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Putting Together Łukasiewicz and Product Logics*

F. Esteva and L. Godo
Institut d'Investigacions en Intel.ligència Artificial (IIIA)
Consejo Superior de Investigaciones Científicas (CSIC)
Campus Universitat Autònoma de Barcelona, s/n
08193 Bellaterra, Spain
{*esteva,godo*}*@iiia.csic.es*

Abstract

In this paper we investigate a propositional fuzzy logical system ŁΠ which contains the well-known Łukasiewicz, Product and Gödel fuzzy logics as sublogics. We define the corresponding algebraic structures, called ŁΠ-algebras and prove the following completeness result: a formula φ is provable in the ŁΠ logic iff it is a tautology for all linear ŁΠ-algebras. Moreover, linear ŁΠ-algebras are shown to be embeddable in linearly ordered abelian rings with a strong unit and cancellation law.

1 Introduction

The idea of this paper is to define a logical system extending both Łukasiewicz (Ł) and Product (Π) logics. Łukasiewicz logic is well-known (see for instance [4, 7]) and Product logic, a fuzzy logic having product as conjunction, was introduced in [8]. In [7] Hájek defines the basic fuzzy logic BL having as main axiomatic extensions Łukasiewicz, Product and Gödel (G) logics. All these logics take the conjunction (&) and the implication ($\rightarrow$) as the only primitive connectives (besides the truth constant 0), and have the Łukasiewicz, product and minimum t-norms and their corresponding residua as truth-functions respectively.

There have been a few attempts to extend these logics with more connectives. Baaz has introduced in [2] a projection (boolean) connective Δ into Gödel logic: the truth value of $\Delta\varphi$ is 1 if the truth-value of φ is 1, 0 otherwise. Afterwards, this connective has also been introduced by Hájek [7] in the rest of the above logics, resulting in the extended logics, BL_Δ, $Ł_\Delta$, Π_Δ and G_Δ. In a recent paper, Esteva et al. [5] extend SBL (an axiomatic extension of BL), Π and G logics with an involutive negation, since in all these logics the definable negation $\neg\varphi = \varphi \rightarrow 0$

*This is a revised version of the paper with the same title appeared in the Proc. of the Estylf'98 Conference, September 8-10, 1998, Pamplona (Spain) is not involutive, in contrast to what happens in Łukasiewicz logic. The resulting logics have been denoted $SBL_\sim$, $\Pi_\sim$ and $G_\sim$ respectively.

On the other hand, the strong system of predicate fuzzy logic of Takeuti-Titani [9] combines the above mentioned three conjunctions and three implications, together with truth-constants. In [7] Hájek also presents a different version of such logic, called TT∀, endowed with an infinitary deduction rule and having the predicate Łukasiewicz, product and Gödel logics as sublogics.

However, unlike the latter logics (Ł, Π, G), the just mentioned Takeuti-Titani-like logics do not present a corresponding algebraic structure as the other ones. Actually, all the completeness results of the first group of logics are obtained via their associated algebraic structures (BL-algebras, MV-algebras, Π-algebras, etc.) by means of decomposition theorems of such algebras into subdirect products of linear algebras and finally relating these with the t-norm based structures of the unit interval [0, 1].

With these results in mind, this paper aims at defining a logical system, and its corresponding algebras, having both Łukasiewicz and product logics as sublogics. As a consequence, we shall also have Gödel logic as sublogic for free. In the second section, after defining the logic ŁΠ, we introduce ŁΠ-algebras and we prove their decomposition as subdirect product of linearly ordered (l.o.) ones and prove completeness of ŁΠ w.r.t. them. Finally, in Section 3 we prove that l.o. ŁΠ-algebras are embeddable in a linearly ordered cancelative ring (they are the interval [0,1] of the ring). We end up with some conclusions. Moreover the paper contains two annexes. Annex 1 contains some necessary background on different systems of fuzzy logic while Annex 2 contains the proof of Proposition 4 of Section 3. After this paper was presented at the ESTYLF'98 conference a standard completeness result has been obtained by the authors in a joint work with Franco Montagna (see [6]).

2 The ŁΠ Logic

Now we introduce the ŁΠ Logic, extending both Łukasiewicz and Product logics. We take three primitive connectives $\to_L$,$\odot$,$\to_\Pi$ and the truth-constant 0. Other definable connectives are $\neg_L$, $\neg_\Pi$, Δ, $\&$, $\underline{\vee}$, $\wedge$, $\vee$ and $\to_G$, where:

$\neg_L\varphi$	is	$\varphi \to_L 0$
$\neg_\Pi\varphi$	is	$\varphi \to_\Pi 0$
$\Delta\varphi$	is	$\neg_\Pi\neg_L\varphi$
$\varphi\&\psi$	is	$\neg_L(\varphi \to_L \neg_L\psi)$
$\varphi \underline{\vee} \psi$	is	$\neg_L\varphi \to_L \psi$
$\varphi \wedge \psi$	is	$\varphi\&(\varphi \to_L \psi)$
$\varphi \vee \psi$	is	$\neg_L(\neg_L\varphi \wedge \neg_L\psi)$
$\varphi \to_G \psi$	is	$\Delta(\varphi \to_L \psi) \vee \psi$
$\varphi \equiv_L \psi$	is	$(\varphi \to_L \psi)\&(\psi \to_L \varphi)$
$\varphi \equiv_\Pi \psi$	is	$(\varphi \to_\Pi \psi) \odot (\psi \to_\Pi \varphi)$

A standard truth-evaluation is any mapping e assigning to each propositional variable p a value of the unit interval [0, 1], and it extends to all formulas by means of the Łukasiewicz and product truth-functions, that is:

$$\begin{array}{lcl} e(\varphi \rightarrow_L \psi) & = & \min(1, 1 - e(\varphi) + e(\psi)), \\ e(\varphi \odot \psi) & = & e(\varphi) \cdot e(\psi), \text{ and} \\ e(\varphi \rightarrow_\Pi \psi) & = & \begin{cases} 1, & \text{if } e(\varphi) \leq e(\psi) \\ e(\psi)/e(\varphi), & \text{otherwise} \end{cases} \end{array}$$

Notice that, with these definitions, we recover the usual truth functions for the above definable connectives: $e(\neg_L\varphi) = 1 - e(\varphi)$; $e(\neg_\Pi\varphi) = 1$ if $e(\varphi) = 0$, 0 otherwise (Gödel negation); $e(\Delta\varphi) = 1$ if $e(\varphi) = 1$, 0 otherwise; $e(\varphi\&\psi) = \max(0, e(\varphi) + e(\psi) - 1)$; $e(\varphi \veebar \psi) = \min(1, e(\varphi) + e(\psi))$; $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$; $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$; and $e(\varphi \rightarrow_G \psi) = 1$ if $e(\varphi) \leq e(\psi)$, $e(\psi)$ otherwise (Gödel implication).

Definition 1 *Axioms of ŁΠ Logic are :*

1. *the axioms of Łukasiewicz logic with projection, $Ł_\Delta$, for the connectives &, $\rightarrow_L$, and Δ (see Annex 1 and [7]),*
2. *the axioms of the product logic with involution, $\Pi_\sim$, for the connectives $\odot$, $\rightarrow_\Pi$, and $\neg_L$ (see Annex 1 and [5]),*
3. *together with the following additional axioms:*

$$\begin{array}{ll} (LP1) & (\varphi\&\psi) \rightarrow (\varphi \odot \psi) \\ (LP2) & (\varphi \rightarrow_\Pi \psi) \rightarrow (\varphi \rightarrow_L \psi) \\ (LP3) & \Delta(\varphi \odot \psi) \rightarrow \Delta(\varphi\&\psi) \\ (LP4) & \Delta(\varphi \rightarrow_L \psi) \rightarrow \Delta(\varphi \rightarrow_\Pi \psi) \\ (LP5) & (\varphi \odot \psi)\veebar(\varphi \odot \neg_L\psi) \equiv \varphi \\ (LP6) & \Delta\neg_L(\psi\&\chi) \rightarrow [(\varphi \odot (\psi\veebar\chi)) \equiv ((\varphi \odot \psi)\veebar(\varphi \odot \chi))] \\ (LP7) & \Delta(\varphi\veebar\psi) \rightarrow [(((\varphi\&\psi) \odot \chi)\veebar\chi) \equiv ((\varphi \odot \chi)\veebar(\psi \odot \chi))] \\ (LP8) & \Delta(\varphi\veebar\psi) \rightarrow [(((\varphi\&\psi) \odot \chi)\&\chi) \equiv ((\varphi \odot \chi)\&(\psi \odot \chi))] \end{array}$$

where $\rightarrow$ and $\equiv$ stand for any of the implications or their corresponding equivalences respectively.

Inference rules of ŁΠ are Modus Ponens for both implications and necessitation for Δ: from φ derive $\Delta\varphi$.

The notion of proof is as usual. We will write $Ł\Pi \vdash \varphi$ and $T \vdash \varphi$ to denote that φ is provable in ŁΠ and provable from a theory T over ŁΠ, respectively.

It is easy to check that ŁΠ is sound w.r.t. the above semantics, that is, each axiom is a 1-tautology and the deduction rules preserve 1-tautologies. Next we introduce the algebraic structure corresponding to our logic.

Definition 2 *A ŁΠ-algebra is an algebra $\mathbf{A} = \langle A, *, \Rightarrow_L, \odot, \Rightarrow_\Pi, \cup, \cap, 0, 1\rangle$ such that:*

- $\langle A, \odot, \Rightarrow_\Pi, \cup, \cap, 0, 1, \neg_L\rangle$ *is a $\Pi_\sim$-algebra where $\neg_L(x) = x \Rightarrow_L 0$ (see Annex 1 and [5])*

- *$\langle A, *, \Rightarrow_L, \cup, \cap, \Delta, 0, 1\rangle$ is a MV_Δ-algebra where $\Delta(x) = (\neg_L x) \Rightarrow_\Pi 0$ (see Annex 1 and [7]).*
- *For all $x, y, z \in A$ the following conditions hold:*
  (1) $x * y \leq x \odot y$
  (2) $\Delta(x \Rightarrow_L y) \leq \Delta(x \Rightarrow_\Pi y)$, $\Delta(x * y) \geq \Delta(x \odot y)$
  (3) If $x * y = 0$ then $z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y)$
  (4) If $x \oplus y = 1$ then:
  $((x * y) \odot z) \oplus z) \quad = (z \odot x) \oplus (z \odot y),$
  $((x * y) \odot z) * z) \quad = (z \odot x) * (z \odot y),$
  where $x \oplus y = \neg_L(\neg_L(x) * \neg_L(y))$.

Lemma 1 *In a ŁΠ-algebra* **A** *the following conditions hold for all $x, y \in A$:*
(i) $x \Rightarrow_\Pi y \leq x \Rightarrow_L y$,
(ii) $\Delta(x \Rightarrow_L y) = \Delta(x \Rightarrow_\Pi y)$, $\Delta(x * y) = \Delta(x \odot y)$,
(iii) $(x \odot y) \oplus (x \odot \neg_L y) = x$,
(iv) $\neg_L(x \odot y) = \neg_L x \oplus (x \odot \neg_L y)$.

Proof: Condition (i) is a consequence of the above property (1) and that $(*, \Rightarrow_L)$ and $(\odot, \Rightarrow_\Pi)$ are adjoint pairs. Equalities (ii) are easy consequences of the property (2) and that from (i) one can derive $\Delta(x \Rightarrow_\Pi y) \leq \Delta(x \Rightarrow_L y)$, taking into account usual properties of the Δ operator. The proof of (iii) is as follows: $x = x \odot 1 = x \odot (y \oplus \neg_L y)$, and since $y * \neg_L y = 0$ we can apply (3) and thus $x \odot (y \oplus \neg_L y) = (x \odot y) \oplus (x \odot \neg_L y)$. The proof of (iv) uses the following result from the theory of MV-algebras (see for instance [4]):

$$\text{if } x * y = 0, \text{ then } (x \oplus y) * \neg y = x.$$

From (iii) we have that $(x \odot y) * (x \odot \neg_L y) = 0$ and therefore,

$$x \odot y = ((x \odot y) \oplus (x \odot \neg_L y)) * \neg_L(x \odot \neg_L y) = x * \neg_L(x \odot \neg_L y).$$

Finally, using the De Morgan laws for negation w.r.t. $*$ and $\oplus$, we have $\neg_L(x \odot y) = \neg_L x \oplus (x \odot \neg_L y)$.

It is easy to check that the ŁΠ logic is sound with respect to ŁΠ-algebras. This means that, for every ŁΠ-algebra **A**, the axioms of ŁΠ are **A**-tautologies and the deduction rules preserve the **A**-tautologies. A formula φ is an **A**-tautology if it gets the value 1 (of A) in each evaluation over the algebra **A**, an evaluation being a mapping e assigning to each propositional variable p an element $e(p) \in A$ and extending to all formulas using the operations of **A** as truth-functions.

It is clear that the real unit interval [0, 1] equipped with the Łukasiewicz and Product logics truth functions is an ŁΠ-algebra. We shall call it the *standard ŁΠ-algebra*. We will prove that the algebra of classes of provable equivalent formulas is also an ŁΠ algebra. But we have to be cautious since in ŁΠ there are two different implications and equivalences. However one can show that if a theory proves one implication or equivalence, it has to prove the other one as well.

Proposition 1 *If T is a theory over ŁΠ, then the following conditions hold:*
(1) $T \vdash \varphi$ iff $T \vdash \Delta\varphi$.
(2) $T \vdash \varphi \rightarrow_{\Pi} \psi$ iff $T \vdash \varphi \rightarrow_L \psi$.
(3) $T \vdash \varphi \equiv_{\Pi} \psi$ iff $T \vdash \varphi \equiv_L \psi$.

Proof: (1) It comes from the rule of necessitation for Δ and axiom (Δ3) (see annex 1).
(2) Axiom (LP2) proves one direction. On the other direction, if $T \vdash \varphi \rightarrow_L \psi$ then, applying the necessitation rule for Δ, $T \vdash \Delta(\varphi \rightarrow_L \psi)$ and so, by (LP4), $T \vdash \Delta(\varphi \rightarrow_{\Pi} \psi)$ and hence, by (1), $T \vdash \varphi \rightarrow_{\Pi} \psi$.
(3) It is an easy consequence of (2).

As a result of this proposition we can define in the usual way the quotient set $\mathcal{L}/\equiv_T$ of equivalence classes of formulas w.r.t. a theory T, where $\varphi \equiv_T \psi$ iff $T \vdash \varphi \equiv \psi$, being $\equiv$ either $\equiv_L$ or $\equiv_{\Pi}$. Then in the quotient set, also as usual, connectives can be interpreted as operations and now the corresponding algebraic structure can be shown to be a ŁΠ algebra.

Lemma 2 *For any theory T over ŁΠ, $\mathcal{L}/\equiv_T$ is a ŁΠ-algebra.*

Proof: The analogous results for MV-algebras and Π-algebras prove that the equivalence relation $\equiv_T$ is a congruence w.r.t. the connectives ($\&, \rightarrow_L$) and ($\odot, \rightarrow_\pi$) respectively, so it remains to prove that the extra-conditions of the ŁΠ-algebras also hold in the quotient algebra. But these easily follow from the axioms (LP1)–(LP8) of the logical system.

Next we have to prove that each ŁΠ-algebra is a subdirect product of linearly ordered ŁΠ-algebras. The proof of this fact is rather standard and the basic definitions and results are given below. We just provide those proofs which are not completely analogous to the standard ones (cf. [7]).

Definition 3 *A subset F of a ŁΠ-algebra A is a* filter *if it satisfies:*
(F1) For all $x, y \in F$, $x * y \in F$
(F2) If $x \in F$ and $y \geq x$, then $y \in F$
(F3) If $x \Rightarrow_{\Pi} y \in F$, then $\neg_L y \Rightarrow_{\Pi} \neg_L x \in F$
Moreover, F is said to be an ultrafilter *(or* prime filter*) iff it is a filter satisfying:*
(F4) For all $x, y \in F$, either $x \Rightarrow_{\Pi} y \in F$ or $y \Rightarrow_{\Pi} x \in F$

Obviously, if F is a filter w.r.t. an ŁΠ-algebra, F is also a filter of the corresponding MV and $\Pi_\sim$ reducts. Moreover, the following properties hold.

Lemma 3 *Let F be a filter in an ŁΠ-algebra A. Then the following equivalences hold:*
(i) $x \in F$ iff $\Delta x \in F$.
(ii) $x * y \in F$ iff $x \odot y \in F$
(ii) $x \Rightarrow_L y \in F$ iff $x \Rightarrow_{\Pi} y \in F$

Proof: (i) Since $\Delta x \leq x$, by (F2) if $\Delta x \in F$ then $x \in F$ too. On the other direction, $x = 1 \Rightarrow_\Pi x$ and thus, by (F3), $\Delta x = (\neg_L x) \Rightarrow_\Pi 0 \in F$ as well.
(ii) and (iii) easily come from (i) taking into account (ii) of Lemma 1.

This lemma allows us to properly define the congruence relation in the next lemma.

Lemma 4 *Let F be a filter in an ŁΠ-algebra $\mathbf{A}$. Define*

$$x \sim_F y \text{ iff both } x \Rightarrow y, y \Rightarrow x \in F,$$

where $\Rightarrow$ is any of the two implications. Then:
(i) $\sim_F$ is a congruence on $\mathbf{A}$ and $\mathbf{A}/\sim_F$ is an ŁΠ-algebra.
(ii) $A/\sim_F$ is linearly ordered iff F is an ultrafilter.

Lemma 5 *If F is a filter and $a \notin F$, then there exists an ultrafilter UF such that $F \subseteq UF$ and $a \notin UF$.*

Proof: Sketch. Suppose that F is a filter not containing a and that there exist two elements $x, y \in A$ such that $x \Rightarrow_\Pi y \notin F$ and $y \Rightarrow_\Pi x \notin F$. Then one can check that the least filters F_1 and F_2 such that contain F and $x \Rightarrow_\Pi y$ and $y \Rightarrow_\Pi x$ respectively are:

$$F_1 = \{u \in A \mid \exists v \in F \text{ and } u \geq (v * \Delta(x \Rightarrow_\Pi y))\}$$
$$F_2 = \{u \in A \mid \exists v \in F \text{ and } u \geq (v * \Delta(y \Rightarrow_\Pi x))\}.$$

Next one proves that either $a \notin F_1$ or $a \notin F_2$. Namely, if $a \in F_1 \cap F_2$ then $a \geq v_1 * \Delta(x \Rightarrow_\Pi y)$ and $a \geq v_2 * \Delta(y \Rightarrow_\Pi x)$ and so $a \geq (v_1 \cap v_2) * \Delta(x \Rightarrow_\Pi y)) \cup ((v_1 \cap v_2) * \Delta(y \Rightarrow_\Pi x)) = (v_1 \cap v_2) * (\Delta(x \Rightarrow_\Pi y) \cup (\Delta(y \Rightarrow_\Pi x)) = (v_1 \cap v_2) * \Delta((x \Rightarrow_\Pi y) \cup (y \Rightarrow_\Pi x)) = (v_1 \cap v_2) * \Delta(1) = v_1 \cap v_2 \in F$ in contradiction with the hypothesis. Finally, one builds an increasing sequence of filters $F_i \subseteq F_{i+1}$ such that $F_0 = \{1\}$ and $a \notin F_i$, for every i. The ultrafilter is then the big union of all the F_i's.

This last lemma is used to show that the intersection of all ultrafilters of an ŁΠ-algebra is just the singleton $\{1\}$, and thus, using (ii) of Lemma 4 together with standard results about subdirect products, we get the following decomposition theorem.

Theorem 1 *Each $L\Pi$-algebra is a subdirect product of linearly ordered $L\Pi$-algebras.*

Moreover l.o. $L\Pi$-algebras are subdirectly irreducible, which is not true for Łukasiewicz and product algebras.

Proposition 2 *An $L\Pi$-algebra is subdirect irreducible iff it is linearly ordered.*

The proof is the same as in [5] for $SBL_\sim$ algebras. The only filters of a l.o. $L\Pi$-algebra are $\{1\}$ and the total algebra. Finally, our completeness result is the following.

Theorem 2 (*Completeness.*) *The logic ŁΠ is complete w.r.t. the class of linearly ordered ŁΠ-algebras. That is, a formula φ is provable in ŁΠ if it is an* **A***-tautology for each linearly ordered ŁΠ-algebra* **A**.

Proof: One direction is soundness. For the other direction, if φ is an **A**-tautology for each linearly-ordered ŁΠ algebra **A** then, by the above decomposition theorem, it is also a tautology for each ŁΠ-algebra, in particular, for the ŁΠ-algebra of the classes of provable equivalent formulas, that is, the logic ŁΠ proves $\varphi \equiv \overline{1}$, that is, Ł$\Pi$ proves φ.

It should be noticed however that this completeness result is only partial, in the sense that ŁΠ has been shown complete with respect to all linearly ordered ŁΠ-algebras and not with respect to the standard algebra defined on $[0,1]$ by means of Łukasiewicz and product t-norms. During the refereeing process a standard completeness for LΠ Logic have been proved (see [6]). Nonetheless, both the representation theorem for l.o. LΠ-algebras as the unit interval of a special class of l.o. rings, proved in the next section, and the Alsina's result referred in the Conclusions section, have been essential steps towards the standard completeness result given in [6].

3 On linearly ordered ŁΠ-algebras

It is known (see for example [4]) that it is possible to embed a linearly ordered MV-algebra into a linearly ordered abelian group (l.o.a.g.). In particular, given an ŁΠ-algebra $\mathbf{A} = \langle A, *, \Rightarrow_L, \odot, \Rightarrow_\Pi, \cup, \cap, 0, 1\rangle$, consider its MV reduct $\mathbf{A}_{MV} = \langle A, *, \Rightarrow_L, \cup, \cap, 0, 1\rangle$ and the l.o.a.g $\mathbf{G_A} = \langle G_A, +, -, 0_G, \leq_G\rangle$ where $G_A = \{(n,x) \mid n \in Z, x \in A, x \neq 1\}$, $0_G = (0,0)$ and

$$(n,x) + (m,y) = \begin{cases} (n+m, x \oplus y), & \text{if } x \oplus y < 1 \\ (n+m+1, x * y), & \text{if } x \oplus y = 1 \end{cases}$$

$$-(n,x) = \begin{cases} (-n, 0), & \text{if } x = 0 \\ (-(n+1), \neg_L x), & \text{if } 0 < x < 1 \end{cases}$$

$$(n,x) \leq_G (m,y) \text{ if } \quad n < m \text{ or } n = m \text{ and } x \leq y.$$

It can be shown that $\mathbf{G_A}$ is a l.o.a.g. with neutral element (0,0) and strong unit (1,0) and that the MV-algebra $\mathbf{A}_{MV}$ is isomorphic to the interval $[(0,0),(1,0)] = \{(n,x) \in G_A \mid (0,0) \leq_G (n,x) \leq_G (1,0)\}$, identifying $(0,x)$ with x and (1,0) with 1.

On the other hand it is possible to define a product operation $\times$ on G_A, extension of the product $\odot$ of the algebra A. The product is defined as follows:

$$(n,x) \times (m,y) = (nm, x \odot y) + m(0,x) + n(0,y),$$

where $m(0,x)$ means the sum of $(0,x)$, m times. It is clear that this product is commutative, $(0,0)$ is absorbent and $(1,0)$ is the unit element. Moreover, it is possible to prove that $\times$, together with $+$, endows G_A with an structure of linearly ordered abelian ring.

Proposition 3 *With the above definitions, the algebraic structure*

$$\mathbf{R_A} = \langle G_A, +, -, \times, (0,0), (1,0), \leq_G \rangle$$

is a linearly ordered abelian ring with unit $(1,0)$ *and with cancellation law for* $\times$.

The proof is given in Annex 2. Therefore, G_A can be embedded in a field of pairs of elements a/b of G_A in the same way the ring of integers can be embedded into the field of rational numbers. Of course the initial algebra is isomorphic to the subalgebra of elements of the type $p/1$, where $p = (0,x)$ for some $x \in A - \{1\}$ or $x = (1,0)$. But, moreover, the converse of Proposition 3 is also true.

Proposition 4 *Let* $(A, +, \times, 0, 1)$ *be a commutative linearly ordered ring with unit 1 and satisfying the cancellation law for* $\times$. *If for each* $x \neq 0$ *the mapping* $f_x : [0,1] \to [0,x]$, *defined as* $f_x(y) = x \times y$, *is onto, then* $[0,1]_A = \{x \in A \mid 0 \leq x \leq 1\}$ *is an ŁΠ algebra with the operations*

$$x * y = \max(0, x + y + (-1))$$
$$x \odot y = x \times y$$

and the corresponding residuated implications.

The proof of this proposition is an easy checking. Condition of f_x being onto is needed to guarantee the existence of the residuum of the product operation. If we take a field instead of a ring this condition is always satisfied due to the existence of inverse elements w.r.t. the product.

4 Conclusions

In this paper we have dealt with an axiomatic approach to a logical system containing Łukasiewicz, product and Gödel logics as sublogics and with its corresponding algebraic structure. We have also generalized existing results for linear MV and product algebras establishing that they can be identified with the interval [0,1] and the negative part of a linearly ordered abelian group respectively. In our case, linear ordered ŁΠ-algebras are the interval [0,1] of a linear ordered abelian ring satisfying the cancellation law. On the other hand the ŁΠ logic, axiomatically defined in section 2, has been shown to be sound and complete w.r.t. linearly ordered ŁΠ-algebras. A step towards the standard completeness is the following Alsina's result [1]: if S is a continuous t-conorm, T is a continuous t-norm and N is a strong negation, the general solution of the functional equation

$$S(T(x,y), T(x, N(y))) = x$$

is given by the following expressions:

$$
\begin{aligned}
S(x,y) &= g^{[-1]}(g(x)+g(y)) \\
T(x,y) &= g^{[-1]}(g(x)\cdot g(y)) \\
N(x) &= g^{[-1]}(1-g(x)),
\end{aligned}
$$

where g is a strictly increasing function $g:[0,1]\to R^+$ such that $g(0)=0$ and $g(1)=1$, and $g^{[-1]}$ is the pseudo-inverse function of g, that is, up to an isomorphism, S is the Łukasiewicz t-conorm, T is the product t-norm and N is the strong Łukasiewicz negation. And we have shown that the above functional equation is verified in any ŁΠ-algebra (see (iii) of Lemma 1). Thus the only ŁΠ-algebra on $[0,1]$ with the natural order (except to isomorphism) is the one defined by Łukasiewicz and product t-norms, together with their corresponding residuated implications and negations.
As we have noticed at the end of the introduction, after the paper was presented at ESTYLF'98 conference, standard completeness result for ŁΠ Logic was obtained (See [6]). This means that a formula φ is provable in ŁΠ if and only if φ is a tautology of the standard ŁΠ-algebra.

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ANNEX 1: Background on some systems of fuzzy logics

Here we summarize some important notions and facts from some systems of propositional fuzzy logics that are used in the paper.

The basic fuzzy logic BL and BL-algebras

The language of the basic logic BL [7] is built in the usual way from a set of propositional variables, a conjunction &, an implication $\rightarrow$ and the constant $\overline{0}$. Further connectives are defined as follows:

$\varphi \wedge \psi$ is $\varphi \& (\varphi \rightarrow \psi)$,
$\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$,
$\neg \varphi$ is $\varphi \rightarrow \bar{0}$,
$\varphi \equiv \psi$ is $(\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$.

The following formulas are the *axioms* of BL:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $(\varphi \& \psi) \rightarrow \varphi$
(A3) $(\varphi \& \psi) \rightarrow (\psi \& \varphi)$
(A4) $(\varphi \& (\varphi \rightarrow \psi) \rightarrow (\psi \& (\psi \rightarrow \varphi))$
(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$
(A5b) $((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A7) $\bar{0} \rightarrow \varphi$

The *deduction rule* of BL is modus ponens.

If one takes a continuous t-norm $*$ for the truth function of & and the corresponding residuum[1] $\Rightarrow$ for the truth function of $\rightarrow$ (and evaluating $\overline{0}$ by 0) then all the axioms of BL become 1-tautologies (have identically the truth value 1). And since modus ponens preserves 1-tautologies all formulas provable in BL are 1-tautologies.

It has been shown [7] that the well-known Łukasiewicz logic, denoted Ł, is the extension of BL by the axiom

(Ł) $\neg\neg\varphi \rightarrow \varphi$,

and Gödel logic, denoted G, is the extension of BL by the axiom

(G) $\varphi \rightarrow (\varphi \& \varphi)$.

[1] The residuum $\Rightarrow$ is the binary function on [0, 1] defined as $x \Rightarrow y = \sup\{z \in [0,1] \mid x * z \leq y\}$.

Finally, product logic, denoted Π, is just the extension of BL by the following two axioms:

(Π1) $\neg\neg\chi \to (((\varphi\&\chi) \to (\psi\&\chi)) \to (\varphi \to \psi))$,

(Π2) $\varphi \wedge \neg\varphi \to \bar{0}$.

A *BL-algebra* is an algebra $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow, 0, 1)$ with four binary operations and two constants such that

(i) $(L, \cap, \cup, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leq$),

(ii) $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e. $*$ is commutative, associative and $1 * x = x$ for all x,

(iii) the following conditions hold:

(1) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z.

(2) $x \cap y = x * (x \Rightarrow y)$

(3) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

Thus, in other words, a BL-algebra is a *residuated lattice* satisfying (2) and (3). The class of all BL-algebras is a variety. Moreover, each BL-algebra can be decomposed as a subdirect product of linearly ordered BL-algebras.

Defining $\neg x = x \Rightarrow 0$, it turns out that MV-*algebras* are BL-algebras satisfying $\neg\neg x = x$, G-*algebras* are BL-algebras satisfying $x * x = x$, and finally, *product algebras* are BL-algebras satisfying

$$x \cap \neg x = 0$$
$$\neg\neg z \Rightarrow ((x * z = y * z) \Rightarrow x = y) = 1.$$

The logic BL is sound with respect to **L**-tautologies: if φ is provable in BL then φ is an **L**-tautology for each BL-algebra **L**.

Theorem 3 *BL is complete, i.e. for each formula φ the following three conditions are equivalent:*

(i) φ is provable in BL,

(ii) for each BL-algebra **L**, *φ is an* **L**- *tautology,*

(iii) for each linearly ordered BL-algebra **L**, *φ is an* **L**-*tautology.*

This theorem also holds if we replace BL by a *schematic extension*[2] $\mathcal{C}$ of BL, and BL-algebras by the corresponding $\mathcal{C}$-algebras (BL-algebras in which all axioms of $\mathcal{C}$ are tautologies). There is also *strong completeness* for provability in theories over BL. Moreover, it has recently been shown in [3] the completeness of BL w.r.t. to the tautologies of BL-algebras in the real unit interval [0, 1], which are exactly the BL-algebras defined by continous t-norms and their residua. This was already conjectured in [7]. For completeness theorems of the three main extensions of BL (Łukasiewicz, Gödel and product logics) see [7].

[2] A calculus which results from BL by adding some axiom schemata.

Extended basic fuzzy logics with Δ and Δ-algebras

Now we expand the language of BL by a new unary (projection) connective Δ whose truth function (denoted also by Δ) is defined as follows:

$$\Delta(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$

The *axioms* of the extended basic logic BL_Δ (first formulated by Baaz in [2]) are those of BL plus:

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$
(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
(Δ3) $\Delta\varphi \to \varphi$
(Δ4) $\Delta\varphi \to \Delta\Delta\varphi$
(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

Deduction rules of BL_Δ are modus ponens and *generalization*: from φ derive $\Delta\varphi$.

A Δ-algebra is a structure $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow, 0, 1, \Delta)$ which is a BL-algebra expanded by an unary operation Δ satisfying the following conditions:

$\Delta x \cup -\Delta x = 1$
$\Delta(x \cup y) \leq \Delta x \cup \Delta y$
$\Delta x \leq x$
$\Delta x \leq \Delta\Delta x$
$(\Delta x) * (\Delta(x \Rightarrow y)) \leq \Delta y$
$\Delta 1 = 1$

The notions of **L**-evaluation and **L**-tautology easily generalize to BL_Δ and Δ-algebras. The decomposition of any BL_Δ algebra as a subdiredt product of linearly ordered ones also holds. Notice that in linearly ordered Δ-algebras we have that $\Delta 1 = 1$ and $\Delta(a) = 0$, for $a \neq 1$. Then the above completeness theorem for BL extends to BL_Δ as follows.

Theorem 4 *BL_Δ is complete, i.e. for each formula φ the following three things are equivalent:*
(i) φ is provable in BL_Δ,
*(ii) for each linearly ordered Δ-algebra **L**, φ is an **L**-tautology;*
*(iii) for each Δ-algebra **L**, φ is an **L**- tautology.*

A *strong completeness* result for provability in theories over BL_Δ is also given in [7].

Moreover, each of the three dintinguished logics, Łukasiewicz, product and Gödel logics, can be added the Δ connective, together with its axioms (Δ1)−−(Δ5), leading to the so denoted logics $Ł_\Delta$, Π_Δ and G_Δ, which are complete w.r.t. their corresponding algebras, i.e. MV_Δ-algebras, Π_Δ-algebras and G_Δ-algebras. See [7] for further details.

The Basic Strict Fuzzy Logic SBL and SBL-algebras

The strict basic logic SBL [5] is an extension of BL logic for which the linearly ordered BL-algebras that satisfy SBL axioms are those having Gödel negation.

The axioms of the basic strict fuzzy logic SBL are those of BL plus the following axiom:

(STR) $(\varphi\&\psi \to 0) \to ((\varphi \to 0) \vee (\psi \to 0))$.

Notice that (STR) is a theorem in both Product and Gödel logics. Moreover, SBL proves $\varphi \wedge \neg\varphi \to 0$.

Definition 4 *A SBL-algebra is a BL-algebra* $(L, \cap, \cup, *, \Rightarrow, 0, 1)$ *verifying this further condition:*

$$(x * y) \Rightarrow 0 \;=\; (x \Rightarrow 0) \cup (y \Rightarrow 0).$$

Examples of SBL-algebras are the algebras in the real unit interval $([0,1], \max, \min, *, \Rightarrow, 0, 1)$, where $*$ is a t-norm without non-trivial zero divisors and $\Rightarrow$ its corresponding residuum, and the quotient algebra SBL/$\equiv$ of provable equivalent formulas.

In linearly ordered SBL-algebras, the above condition turns into

$$x * y = 0 \text{ iff } x = 0 \text{ or } y = 0.$$

Moreover, this condition identifies linear SBL-algebras with algebras which have Gödel negation.

Theorem 5 (Completeness.) *The logic SBL is complete w.r.t. the class of linearly ordered SBL-algebras.*

In [3] it is also shown the completeness of SBL w.r.t. the class of SBL-algebras on [0, 1].

Strict basic fuzzy logics extended with an involutive negation

Now we extend SBL with a unary connective $\sim$ (See [5]). The *semantics* of $\sim$ is an arbitrary strong negation function $n : [0,1] \to [0,1]$, which is nothing but a decreasing involution, i.e. $n(n(x)) = x$ and $n(x) \leq n(y)$ whenever $x \geq y$. With both negations, $\neg$ and $\sim$, the projection connective Δ is now definable: $\Delta\varphi$ is $\neg\sim\varphi$.

Definition 5 *Axioms of* $\mathrm{SBL}_\sim$ *are those of SBL plus*

(~1) $(\sim\sim\varphi) \equiv \varphi$

(~2) $\neg\varphi \to \sim\varphi$

(~3) $\Delta(\varphi \to \psi) \to \Delta(\sim\psi \to \sim\varphi)$

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$

(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$

(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

where $\Delta\varphi$ *is* $\neg\sim\varphi$. *Deduction rules of* $\mathrm{SBL}_\sim$ *are those of* BL_Δ, *that is, modus ponens and necessitation for* Δ.

Definition 6 *A* SBL$_\sim$*-algebra is a structure* $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow, \sim, 0, 1)$ *which is a SBL-algebra expanded with a unary operation* $\sim$ *satisfying the following conditions:*

$(A_\sim 1)$ $\sim\sim x = x$
$(A_\sim 2)$ $\neg x \leq \sim x$
$(A_\sim 3)$ $\Delta(x \Rightarrow y) = \Delta(\sim y \Rightarrow \sim x)$
$(A_\sim 4)$ $\Delta x \cup \neg\Delta x = 1$
$(A_\sim 5)$ $\Delta(x \cup y) \leq \Delta x \cup \Delta y$
$(A_\sim 6)$ $\Delta x * (\Delta(x \Rightarrow y)) \leq \Delta y$

where $\neg x = x \Rightarrow 0$ *and* $\Delta x = (\sim x \Rightarrow 0)$.

The decomposition of any SBL$_\sim$-algebra as a subdirect product of linearly ordered ones also holds.

Theorem 6 SBL$_\sim$ *is complete w.r.t. the class of* SBL$_\sim$ *algebras.*

Remarkable extensions of SBL$_\sim$ are the product logic with involution, $\Pi_\sim$, and Gödel's logic with involution G$_\sim$, obtained by adding the corresponding axioms to SBL$_\sim$, that is, axioms (Π1) and (Π2) for the product logic, and axiom (G) for Gödel logic. For these particular extensions there are stronger completeness theorems:

- $\Pi_\sim$ is complete w.r.t. the semi-standard SBL$_\sim$ algebras $([0,1], \min, \max, \cdot, \Rightarrow_\pi, n, 0, 1)$, where $\cdot$ is usual product, $\Rightarrow_\pi$ is the residuum of product (Goguen's implication) and n is a strong negation in [0, 1].

- G$_\sim$ is complete w.r.t. to the standard G$_\sim$-algebra $([0,1], \min, \max, 1-x, 0, 1)$.

ANNEX 2: Proof of Proposition 3

Proposition 3 With the above definitions,
$\mathbf{R_A} = \langle G_A, +, -, \times, (0,0), (1,0), \leq_G \rangle$ is a linearly ordered abelian ring with unit $(1,0)$ and with cancellation laws for $\times$.

We only need to prove the distributivity and the associativity and cancellation laws for the product.

Proof of the distributivity law. We must prove that $(m,z) \times ((n,x) + (k,y)) = ((m,z) \times (n,x)) + ((m,z) \times (k,y))$ and the proof needs to study different cases.

A *Proof for positive pairs*: $n, k, m \geq 0$. In turn, the proof is divided in different sub-cases:

1. If $x * y = 0$ the law is given by (3) of definition 2.
2. If $x \oplus y = 1$, and $(x \odot z) * (y \odot z) = 0$, then remembering condition (4) of definition 2 we have
$(m,z) \times ((n,x) + (k,y)) = (m,z) \times (n + k + 1, x * y) =$

$= (m(n+k+1), ((x*y) \odot z) \oplus z) + m(1, x*y) + (n+k)(0,z)$.
On the other side,$((m,z) \times (n,x)) + ((m,z) \times (k,y)) =$
$(mn, z \odot x) + m(0,x) + n(0,z) + (mk, z \odot y) + m(0,y) + k(0,z) =$
$((m(n+k), (z \odot x) \oplus (z \odot y)) + m(1, x*y) + (n+k)(0,z) =$
$((m(n+k+1), (z \odot x) \oplus (z \odot y)) + m(0, x*y) + (n+k)(0,z)$
which taking into account the condition (4) of definition 2 proves the equality of the two expressions.

3. If $(z \odot x) \oplus (z \odot y) = 1$, then remembering condition (4) of definition 2 we have:
$(m,z) \times ((n,x) + (k,y)) = (m,z) \times (n+k+1, x*y) =$
$(m(n+k+1), ((x*y) \odot z)) + m(1, x*y) + (n+k+1)(0,z) =$
$(m(n+k+1)+1, ((x*y) \odot z) * z)$.
On he other hand,$((m,z) \times (n,x)) + ((m,z) \times (k,y)) =$
$(mn, z \odot x) + m(0,x) + n(0,z) + (mk, z \odot y) + m(0,y) + k(0,z) =$
$((m(n+k)+1, (z \odot x) * (z \odot y)) + m(1, x*y) + (n+k)(0,z) =$
$((m(n+k+1)+1, (z \odot x) * (z \odot y)) + m(0, x*y) + (n+k)(0,z)$
which proves the equality using condition (4) of definition 2.

B *Proof for negative pairs*:

1. First we prove the sign rule, that is, $(-(m,z)) \times (n,x) = -[(m,z) \times (n,x)]$. The proof is as follows: $(-(m,z)) \times (n,x) = (-(m+1), \neg_L z) \times (n,x) = (-(m+1)n, \neg_L z \odot x)$ which is equal to $-(mn, z \odot x)$ taking into account (iii) of Lemma 1. As a consequence it also holds that $(-(m,z)) \times (-(n,x)) = (m,z) \times (n,x)$.

2. We will prove directly the following case:
$(m,z) \times ((-(n,x)) + (k,y)) =$
$((m,z) \times (-(n,x))) + ((m,z) \times (k,y))$. The proof is by cases (suppose $k > n$):

   - If $\neg x * y = 0$, then:
   $(m,z) \times ((-(n,x)) + (k,y)) = (m,z) \times (k-(n+1), \neg x \oplus y)$
   $= (m((k-(n+1)), z \odot (\neg x \oplus y)) + m(0, \neg x \oplus y) + (k-(n+1))(0,z)$. (1)
   On the other hand,
   $((m,z) \times (-(n,x))) + ((m,z) \times (k,y)) = (-m(n+1), z \odot \neg x) + m(0, \neg x) + (n+1)(-1, \neg z) + (mk, z \odot y) + m(0,y) + k(0,z) =$
   (taking into account that $(-1, \neg z) + (0,z) = (0,0)$)
   $= (m(k-(n+1), (z \odot \neg x) \oplus (z \odot y)) + (k-(n+1))(0,z)$ (2)
   Of course, by condition (4), the equality is easily proved.
   - If $\neg x \oplus y = 1$ and $(z \odot \neg x) \oplus (z \odot y) \leq 1$.
   Then $(m,z) \times ((-(n,x)) + (k,y)) = (m,z) \times (k-n, \neg x * y) =$
   $= (m(k-n), z \odot (\neg x * y)) + m(0, \neg x * y) + (k-n)(0,z)$.
   On the other hand,
   $((m,z) \times (-(n,x))) + ((m,z) \times (k,y)) = (-m(n+1), z \odot \neg x) + m(0, \neg x) + (n+1)(-1, \neg z) + (mk, z \odot y) + m(0,y) + k(0,z) =$

$= (m(k-n)-m, (z \odot \neg x) \oplus (z \odot y)) + m(1, \neg x * y) + (n+1)(-1, \neg z) + k(0, z) =$
taking into account that $(-1, \neg z) + (0, z) = (0, 0)$
$= (m(k-n), (z \odot \neg x) \oplus (z \odot y)) + m(0, \neg x * y) + (k-(n+1))(0, z)$.
Thus an easy computation shows the desired equality using (4) of definition 2.

- If $\neg x \oplus y = (z \odot \neg x) \oplus (z \odot y) = 1$, then:
$(m, z) \times ((-(n, x)) + (k, y)) = (m, z) \times (k-n, \neg x * y) =$
$= (m(k-n), z \odot (\neg x * y)) + m(0, \neg x * y) + (k-n)(0, z)$.
On the other hand,
$((m, z) \times (-(n, x))) + ((m, z) \times (k, y)) = (-m(n+1), z \odot \neg x) + m(0, \neg x) + (n+1)(-1, \neg z) + (mk, z \odot y) + m(0, y) + k(0, z) =$
$= (m(k-(n+1)) + 1, (z \odot \neg x) * (z \odot y)) + m(1, \neg x * y) + (n+1)(-1, \neg z) + k(0, z) =$
taking into account that $(-1, \neg z) + (0, z) = (0, 0)$
$= (m(k-n)+1, (z \odot \neg x) * (z \odot y)) + m(0, \neg x * y) + (k-(n+1))(0, z)$
An easy computation shows the desired equality using (4) of definition 2.

Thus the proof is completed.

C All other cases can be proved by A and B.2 using the sign rule B.1.

Proof of the associativity law for $\times$. From distributivity, associativity of $\times$ can be easily checked.

Proof of the cancellation law for $\times$. The cancellation law is a consequence of the preservation of strict inequality by products which, in turn, is a direct and obvious consequence of the definition of $\times$ in G_A.