$$

This means, the stability of the following polynomials should be analyzed:

$$\left(1 - \frac{b_0^2 - b_0^1}{x_2 - x_1}\right)\lambda + \; b_1^1 = 0 \tag{38}$$

$$\left(1 - \frac{b_0^2 - b_0^1}{x_2 - x_1}\right)\lambda + \; b_1^2 = 0 \tag{39}$$

which become

$$-\lambda + 1 = 0 \tag{40}$$

$$-\lambda + 1 = 0 \tag{41}$$

Both polynomials are unstable, so the asymptotic stability of the fuzzy system can not be verified.

5 Conclusion

Further steps towards the stability analysis of continuous fuzzy systems based on Takagi-Sugeno model have been presented in this work.

The solution presented corresponds to a realistic model, in which constant term in the rules consequent part exists. Furthermore, the application of the presented theorems is easy, as has been shown by an example.

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