Mathware & Soft Computing 6 (1999) 155-161

The Limits of Fuzzy Logic

J.L. Castro*
Dept. Ciencias de la Computación e Inteligencia Artificial.
Universidad de Granada. ETSI Informática.
18071 Granada. España.
castro@decsai.ugr.es

Abstract

In this paper we try to answer to the following questions: What can be made by applying fuzzy logic? and What can not be made by applying fuzzy logic? The question will be analyzed from both theoretical and applied point of view. A (partial) answer will be given for three topics: a) as calculus procedure b) as reasoning mechanism and c) as engineering tool.
Keywords: Fuzzy Logic. Fuzzy Rule Based Systems..

1 Fuzzy Logic as calculus procedure

In this section we analyze fuzzy logic as a calculus procedure. Usual classical logic based systems are reasoning systems which derivate propositional conclusions from some propositional premises. On the contrary, most of actual Fuzzy Logic Based systems are systems calculating a real valued function instead of a reasoning system. That is, they have real values as inputs and they obtain real values outputs. In fact, in many cases the inference is made with propositions like X=a, where a is real value instead of a fuzzy or symbolic value, and a defuzzification procedure is added to fuzzy inference in order to obtain a numerical value.

In this way, we study the effectiveness of fuzzy logic as calculus procedure by analyzing which are the functions that can be calculated such a system. We must bear in mind that with this procedure it is possible to calculate two kind of functions: continuous variable functions, and discrete variable functions.

We start this analysis with a reflection about the account of functions that can be approximates:

Proposition. If we restrict us to use:
a) trapezoidal membership functions

*This paper was supported by the spanish CICYT under the project TIC98-0996

b) a finite number of fuzzy rules,

then the number of functions that can be calculates by means of fuzzy rule based systems has cardinal $\aleph_1$.

Proof. It is enough with observe that: if a set of input variables, and the labels for any variable are fixed, then the set of functions calculated by this kind of fuzzy rules based systems is a finite dimensional vectorial space, where the dimension is the product of the number of labels of each variable (see [castro97]). Thus, since a trapezoidal fuzzy set is defined by 4 real numbers, we obtain the result.

Bear in mind that the set of real functions with real variables has, even restricted to continuous functions, cardinal $\aleph_2$, we have that it is not possible to calculate with fuzzy-rule-based-systems all continuous real functions with real variables functions.

On the other hand, we can hope that they can approximate every continuous function to any accuracy degree over every compact set, in the same way that any continuous function can be approximate over every compact to any accuracy degree by a polynomial function. This property is known as "Universal Approximation" property:

Definition. It is said that a family of real functions with real variables satisfy the universal approximation property if it is hold the following condition: for every continuous function defined on a compact set, and for every $\epsilon > 0$, there exits a function in this family such that the distance between both functions is lower than ϵ uniformly over all the compact set.

The following theorem ([1]) indicates that the family of fuzzy-rule-based-systems verifies the universal approximation property, and at least from a theoretical point of view, for every accuracy degree they can calculate an approximate function for every continuous function.

Theorem The family of all functions which can be calculated by a fuzzy-rule-based-system verifies the universal approximation property.

This result about the theoretical effectiveness of fuzzy-rule-based-systems as calculus procedure is interesting, but it is not so much strong from a practical point of view. For instance, there are other families of functions verifying the universal approximation property: polynomials, piecewise lineal functions, splines, neural networks, etc... And it is clear that this list include families which are not operative in practice, because they can need too much resources (time, space, money, ...) when they are applied.

In this way it will be interesting to analyze from a practical point of view the effectiveness of fuzzy logic as calculus procedure. Obviously, the main problem of the application of fuzzy-rule-based-systems is the big number of rules that could be necessary to approximate the function, and the computational complexity of the algorithms searching the rules. We will only consider the first question. Concretely, we will give an upper bound of error obtained when we approximate a function by means of a fuzzy rule based system. This bound will depend on the number of rules, and thus, we will have an upper bound of the number of rules that we need to obtain a fixed error level ([3]).

Proposition. Let $\mathcal{F}$ be the family of all fuzzy rule based with only a antecedent variable and with triangular membership functions. For every continuous function f on a compact K, there exists a function F in $\mathcal{F}$ verifying:

a) If $f \in C^2(K)$, then

$$||f - F||_{[a_i,a_{i+1}]} \leq \frac{||f''||_{[a_i,a_{i+1}]}}{4}(a_{i+1} - a_i)^2$$

$$||f - F|| \leq \frac{||f''||}{4}h^2$$

being $h = \max_i a_i - a_{i-1}$ and a_i the points of the peaks of the triangles..

b) If $f \in C^1(K)$, then

$$||f - F||_{[a_i,a_{i+1}]} \leq \frac{27||f'||_{[a_i,a_{i+1}]}}{2}(a_{i+1} - a_i)$$

$$||f - F|| \leq \frac{27||f''||}{2}h$$

.

c) If $f \in C^0(K)$, then

$$||f - F||_{[a_i,a_{i+1}]} \leq 54w_{[a_i,a_{i+1}]}(f, \frac{a_{i+1} - a_i}{4})$$

$$||f - F|| \leq 54w_{U_v}(f, \frac{h}{4})$$

then $w_I(f, s) = sup\{|f(x + \epsilon) - f(x)| : x, x + \epsilon \in I, \epsilon \leq s\}$.

Proposition. Let $\mathcal{F}$ be the family of all fuzzy rule based with two antecedent variables and with triangular membership functions. For every continuous function f on a compact K, there exists a function F in $\mathcal{F}$ verifying:

i) Local error formula: being $I = [a_i, a_{i+1}]$ and $J = [b_j, b_{j+1}]$,

$$||f - F||_{I \times J} \leq \beta_{I \times J} \max\{(a_{i+1} - a_i)^2, (b_{j+1} - b_j)^2\}$$

where

$$\beta_{I,J} = \max\{A, B\}$$

and

$$A = \frac{||D^2{}_w f||_{I \times J}}{8}$$

and

$$B = \frac{1}{8}[||D^2{}_v f||_{I \times J} + \frac{||D^2{}_w D^2{}_v f||_{I \times J}}{8}](length(I))^2$$

ii) Error formula:

$$||f - F|| \leq \beta \max\{{h_x}^2, {h_y}^2\}$$

being $h_x = \max_i a_i - a_{i-1}$, $h_y = \max_i b_j - b_{j-1}$ and
$\beta = \max\{\frac{||D^2{}_w f||}{8}, \frac{1}{8}[||D^2{}_v f|| + \frac{||D^2{}_w D^2{}_v f||}{8}]{h_x}^2\}$

These error formulas are the same as those of piecewise Lagrange interpolation [5], in fact the proof is obtained by proving that this kind of fuzzy systems include this kind of interpolation. Moreover, it can be proved that if we use cubic membership functions, then fuzzy rule based systems include cubic spline interpolation.

Conclusion From a theoretical point of view, fuzzy systems can approximate over compacts any continuous function to any accuracy degree. Thus, fuzzy systems are so powerful as interpolation methods in approximation problems, but with the advantage that it allows a linguistic interpretation of the calculations.

2 Fuzzy Logic as Reasoning Mechanism

In this section we will analyze the effectiveness of fuzzy logic as reasoning mechanism.. First, we show that fuzzy logic include, as a particular case, classical logic, and thus, it will have at least the same power.

Proposition. Given a set of propositions of propositional classical logic P, there exists a equivalent set of fuzzy propositions P', in the sense that if something is derivated from P using the propositional logic inference, then this one can be derived from P' using the fuzzy logic inference.

Proof. P' will be the result of changing each proposition q in P by the proposition "q is true", being true the fuzzy set on [0,1] whose membership function is equal to 1 in the number 1 and it is equal to 0 in the other cases. Then, q can be derivated from P in the propositional logic if and only if sera "q is true" can be derivated from P' in the fuzzy logic.

Now, we will study the case of first order logic. We will prove a similar result:

Proposition. Given a set of propositions of first order logic P, there exists a equivalent set of fuzzy propositions P', in the sense that if something is derivated from P using the first order inference, then this one can be derived from P' using the fuzzy logic inference.

Proof. In order to P', first we assign the obvious membership function to each universal quantifier. Then, we assign for every relation symbol the respective membership function. Finally, we will paste the above proof to this case.

Then, we have that fuzzy logic is an extension of classical logic. Now we will analyze its advantages.. These ones are defined by two important facts:

a) Fuzzy logic allows deal with vague predicates and vague terms, and it includes the capacity of representing a great account of knowledge in a compact way, and

b) Fuzzy logic allows to implement both exact and approximate reasoning.

Because of first property, fuzzy logic is a very general knowledge representation model with a very interesting property, the capacity of representing a great account of information with a small number of propositions. On my view, it is one of the most important powers of fuzzy logic.

For instance, if we consider a set of rules with the form:

$$R_j\text{: If } v_1 \text{ is } A_1^j \text{ [and } v_2 \text{ is } A_2^j\text{], then } y \text{ is } B^j \ ,$$

with only a few rules we can describe a complex relation $y = f(v_1, v_2)$, (it will be only a rough approximation) ordering the values which can take according to a possibility distribution. This one is made in a conservative way, since many values are possible and with high possibility degrees (see figure 1). On the other hand, the model has inconveniences from an applied point of view, since if it is not restricted to only simple fuzzy rules, then the time complexity of the inference algorithms is hard. Moreover, the vagueness grows with every inference, and it is a serious limit for chaining inferences.

Second property, the possibility of implement an approximate reasoning, is without doubt one of the biggest appeal of fuzzy logic. Nevertheless, we have not a good application yet. It is due to the complexity problem previously described. I do not know any expert system which really use fuzzy logic as reasoning mechanism. It is in this topic where fuzzy logic has a great future and where, paradoxically, it has been no so developed.

3 Fuzzy Logic as a tool for engineering

On the application of fuzzy logic to engineering, our comments will be necessarily a review of the previous sections. It is a tool so powerful as interpolation, but moreover it has an important advantage: the vague and linguistic human knowledge can be modeled by means of this tool.

This one make it appropriate for a great number of problems, specially those mathematically ill-defined those without a good "classical" solution. Even if that is not the case, it can be an adequate tool because of its easy management, its low cost and its acceptable average results. From my point of view, the reason is the capacity of compressing information in a few rules with a proved efficacy [4].

Among its limits in this topic, we have the restriction (to avoid complexity algorithm) to fuzzy rules based systems, and without chaining inferences. It is a serious limit for its capacity of acquiring linguistic information. Moreover, we need tuning the rules because of processing of compressing-decompressing..

4 Conclusion

In summary, fuzzy logic is a powerful tool for functional calculus (control, approximation, etc. ...). In fact, it is an extension of classical interpolation. Moreover

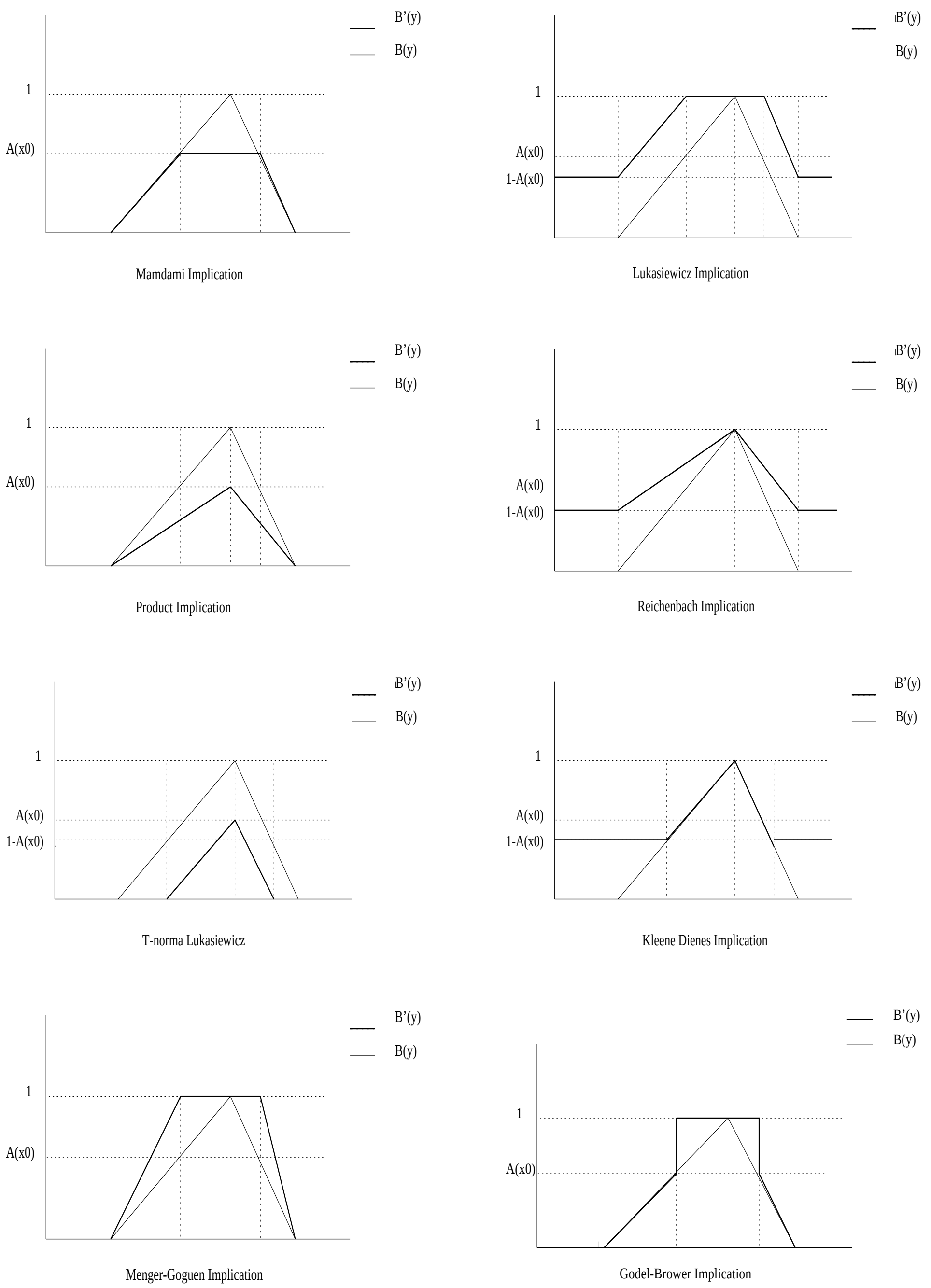

Figure 1: Inference with some implication functions

it has an easy programming since it allows represent linguistic information, and we have not need a deep mathematical knowledge of the function for calculating. Even it can be applied in cases where the function is not well-defined. The main limit in this topic is that the number of rules grows exponentially inverse with the accuracy level.

As mechanism model, from its beginning fuzzy logic has have many expectative, because of its capacity of modeling linguistic knowledge. Nevertheless, it still is only a challenger because of their complexity and rule-chaining problems.

As engineering tool, it is enough to adjoint the before comments.

References

[1] Castro, J.L., 1995. Fuzzy Logic Controllers are Universal Approximators, *IEEE Trans. on Syst, Man. and Cybern. 25-4*, pp. 629-635.

[2] Castro, J.L., Castro, J.J., Zurita, J.M., 1997. The influence of Implication functions in Fuzzy Logic Controllers with Defuzzification First and Aggregation After, *Proceedings de IFSA97.*

[3] Castro, J.L., 1997. How many rules are necessary to get a "good" fuzzy controller for a control problem?, *Proceedings de Fuzz-IEEE97.*

[4] Hirota, K, 1993. *Industrial Applications of Fuzzy Technologies*, Springer Verlag.

[5] Prenter, P.M., 1975. Splines and Variational Methods, John Wileys & Sons.