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Method of Least Squares Applied to the Generalized Modus Ponens with Interval-Valued Fuzzy Sets

E. Agustench, H. Bustince* and V. Mohedano
Departamento de Automática y Computación.
Universidad Pública de Navarra
Campus Arrosadía, 31006 Pamplona, Spain,
bustince@unavarra.es

Abstract

Firstly we present a geometric interpretation of interval-valued fuzzy sets. Secondly, we apply the method of least squares to the fuzzy inference rules when working with these sets. We begin approximating the lower and upper extremes of the membership intervals to axb type functions by means of the method of least squares. Then we analyze a technique for evaluating the conclusion of the generalized modus ponens and we verify the fulfillment of Fukami and alumni axioms [9].

Keywords: Key words: Approximate reasoning, fuzzy inference rules, generalized modus ponens, interval-valued fuzzy set, method of least squares.

1 Introduction

Approximate reasoning is, informally speaking, as I.B. Turksen says in [21], the process or processes by which it is possible to deduce an imprecise conclusion from a collection of imprecise premises. The classic modus ponens is expressed by:

$$\begin{array}{l} A \longrightarrow B \\ \underline{A \qquad\quad} \\ \qquad\quad B \end{array}$$

This means that if,

A implies B and A is true, then
B is also true.

This line of reasoning was extended to fuzzy reasoning by L.A. Zadeh [26, 27] as follows:

The implication $A \longrightarrow B$ is replaced by the fuzzy inference rule:

Corresponding author

If x is A then y is B

Where A and B are fuzzy sets. A on a universe of discourse X and B on a universe of discourse Y, x is a variable that takes values in X, and y is a variable that takes values in Y. The fuzzy rule represents the existing relation between the variables x and y.

In the same fashion, the premise A is replaced by the fuzzy premise: x *is* A' where A' is a fuzzy set on a universe of discourse X, expressing the knowledge we have of the value x.

Combining the rule and the premise, it is possible to deduce new information, writing: y *is* B', where B' is a fuzzy set on the universe of discourse Y.

From the modus ponens and using the fuzzy set theory we can obtain representations of the imprecisions inherent to the human language. This way, the generalized modus ponens (GMP) was introduced in order to obtain a conclusion when starting from fuzzy premises. These rules can be expressed as follows:

$$\begin{array}{l} \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \underline{x \text{ is } A' \qquad\qquad\qquad} \\ y \text{ is } B' \qquad \text{(GMP)} \end{array}$$

The main advantage to this extension to fuzzy reasoning is that we can deduce new information, even when the membership is not exactly identical to the rule is condition or when the information we are considering is not exact. It is known that in the (GMP) when $A' = A$, then the generalized modus ponens is reduced to the case of the classic modus ponens.

Different methods have been suggested by authors such as Zadeh [26, 27], Fukami [9] Mizumoto and Zimmermann [12, 13], Ezawa and Kandel [8], for the study of the rules of fuzzy inference. In 1980 Fukami and alumni [9, 7] suggested the following set of axioms for the generalized modus ponens:

$(F1)$ If $A' = A$, then $B' = B$; (coincidence with classical modus ponens)

$$(F2) \text{ Either } \begin{cases} \text{(i) If } A' = A^2, \text{ then } B' = B \\ \qquad\qquad \text{or} \\ \text{(ii) If } A' = A^2, \text{ then } B' = B^2; \end{cases}$$

$(F3)$ If $A' = A^{\frac{1}{2}}$, then $B' = B^{\frac{1}{2}}$;

$$(F4) \text{ Either } \begin{cases} \text{(i) If } A' = A_c, \text{ then } B' = Y \\ \qquad\qquad \text{or} \\ \text{(ii) If } A' = A_c, \text{ then } B' = B_c. \end{cases}$$

where Ac is the complementary of A.

Almost simultaneously, Baldwin and Bilsworth [2] established another set of six axioms for the GMP, some of them in complete contradiction with those demanded by Fukami and alumni. For example Baldwin and Pilsworth demand that $B' \geq B$, something impossible if $(F2)(ii)$ is demanded at the same time. These contradictions arise because Baldwin and Pilsworth carry out studies that are led by classic logic, while Fukami and al. focus their studies on the conservation of the linguistic labels.

In this paper we are going to study the generalized modus ponens using interval-valued fuzzy sets and following a reasoning different from the one used by the above-mentioned authors [8, 9, 12, 13].

Our method is based on the ideas established by J.F. Baldwin [1, 2] A. Nafarie [14] Gorzalczany [10], etc$\cdots$ which can be summarized in the two following steps:

1) First relate A with A'

2) Construct the consequence B' using the result of the comparison above and B.

2 Interval-valued fuzzy sets

In this section we will recall the notion of interval-valued fuzzy set or Φ-fuzzy set introduced by L.A. Zadeh [26, 27] and R. Sambuc [19]. We begin presenting the notation we are going to use.

$D[0,1]$ will stand for the set of all the closed subintervals of the interval $[0,1]$; the elements of this set well be represented by capitals $M, N, \cdots$ it is known that $M = [M_L, M_U]$ where M_L and M_U are the lower and upper extreme respectively. $W_M = M_U - M_L$ will represent the amplitude of the interval M. We will say that $M \leq N$ if $M_L \leq N_L$ and $M_u \leq N_U$, this relation is transitive, reflexive, and antisymmetric and expresses that M is contained in N, that is, for each point $x \in M$ there is a point $Y \in N$ such that $y \geq x$. It is necessary to note that in interval-valued fuzzy set literature there have been other orders [4, 11, 16, 19] as well, however the relation we present herein is the most common and it is the one will use hereinafter in the rest of the paper.

We know that [4] $M = N$ if and only if $M_L = N_L$ and $M_u = N_U$. We will call the complimentary of M, $c(M) = M_c$ that is, $c(M) = [c(M_U), c(M_L)]$, c being any fuzzy complementation.

Let $X \neq \emptyset$ a given set. [3, 4, ...] *An interval-valued fuzzy set in X is an expression A given by:*

$$A = \{< x, M_A(x) > | x \in X\}$$

where the function

$$M_A : X \longrightarrow D[0,1]$$
$$x \longrightarrow M_A(x) = [M_{AL}(x), M_{AU}(x)]$$

defines the degree of membership of an element x to set A.

$IVFSs(X)$ will represent the set of all interval-valued fuzzy sets on X. We should insist on the fact that we will always take finite and not empty X, so that $Cardinal(X) = n$.

We will say that an interval-valued fuzzy set A is normal if there is at least one $x \in X$ such that $M_A(x) = [1,1]$.

The following expressions are defined in [3, 4] for all $A, B \in IVFSs(X)$

1. $A \leq B$ if and only if $M_{AL}(x) \leq M_{BL}(x)$ and $M_{AU}(x) \leq M_{BU}(x)$ $\forall x \in X$
2. $B \preceq A$ if and only if $M_{AL}(x) \leq M_{BL}(x)$ and $M_{AU}(x) \geq M_{BU}(x)$ for all $x \in X$
3. $A \sqsubseteq B$ if and only if $M_{AU}(x) \leq M_{BL}(x)$ $\forall x \in X$

4. $A = B$ if and only if $M_{AL}(x) = M_{BL}(x)$ and $M_{AU}(x) = M_{BU}(x)\ \ \forall x \in X$
5. $A_c = \{< x, c(M_A(x)) > | x \in X\} = \{< x, [c(M_{AU}(x)), c(M_{AL}(x))] > | x \in X\}$.

Besides, in [4] the following theorem is proved:

Theorem 1. [4] *Let β and α be t-norm and t-conorm respectively, we define*

$$\beta(A, B) \equiv \{< x, [\beta(M_{AL}(x), M_{BL}(x)), \beta(M_{AU}(x), M_{BU}(x))] > | x \in X\}$$
$$\alpha(A, B) \equiv \{< x, [\alpha(M_{AL}(x), M_{BL}(x)), \alpha(M_{AU}(x), M_{BU}(x))] > | x \in X\}$$

for all $A, B \in IVFSs$. Then, it is verified that:

a) If $\beta = \wedge$ and $\alpha = \vee$ then $\{IVFS(X), \wedge, \vee\}$ is a distributive lattice,which is bounded, not complemented and satifies Morgan's laws.

b) For any β and α (α dual of β) , the conmutative, associative properties and $\beta(A_c, B_c) = (\alpha(A, B))_c$, $\alpha(A_c, B_c) = (\beta(A, B))_c$. are satisfied.

3 Geometric interpretation

In figure 1 we present a geometric interpretation of the notion of interval-valued fuzzy sets. Basically it means the following. Since the lower extreme, the upper extreme and the amplitude of all of the intervals are $[0, 1]$ numbers, we can imagine a unit cube with its three axis given by these parameters.

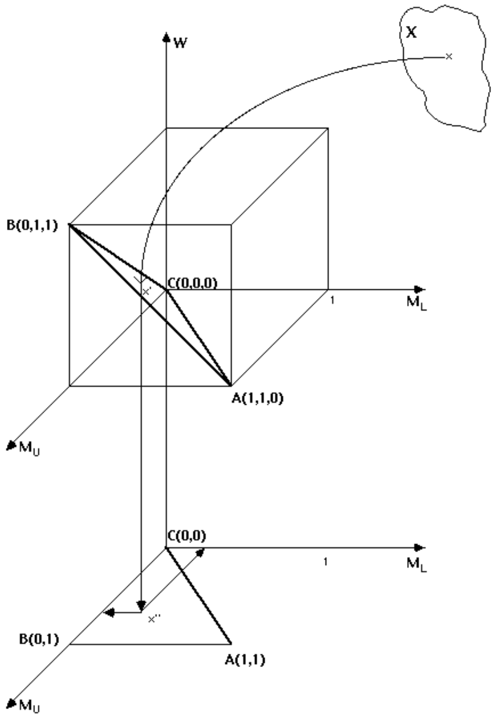

Figure 1

Since $M_L(x) \leq M_U(x)$ and $W(x) = M_U(x) - M_L(x)$ for all $x \in X$, the values of the parameters characterize an interval-valued fuzzy set so that the values the membership functions take are points of the triangle ACB. Thus an interval-valued fuzzy set can be interpreted as a mapping that goes from X to the triangle ACB so that each element of X has a corresponding element in the triangle ACB. For example, a $x \in X$ has a corresponding $x' \in ACB$ point characterized by these three values $(M_L(x), M_U(x), W(x))$.

When $W(x) = 0$ then $M_L(x) = M_U(x)$, in figure 1 this condition is represented in the segment CA. Therefore, the segment CA can be considered the representation of the fuzzy sets.

The orthogonal projection of the triangle ACB gives us the representation of a interval-valued fuzzy set on a drawing, such that in this drawing the interior of the triangle ACB is the area where $W > 0$.

4 Method of least squares applied to interval-valued fuzzy sets

In this section we present a method for obtaining the conclusion of the GMP when we are working with interval-valued fuzzy sets and we apply the method of least squares in order to, on the one hand approximate the lower extremes of the membership functions to $a_1 x^{b_1}$ type functions, and on the other, approximate the upper extremes to the same type of functions.

Since T.C. Chang, K. Hasegawa and C. W. Ibbs. [6], for convenience, we take the universes of discourse as follows:

1) $X = \{x_1, \cdots, x_n\}$, $Y = \{y_1, \cdots, y_m\}$

2) $x_i, y_j, \in (0, 1]$ for all $i = 1, \cdots, n,\ j = 1, \cdots, m,$

3) $x_i < x_{i+1}$, and $y_j < y_{j+1}$ for all i, j

L.A. Zadeh [24] presented the operators very, greater than, less than, more, less, etc$\cdots$ expressed in terms of the membership functions, and so associated numerical values to them, which allowed to work in an easier way.

From these considerations we present the following method for obtaining the conclusion of the GMP when we work with interval-valued fuzzy sets.

(i) By means of the method of least squares, approximate the lower extremes of the membership intervals to set A to functions of type $a_1 x^{b_1}$, so that we can write: $M_{\tilde{A}L}(x) = a_1 x^{b_1}$.

(ii) To do the same as the item above with the lower extremes of the membership intervals to set A', that is, $M_{\tilde{A}'L}(x) = a'_1 x^{b'_1}$.

(iii) Take x of $M_{\tilde{A}L} = a_1 x^{b_1}$ and substitute in $M_{\tilde{A}'L} = a'_1 x^{b'_1}$, that is, $M_{\tilde{A}'L} = \frac{a'_1}{a_1^{\frac{b'_1}{b_1}}} M_{\tilde{A}L}^{\frac{b'_1}{b_1}}$, (with $b_1 \neq 0$).

(iv) Build fuzzy set $\Lambda_1 = \{< y, M_{\Lambda_1}(y) = \min(1, \frac{a_1'}{a_1^{\frac{b_1'}{b_1}}} M_{BL}^{\frac{b_1'}{b_1}}(y)) > |y \in Y\}$.

(v) Repeat the four steps above for the upper extreme so that:

$$\Lambda_2 = \{< y, M_{\Lambda_2}(y) = \min(1, \frac{a_2'}{a_2^{\frac{b_2'}{b_2}}} M_{BU}^{\frac{b_2'}{b_2}}(y)) > |y \in Y\}.$$

(vi) Build conclusion B' as follows:

$$B' = \{< y, M_{B'}(y) = [\min(M_{\Lambda_1}(y), M_{\Lambda_2}(y)), \max(M_{\Lambda_1}(y), M_{\Lambda_2}(y))] > |y \in Y\}.$$

We carry out the approximation indicated in item (i) by resolving the following algebraic equations:

$$(1) \begin{cases} n\text{Ln}a_1 + (\sum_{i=1}^{n} \text{ln}x_i)b_1 = \sum_{i=1}^{n} \text{Ln}M_{AL}(x_i) \\ (\sum_{i=1}^{n} \text{Ln}x_i)\text{Ln}a_1 + (\sum_{i=1}^{n} \text{Ln}^2 x_i)b_1 = \sum_{i=1}^{n} \text{Ln}x_i \text{Ln}M_{AL}(x_i) \end{cases}$$

obtained from taking logarithms in the expression $M_{AL}(x_i) = a_1 x_i^{b_1}$ and applying the method of least squares. *In the construction of equations (1) we do not take into account elements like $M_{AL}(x_i) = 0$, that is, these elements are ignored in order to obtain the systems of linear algebraic equations (1).*

Theorem 2. *Let $p \in \mathbb{R}^+ \cup \{0\}$ and let $A \in IVFSs(X)$. In the conditions above, (of the six item method), if $A' = A^p$, then $B' = B^p$.*

Proof. Let $A', A \in IVFSs(X)$ we will represent as $M_{\tilde{A}L}(x_i) = a_1 x_i^{b_1}$ and $M_{\tilde{A}'L}(x_i) = a_1' x_i^{b_1'}$ the approximations obtained of the lower extremes by the method of least squares when we approximate to functions of the type ax^b. From (7) we deduce that:

$$n\text{Ln}a_1' + (\sum_{i=1}^{n} \text{Ln}x_i)b_1' = \sum_{i=1}^{n} \text{Ln}M_{A'L}(x_i) = \sum_{i=1}^{n} \text{Ln}M_{AL}^p(x_i) = p\sum_{i=1}^{n} \text{Ln}M_{AL}(x_i)$$
$$= pn\text{Ln}a_1 + p(\sum_{i=1}^{n} \text{Ln}x_i)b_1 \quad (8).$$

$$(\sum_{i=1}^{n} \text{Ln}x_i)\text{Ln}a_1' + (\sum_{i=1}^{n} \text{Ln}^2 x_i)b_1'$$
$$= \sum_{i=1}^{n} \text{Ln}x_i \text{Ln}M_{A'L}(x_i) = \sum_{i=1}^{n} \text{Ln}x_i \text{Ln}M_{AL}^p(x_i) = p\sum_{i=1}^{n} \text{Ln}x_i \text{Ln}M_{AL}(x_i) =$$
$$= p(\sum_{i=1}^{n} \text{Ln}x_i)\text{Ln}a_1 + p(\sum_{i=1}^{n} \text{Ln}^2 x_i)b_1 \quad (9).$$ Solving in (8) we have

$$\text{Ln}\frac{a_1'}{a_1^p} = \frac{(\sum_{i=1}^{n} \text{Ln}x_i)(p \cdot b_1 - b_1')}{n}$$

from (9) we have $(\sum_{i=1}^{n} \text{Ln}x_i)\text{Ln}\frac{a_1'}{a_1^p} = (\sum_{i=1}^{n} \text{Ln}^2 x_i)(p \cdot b_1 - b_1')$, substituting $\text{Ln}\frac{a'}{a^p}$ in this expression and taking into account that $(\sum_{i=1}^{n} \text{Ln}x_i)^2 - n(\sum_{i=1}^{n} \text{Ln}^2 x_i) \neq 0$, we have $b_1' = p \cdot b_1$ and $a_1' = a_1^p$, therefore $\frac{b_1'}{b_1} = p$ and $\frac{a_1'}{a_1^{\frac{b_1'}{b_1}}} = \frac{a_1'}{a_1^p} = \frac{a_1^p}{a_1^p} = 1$.

The upper extremes are proven in a similar way. □

The importance of the theorem is made clear in the following particular cases:
(a) If $A' = A$, then $B' = B$.
(b) If $A' = A^2$, then $B' = B^2$.
(c) If $A' = A^{1/2}$, then $B' = B^{1/2}$.
(d) If $A' = A^4$, then $B' = B^4$.
Where
$A^2 = \{< x, M_{A^2}(x) = [M_{AL}^2(x), M_{AU}^2(x)] > | x \in X\}$
$A^{1/2} = \{< x, M_{A^{1/2}}(x) = [M_{AL}^{1/2}(x), M_{AU}^{1/2}(x)] > | x \in X\}$
$A^4 = \{< x, M_{A^4}(x) = [M_{AL}^4(x), M_{AU}^4(x)] > | x \in X\}$.

The mechanism of inference we present in the 6 items above has the following two advantages:

1) It can be applied when the membership functions in the premises are characterized by the intervals whose extremes represent the fuzzy linguistic operators as defined by L.A. Zadeh.

2) The mathematical operations for the GMP are simple and appropriate for implementation.

5 Conclusions and future research

Besides the geometric interpretation of the interval-valued fuzzy sets, in this paper it is made clear that the method that we develop for obtaining the conclusion of the GMP when using interval-valued fuzzy sets gives good results when applied to the linguistic labels just as they were introduced by L.A. Zadeh in [24].

Besides, with this method the axioms (F1), (F2) and (F3) of Fukami and alumni are satisfied, not satisfying (F4).

The reason for which (F4) is not satisfied is due to the fact that with IVFSs total ignorance is obtained when the membership interval of every element of the set is always $[0, 1]$, and evidently, these intervals do not always coincide with the complementary just as we have defined it in section 2.

It is necessary to note that it is not always advisable to use the method we present for obtaining the conclusion of the GMP, because being the functions $a_1 x^{b_1}$ monotonous, it can occur that the lower extremes $M_L(x)$ or the upper extremes

$M_U(x)$ could not be monotonous. For example, $M_L(x)$ can be of the *medium* type and then the approximation of $M_L(x)$ to functions of type $a_1x^{b_1}$ is not advisable.

We conclude by saying that the fact that working with interval-valued fuzzy sets and not working with fuzzy sets does not lessen generality to the developments made in this paper but rather the contrary, since everything exposed here is also valid for fuzzy sets by just taking intervals with 0 amplitude.

It is evident that the method presented in Section 4 is based on the functional relations, thus, it is necessary to indicate that our future research will be focused on obtaining the conclusion of the GMP when we are working:
a) with interval-valued fuzzy sets and,
b) when the input-output relation is defined by one of the classic implication operators in fuzzy literature.

Evidently, next we will have to compare the results obtained in this paper (relative to the conservation of the linguistic labels) with the results we obtain when we do not use functional relations between the sets, but implication operators instead.

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