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BL-algebras of Basic Fuzzy Logic

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Abstract

BL-algebras [7] rise as Lindenbaum algebras from certain logical axioms familiar in fuzzy logic framework. BL-algebras are studied by means of deductive systems and co-annihilators. Duals of many theorems known to hold in MV-algebra theory remain valid for BL-algebras, too.
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1 Introduction

BL-algebras have been invented recently by Hájek [7] in order to provide an algebraic proof of the completeness theorem of a class of $[0, 1]$-valued logics familiar in fuzzy logic fremework. BL-algebras [1] rise as Lindenbaum algebras from certain logical axioms in a similar manner as MV-algebras (cf. [1], [2], [3], [4], [6], [9]) do from the axioms of Lukasiewicz logic. In fact, MV-algebras are BL-algebras. The converse, however, is not true. It follows from a result of Höhle [10] that BL-algebras with involutory complement are MV-algebras. In this study we start a similar study of BL-algebras as Belluce [1], [2], Chang [3], [4], Gluschankof [6], Hoo [9] and others have done in the theory of MV-algebras; there the basic tool is ideal theory while in BL-algebras, because of lack of a suitable algebraic addition, we have to deal with deductive systems. Moreover, in logic context deductive systems have a natural interpretation as sets of provable formulas. In MV-algebra theory, deductive systems and ideals are dual notions; there deductive systems are also called filters but, in order to avoid confusion, we prefer to talk about deductive systems. We introduce locally finite BL-algebras and prove that such algebras are MV-algebras. As one may expect, there is a one-to-one correspondence between deductive systems and congruence relations of a BL-algebra. We prove that a deductive system is maximal if, and only if the corresponding quotient algebra is a locally finite MV-algebra. This fact implies one of the main result of our study: semisimple MV-algebras are, in the sense of Chang [3] and Belluce [1], the only BL-algebras that are representable by a system of fuzzy subsets of a set. However, as proved by Hájek [7], all BL-algebras are representable by linear BL-algebras.

[1] The letters BL stand for basic logic

It remains an open problem to chracterize all linear BL-algebras. We introduce co-annihilators and prove some of their elementary properties; all these results will be an introduction for a future, more detailed analysis on BL-algebras.

2 Preliminaries

Recall from [1], [2], [3], [4], [6], [9] the definition and basic properties of an *MV-algebra* $A = \langle A, \odot, \oplus, ^*, \mathbf{0}, \mathbf{1}\rangle$.

Definition 1 *A* residuated lattice $L = \langle L, \leq, \wedge, \vee, \odot, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$ *is a lattice* L *containing the least element* $\mathbf{0}$ *and the largest element* $\mathbf{1}$, *and endowed with two binary operations* $\odot$ *(called* product*) and* $\rightarrow$ *(called* residuum*) such that* (i) $\odot$ *is associative, commutative and isotone and, for all elements* $x \in L$, $x \odot \mathbf{1} = x$, (ii) *for all* $x, y, z \in L$, *the Galois correspondence*

$$x \odot y \leq z \textit{ iff } x \leq y \rightarrow z$$

holds.

Residuated lattices are known also under other names, e.g. Höhle [10] calls them *integral, residuated, commutative l-monoids*. The following equations are valid in any residuated lattice L [13]:

$$x \odot \bigvee_{i \in \Gamma} y_i = \bigvee_{i \in \Gamma} (x \odot y_i), \tag{1}$$

$$\text{if } y \leq x \text{ then } x \rightarrow z \leq y \rightarrow z \text{ and } z \rightarrow y \leq z \rightarrow x, \tag{2}$$

$$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z, \tag{3}$$

$$x \leq y \text{ iff } x \rightarrow y = \mathbf{1}, \tag{4}$$

$$z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y), \tag{5}$$

$$(\bigvee_{i \in \Gamma} y_i) \rightarrow x = \bigwedge_{i \in \Gamma} (y_i \rightarrow x), \tag{6}$$

$$\mathbf{0} \rightarrow y = \mathbf{1} \text{ and } x = \mathbf{1} \rightarrow x. \tag{7}$$

We define $x^* = \bigvee\{y \in L \mid x \odot y = \mathbf{0}\}$, equivalently, $x^* = x \rightarrow \mathbf{0}$. Then

$$\mathbf{0}^* = \mathbf{1}, \mathbf{1}^* = \mathbf{0} \text{ and } x \leq x^{**}, x^* = x^{***}. \tag{8}$$

Moreover,

$$\bigvee_{i \in \Gamma} (y_i \rightarrow x) \leq (\bigwedge_{i \in \Gamma} y_i) \rightarrow x, \tag{9}$$

$$x \rightarrow \bigwedge_{i \in \Gamma} y_i = \bigwedge_{i \in \Gamma} (x \rightarrow y_i), \tag{10}$$

$$y \leq (y \rightarrow x) \rightarrow x, \tag{11}$$

where Γ is a finite or infinite index set and assuming, of course, that the corresponding infinite meets and joins exist in L. For sake of completeness, we recall that

$$x^* \wedge y^* = (x \vee y)^*. \tag{12}$$

Indeed, for all $x, y \in L$, $x, y \leq x \vee y$, thus $(x \vee y)^* \leq x^* \wedge y^*$, and $(x^* \wedge y^*) \odot (x \vee y)$ $= [(x^* \wedge y^*) \odot x] \vee [(x^* \wedge y^*) \odot y] \leq (x^* \odot x) \vee (y^* \odot y) = \mathbf{0}$, hence $x^* \wedge y^* \leq (x \vee y)^*$. Then, in particular, if $x \vee x^* = \mathbf{1}$,

$$x^* \wedge x = \mathbf{0}, \tag{13}$$

since $x^* \wedge x \leq x^* \wedge x^{**} = \mathbf{0}$.

Definition 2 *A residuated lattice L is called an* BL-algebra *if the following three identities hold for all $x, y \in L$:*

$$\begin{aligned} x \wedge y &= x \odot (x \to y), & (14)\\ x \vee y &= [(x \to y) \to y] \wedge [(y \to x) \to x], & (15)\\ (x \to y) \vee (y \to x) &= \mathbf{1}. & (16) \end{aligned}$$

In [7] it is shown that any *continuous t-norm* generates an BL-algebra, and that a linear residuated lattice is a BL-algebra iff (14) holds. The following three structures are main examples of BL-algebras on the real unit interval [11].

Example 1 Gödel structure: $x \odot y = \min\{x, y\}$, $x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{elsewhere.} \end{cases}$
It is well-known that $\min\{x, y\}$ is the greatest t-norm on $[0, 1]$.

Example 2 Product structure: $x \odot y = xy$, $x \to y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{elsewhere.} \end{cases}$
W.M. Faucett proved in [5] that any continuous t-norm with no idempotents, except $0, 1$ and no nilpotents i.e. non-zero elements x such that $x^n = 0$ for some n, is equivalent to $\odot$.

Example 3 Lukasiewicz structure: $x \odot y = \max\{0, x + y - 1\}$,

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ 1 - x + y & \text{elsewhere.} \end{cases}$$

As proved in [12], any continuous t-norm on $[0, 1]$ with no idempotents, except $0, 1$ and at least one nilpotent, is equivalent to $\odot$.

From [8] we learn that any BL-algebra defined on the real unit interval is a 'mixture' of the three above BL-algebras. Not all the residuated lattices, however, are BL-algebras. Indeed,

Example 4 For all $x, y \in [0, 1]$, define
$$x \odot y = \begin{cases} 0 & \text{if } x + y \leq \frac{1}{2} \\ x \wedge y & \text{elsewhere} \end{cases}, \quad x \to y = \begin{cases} 1 & \text{if } x \leq y \\ \max\ \{\frac{1}{2} - x, y\} & \text{elsewhere.} \end{cases}$$

Then we obtain a residuated lattice which is not an BL-algebra as (14) does not, in general, hold. An MV-algebra generates an BL-algebra, where the residuum is defined by $x \to y = x^* \oplus y$. Höhle [10] defined that a residuated lattice L is an MV-algebra if, and only if the additional condition

$$\text{for all } x, y \in L, x \vee y = (x \to y) \to y \tag{17}$$

is satisfied. Then the MV-operation $\oplus$ is given via $x \oplus y = x^* \to y$. Höhle [10] also showed that an equivalent condition with (17) is that the residuated lattice L fulfils (14) and the operation * is an involution, that is

$$\text{for all } x \in L, x = x^{**}. \tag{18}$$

We therefore have that an BL-algebra generates an MV-algebra if, and only if (18) holds.

Our basic observation on BL-algebras is the following

Proposition 1 *BL-algebras are distributive lattices.*

Proof. Let a, b, c be elements of an BL-algebra L. Then

$$\begin{aligned} a \wedge (b \vee c) &= (b \vee c) \odot [(b \vee c) \to a] \\ &= \{b \odot [(b \vee c) \to a]\} \vee \{c \odot [(b \vee c) \to a]\} \\ &\leq [b \odot (b \to a)] \vee [c \odot (c \to a)] \\ &= (b \wedge a) \vee (c \wedge a) \\ &= (a \wedge b) \vee (a \wedge c). \end{aligned}$$

The converse holds, too, since $(a \wedge b)$, $(a \wedge c) \leq a \wedge (b \vee c)$.□

3 Deductive systems of BL-algebras

Definition 3 *A* deductive system *D of an BL-algebra L* (ds, *in short*) *is a subset of L such that* (i) $\mathbf{1} \in D$, (ii) *if $a, a \to b \in D$, then $b \in D$.*

Hájek [7] defined a *filter* of an BL-algebra L to be a such non-void subset of L that (i) if $a, b \in F$ then $a \odot b \in F$ and (ii) if $a \in F, a \leq b$, then $b \in F$.

Proposition 2 *A subset D of an BL-algebra L is a ds of L if, and only if D is a filter of L.*

Proof. Let D be a ds. Then D is non-void since $\mathbf{1} \in D$. Moreover, if $a, b \in D$, then $\mathbf{1} = a \to [b \to (a \odot b)] \in D$, so $b \to (a \odot b) \in D$ and therefore also $(a \odot b) \in D$, and if $a \in D, a \leq b$, then $a \to b = \mathbf{1} \in D$, thus $b \in D$, hence D is a filter. Conversely, if D is a filter of L then there is an element x in D. Since $x \leq \mathbf{1}$, $\mathbf{1} \in D$. Assume $a, a \to b \in D$. Then $a \odot (a \to b) \in D$, and since $a \odot (a \to b) \leq b$, we have that $b \in D$. Consequently, D is a ds of L.□

Clearly $\{\mathbf{1}\}$ and L are deductive systems of L. If D is a *ds* of L, then for any $a \in L$ we have $a \in D$ if, and only if $a^n \in D$ for any natural number n. A *ds* D is called *proper* if $D \neq L$. It is easy to see that D is proper iff $\mathbf{0} \notin D$ iff there is no such element $a \in L$ that both $a \in D$ and $a^* \in D$. A proper deductive system D is called *prime* if $a \vee b \in D$ always implies $a \in D$ or $b \in D$. We can say that D is prime iff

$$\text{for all } a, b \in L,\ a \to b \in D \text{ or } b \to a \in D. \tag{19}$$

Indeed, by (16), $(a \to b) \vee (b \to a) \in D$ for any $a, b \in L$, thus if D is prime then either $(a \to b) \in D$ or $(b \to a) \in D$. Conversely, let (19) hold, D being a *ds*. Assume $a \vee b \in L$ and let, say, $(a \to b) \in D$. By (15), $a \vee b \leq (a \to b) \to b$, thus $(a \to b) \to b \in D$ and therefore $b \in D$. Similarly, $(b \to a) \in D$ implies $a \in D$. Thus D is prime. We also realize that any *ds* D of an BL-algebra L is a lattice filter of L; indeed, if $a, b \in D$, then $a \odot b \in D$ and $a \odot b \leq a \wedge b$, thus $a \wedge b \in D$ and conversely, if $a \wedge b \in D$, then $a, b \in D$ since $a \wedge b \leq a, b$.

Proposition 3 *If X is a non-void subset of an BL-algebra L, then*

$$\langle X \rangle = \{a \in L \mid x_1 \odot \cdots \odot x_n \leq a \textit{ for some } x_1, \cdots, x_n \in X\}$$

is a ds of L and $X \subseteq \langle X \rangle$.

Proof. Trivially $\mathbf{1} \in \langle X \rangle$ and if $a, a \to b \in \langle X \rangle$ then there are $x_1, \cdots, x_n \in X$, $y_1, \cdots, y_m \in X$ such that $x_1 \odot \cdots \odot x_n \leq a$, $y_1 \odot \cdots \odot y_m \leq a \to b$. Since

$$x_1 \odot \cdots \odot x_n \odot y_1 \odot \cdots \odot y_m \leq a \odot (a \to b) \leq b,$$

we have $b \in \langle X \rangle$. Therefore $\langle X \rangle$ is a *ds* of L. As $y \leq y$ for any $y \in X$, we have $X \subseteq \langle X \rangle$.□

The following four Propositions generalize the corresponding results of Gluschankof [6].

Proposition 4 *Any BL-algebra L contains a prime deductive system.*

Proof. Since an BL-algebra L contains the elements $\mathbf{0}, \mathbf{1}$ and since L is a distributive lattice, it follows from general lattice theory that L contains a maximal, prime lattice filter P. Set $P^c = L \setminus P \neq \emptyset$ and define

$$\hat{P} = \textstyle\bigcap_{y \in P^c} \{x \in L \mid x \to y \in P^c\}.$$

We show that $\hat{P}$ is a prime *ds* of L. If $y \in P^c$ then, since $y = \mathbf{1} \to y$, we conclude that $\mathbf{1} \in \hat{P}$. Assume $x, x \to z \in \hat{P}$. Then for any $y, w \in P^c$, we have $x \to y \in P^c$, $(x \to z) \to w \in P^c$, so in particular $(x \to z) \to (x \to y) \in P^c$ for all $y \in P^c$. By (5), $(z \to y) \leq (x \to z) \to (x \to y)$, thus the assumption $(z \to y) \in P$ would imply the contradiction $(x \to z) \to (x \to y) \in P$. Thus $(z \to y) \in P^c$ for all $y \in P^c$, therefore $z \in \hat{P}$ and so $\hat{P}$ is a *ds* of L. To see $\hat{P}$ is proper it is enough to realize that, for any $y \in P^c$, we have $y \to y = \mathbf{1} \in P$ so $y \notin \hat{P}$. By a similar argument we also see that $\hat{P} \subseteq P$. Indeed, it there would be an $x \in \hat{P}$ with $x \notin P$, then $x \in P^c$

and since $x \to x = \mathbf{1} \in P$, we should have $x \notin \hat{P}$. It remains to demonstrate that $\hat{P}$ is prime. Assume $x \vee y \in \hat{P}$ but $x \notin \hat{P}$, $y \notin \hat{P}$. Then there are $z, w \in P^c$ such that $x \to z \in P$, $y \to w \in P$. If $z \vee w$ would be in P, which is a prime lattice filter, then we should have $z \in P$ or $w \in P$, a contradiction. Therefore $z \vee w \in P^c$, thus $(x \vee y) \to (z \vee w) \in P^c$. On the other hand, as $x \to z \leq x \to (z \vee w)$, $y \to w \leq y \to (z \vee w)$, we have $x \to (z \vee w)$, $y \to (z \vee w) \in P$. Applying now (6), we have

$$[x \to (z \vee w)] \wedge [y \to (z \vee w)] = (x \vee y) \to (z \vee w) \in P.$$

This contradiction proves that $x \in \hat{P}$ or $y \in \hat{P}$. The proof is complete.□

Proposition 5 *If P is a prime ds and $P \subseteq D$ is a proper ds, then also D is prime.*

Proof. Assume $a \vee b \in D$. Since P is prime, either $a \to b \in P$ or $b \to a \in P$. Let, say, $b \to a \in P \subseteq D$. By (15), we have $a \vee b \leq (b \to a) \to a$, hence $(b \to a) \to a \in D$ and so $a \in D$. Thus D is prime.□

Proposition 6 *If P is a prime ds, then the set*

$$\mathcal{F} = \{D \mid P \subseteq D, D \textit{ is a proper ds}\}$$

is linearly ordered with respect to set-theoretical inclusion.

Proof. Let $E, D \in \mathcal{F}$. Assume $E \not\subseteq D$, $D \not\subseteq E$. Then there are $a, b \in L$ such that $a \in D$, $a \notin E$, $b \in E$, $b \notin D$. Since P is prime, either $a \to b \in P$ or $b \to a \in P$. If $a \to b \in P \subseteq D$, then $b \in D$, a contradiction, and if $b \to a \in P \subseteq E$ then $a \in E$, another contradiction. Thus $E \subseteq D$ or $D \subseteq E$.□

Proposition 7 *Any proper ds D of an BL-algebra L can be extended to a prime one.*

Proof. If D is a proper *ds* then it is a lattice filter and can therefore be extended to a maximal, prime lattice filter P. Similarly to the proof of Proposition 4 we show that $\hat{P}$ is a prime *ds* and $\hat{P} \subseteq P$. By Proposition 3, $\langle D \cup \hat{P} \rangle$ is a *ds* and $D \subseteq \langle D \cup \hat{P} \rangle$. We verify that $\langle D \cup \hat{P} \rangle$ is proper by showing $\langle D \cup \hat{P} \rangle \subseteq P$. Let therefore $x \in \langle D \cup \hat{P} \rangle$. Then $z_1 \odot \cdots \odot z_n \odot y_1 \odot \cdots \odot y_m \leq x$ where, by the commutativity of $\odot$, we may assume $z_1, \cdots, z_n \in D$, $y_1, \cdots, y_m \in \hat{P}$. Since

$$z_1 \to (\cdots \to (z_n \to (y_1 \to (\cdots \to (y_m \to x) \cdots))) \cdots) = \mathbf{1} \in D,$$

we conclude $y_1 \to (\cdots \to (y_m \to x) \cdots) \in D \subseteq P$. Assume now $x \in P^c$. Since $y_m \in \hat{P}$, $y_m \to x \in P^c$, thus also $y_{m-1} \to (y_m \to x) \in P^c$, etc. and finally $y_1 \to (\cdots \to (y_m \to x) \cdots) \in P^c$, a contradiction. Therefore $x \in P$. Thus $\langle D \cup \hat{P} \rangle$ is a proper *ds*. Since $\hat{P} \subseteq \langle D \cup \hat{P} \rangle$ and $\hat{P}$ is prime, also $\langle D \cup \hat{P} \rangle$ is prime.□

As usually, a proper *ds* is called *maximal* if it is not contained in any other proper *ds*. There are prime deductive systems which are not maximal. Maximal deductive systems, however, are prime since we have

Proposition 8 *Any proper ds D of an BL-algebra L can be extended to a maximal, prime ds.*

Proof. Let D is a proper ds. By Proposition 7, D can be extended to a prime ds E and, by Proposition 6, the set $\mathcal{F} = \{G \mid E \subseteq G, G \text{ is a proper } ds\}$ is linearly ordered. Define

$$M = \bigcup_{G \in \mathcal{F}} G$$

Then trivially $\mathbf{1} \in M$ and if $a, a \to b \in M$ then $a, a \to b \in G$ for some $G \in \mathcal{F}$, thus $b \in G \subseteq M$. Therefore M is a ds and is also proper; indeed, since no $G \in \mathcal{F}$ contains $\mathbf{0}$, we have $\mathbf{0} \notin M$. By Proposition 5, M is prime and obviously maximal.□

It is easy to see that there is one-to-one correspondence between (maximal) deductive systems of an BL-algebra L and (maximal) congruence relations on L, namely

Proposition 9 *Let L be an BL-algebra. Then*
(i) *if $\sim$ is a (maximal) congruence relation on L, then $D = \{a \in L \mid a \sim \mathbf{1}\}$ is a (maximal) ds of L.*
(ii) *if D is a (maximal) ds of L, then $x \sim y$ iff $(x \to y) \odot (y \to x) \in D$ is a (maximal) congruence relation on L.*

Hájek [7] proved that, given a ds D, the corresponding quotient algebra L/D is a BL-algebra and is linear if, and only if D is prime. As in the case of MV-algebras, we have

Proposition 10 *An BL-algebra L is linear if, and only if any proper ds of L is prime.*

Proof. If L is linear then, for all $a, b \in L$, $a \vee b = a$ or $a \vee b = b$. Thus, $a \vee b \in D$ iff $a \in D$ or $b \in D$, where D is any proper ds. Conversely, assume any proper ds is prime. Then, in particular, $\{\mathbf{1}\}$ is prime. Since, for any $a, b \in L$, $(a \to b) \vee (b \to a) \in \{\mathbf{1}\}$, we have that $(a \to b) \in \{\mathbf{1}\}$ or $(b \to a) \in \{\mathbf{1}\}$. Therefore $a \leq b$ or $b \leq a$.□

Definition 4 *The* order *of an element x of an BL-algebra L, in symbols $ord(x)$, is the least integer m such that $x^m = \overbrace{x \odot \ldots \odot x}^{m \text{ terms}} = \mathbf{0}$. If no such m exists then $ord(x) = \infty$. An BL-algebra L is called* locally finite *if all non-unit elements are of finite order.*

Notice that the order of an element x of an BL-algebra L does *not*, in general, coincide with the MV-order of x, if L happens to be simultaniously an MV-algebra. In MV-algebra theory, the order of an element x is defined to be the least integer n such that $nx = \overbrace{x \oplus \ldots \oplus x}^{n \text{ terms}} = \mathbf{1}$, in symbols $O(x) = n$, and if no such integer n exists, then $O(x) = \infty$. In the Lukasiewicz structure, for example, $ord(0.6) = 3$, while $O(0.6) = 2$.

Proposition 11 *Locally finite BL-algebras are linear.*

Proof. Assume $a \vee b = \mathbf{1}$. Then $\mathbf{1} = [(a \to b) \to b] \wedge [(b \to a) \to a] \leq [(a \to b) \to b]$, thus $(b \leq)\ a \to b \leq b$, hence $b = a \to b$. Let now $a \neq \mathbf{1}$. Since the BL-algebra L under consideration is locally finite, there is an m such that $a^m = \mathbf{0}$. Now $b = a \to b = a \to (a \to b) = a^2 \to b = \ldots = a^m \to b = \mathbf{0} \to b = \mathbf{1}$. Thus, $a \vee b = \mathbf{1}$ iff $a = \mathbf{1}$ or $b = \mathbf{1}$. Since, for all elements $a, b \in L$, $(a \to b) \vee (b \to a) = \mathbf{1}$, we have that $a \to b = \mathbf{1}$ or $b \to a = \mathbf{1}$. Therefore $a \leq b$ or $b \leq a$.□

Proposition 12 *In a locally finite BL-algebra L, for all $x \in L$,*

$$\mathbf{0} < x < \mathbf{1} \quad \textit{iff} \quad \mathbf{0} < x^* < \mathbf{1}, \tag{20}$$

$$x^* = \mathbf{0} \quad \textit{iff} \quad x = \mathbf{1}, \tag{21}$$

$$x^* = \mathbf{1} \quad \textit{iff} \quad x = \mathbf{0}. \tag{22}$$

Proof. Assume $\mathbf{0} < x < \mathbf{1}$, $ord(x) = m(\geq 2)$. Then $x^{m-1} \odot x = \mathbf{0}$, $x^{m-2} \odot x \neq \mathbf{0}$ so, by the definition of x^*, $\mathbf{0} < x^{m-1} \leq x^* < x^{m-2} \leq \mathbf{1}$. Conversely, let $\mathbf{0} < x^* < \mathbf{1}$, $ord(x^*) = n(\geq 2)$. Then, by a similar argument, $\mathbf{0} < (x^*)^{n-1} \leq x^{**} < (x^*)^{n-2} \leq \mathbf{1}$. If now $x = \mathbf{0}$, then $x^* = \mathbf{1}$, a contradiction. Therefore $\mathbf{0} < x \leq x^{**} < \mathbf{1}$ and (20) is proved. If $x^* = \mathbf{0}$ but $x \neq \mathbf{1}$, then $\mathbf{0} < x < \mathbf{1}$, which leads to a contradiction $x^* \neq \mathbf{0}$. Thus $x = \mathbf{1}$, which proves (21). The verification of (22) is similar.□

Proposition 13 *In any BL-algebra L, for all $x, y, z \in L$,*

$$\textit{if } z \to x = z \to y \textit{ and } x, y \leq z \quad \textit{then} \quad x = y, \tag{23}$$

$$\textit{if } L \textit{ is linear and } z \to x = z \to y \neq \mathbf{1} \quad \textit{then} \quad x = y. \tag{24}$$

Proof. If $x, y \leq z$ then $x = (z \wedge x) = z \odot (z \to x) = z \odot (z \to y) = (z \wedge y) = y$, thus (23) holds and, if $z \to x = z \to y \neq \mathbf{1}$, then $z \not\leq x$, $z \not\leq y$ therefore, if L is linear then $x, y \leq z$ and (24) now follows by (23).□

Proposition 14 *Locally finite BL-algebras are MV-algebras.*

Proof. It is enough to show that $a^{**} = a$ holds for any element $\mathbf{0} < a < \mathbf{1}$ of a locally finite BL-algebra L; for such an element a we have $\mathbf{0} < a^* < \mathbf{1}$ and $\mathbf{0} < a^{**} < \mathbf{1}$. By setting $x = a$, $y = b$ and $c = \mathbf{0}$ in (3) we see, for any $b \in L$, that $(a \odot b)^* = a \to b^*$. Since $a \leq a^{**}$, we have $a = a \wedge a^{**} = a^{**} \odot (a^{**} \to a)$, thus $a^* = [a^{**} \odot (a^{**} \to a)]^* = a^{**} \to (a^{**} \to a)^*$. On the other hand, $a^* = a^{***} = a^{**} \to \mathbf{0}$. Since L is linear and $a^{**} \to \mathbf{0} = a^{**} \to (a^{**} \to a)^* \neq \mathbf{1}$ we have, by (24), that $(a^{**} \to a)^* = \mathbf{0}$ and, by (21), $a^{**} \to a = \mathbf{1}$. Thus, $a^{**} = a$.□

Let L be the MV-algebra generated by a locally finite BL-algebra. If L would contain an element $\mathbf{0} < x < \mathbf{1}$ such that $mx < \mathbf{1}$ for all natural numbers m, then the element $\mathbf{0} < x^* < \mathbf{1}$ should have the property $(x^*)^m = (mx)^* \neq \mathbf{0}$ for all natural numbers m. This contradition proves that L is a locally finite MV-algebra. By a similar argument we easily see that also the converse holds. Summerizing,

Theorem 1 *Locally finite MV-algebras and locally finite BL-algebras coincide.*

The following theorem generalizes a result proved by Chang [3]. It also follows by a more general argument given by Höhle [10].

Proposition 15 *Let M be a ds of an BL-algebra L. Then the following conditions are equivalent:*

$$M \text{ is a maximal ds.} \tag{25}$$

$$\forall x \notin M : \exists n \in \mathcal{N} \text{ such that } (x^n)^* \in M. \tag{26}$$

$$L/M \text{ is a locally finite MV-algebra.} \tag{27}$$

Proof. Assume (25). Let $x \notin M$. Define a subset D of L by

$$D = \{z \in L \mid \text{for some } y \in M, n \in \mathcal{N}, y \odot x^n \leq z\}.$$

Then trivially $\mathbf{1} \in D$. If $a, a \to b \in D$ then, for some $y, y' \in M$, $n, m \in \mathcal{N}$, holds $y \odot x^n \leq a$, $y' \odot x^m \leq a \to b$. Since $y \odot y' \in M$ and $(y \odot x^n) \odot (y' \odot x^m) = (y \odot y') \odot x^{n+m} \leq a \odot (a \to b) \leq b$, we conclude that $b \in D$ and, therefore, D is a *ds*. Since, for any $y \in M$, $y \odot x \leq y$, we have $M \subseteq D$. But, as $\mathbf{1} \in M$ and $\mathbf{1} \odot x \leq x$, we also have $x \in D$. Since M is maximal, this implies $D = L$. Therefore $\mathbf{0} \in D$, i.e. there exists $y \in M$, $n \in \mathcal{N}$ such that $y \odot x^n \leq \mathbf{0}$, in other words $y \leq (x^n)^*$. Hence $(x^n)^* \in M$. Thus, (26) holds. Assume now (26). Let $x/M \in L/M$ be such that $x/M \neq \mathbf{1}/M$, so $x \notin M$. Then there exists a natural number n such that $(x^n)^* \in M$ and therefore $(x^n)^*/M = \mathbf{1}/M$, so that $x^n/M \leq (x^n)^{**}/M = \mathbf{1}^*/M = \mathbf{0}/M$. Therefore $x^n/M = \mathbf{0}/M$, hence L/M is a locally finite MV-algebra. Finally, assume (27). Let D be a *ds* such that $M \subseteq D$. Assume there is an element $x \in L$ such that $x \in D$, $x \notin M$. Then $x/M \neq \mathbf{1}/M$ and therefore $x^n/M = \mathbf{0}/M$ for some n, i.e. $\mathbf{0} \sim_M x^n$. Since $M \subseteq D$, also $\mathbf{0} \sim_D x^n$, i.e. $x^n/D = \mathbf{0}/D$. On the other hand, $x \in D$ so $x^n \in D$, thus $x^n/D = \mathbf{1}/D$, therefore $\mathbf{0}/D = \mathbf{1}/D$, which implies $\mathbf{0} \in D$, whence M is maximal.□

An MV-algebra is called *semisimple* if the intersection of all it's maximal ideals contains only the element $\mathbf{0}$, or dually, if the intersection of all it's maximal deductive systems contains only the element $\mathbf{1}$. In the same manner we define an BL-algebra to be *semisimple*. Let L be such an BL-algebra and $\mathcal{M}$ the set of all maximal deductive systems of L. Then L is a subalgebra of the direct product of the quotient algebras L/M, $M \in \mathcal{M}$. By Proposition 15, each L/M is a locally finite MV-algebra. By a well-known theorem of Chang [4], L is a semisimple MV-algebra, hence isomorphic to a subalgebra of the Bold MV-algebra of fuzzy sets $[0,1]^{\mathcal{M}^*}$, where $\mathcal{M}^* = \{M^* \mid M \in \mathcal{M}\}$ is the set of all maximal ideals M^* of L. This justifies the following:

Theorem 2 *In the class of BL-algebras, semisimple MV-algebras are the unique algebras representable by fuzzy sets, i.e. isomorphic to a subalgebra of* $[0,1]^{\mathcal{M}^*}$

A general BL-algebra has, however, another kind of representation as Hájek [7] proved that any BL-algebra is isomorphic to a subalgebra of a direct product of linear BL-algebras. Characterizing all linear BL-algebras is therefore an important and interesting problem. BL-algebras on the real unit interval are well-known, however, generally this problem remains open.

4 Co-annihilators in BL-algebras

Annihilators offer a powerful tool in MV-algebra theory. Dealing with BL-algebras, we use a dual notion and define

Definition 5 *Given a non-void set A of an BL-algebra L, a set*

$$^{\perp}A = \{x \in L \mid a \vee x = \mathbf{1} \text{ for all } a \in A\}$$

is a co-annihilator *of A.*

The following Propositions generalize some results of Hoo [9].

Proposition 16 *$^{\perp}A$ is a ds of L. If $A \neq \{\mathbf{1}\}$ then $^{\perp}A$ is proper.*

Proof. Trivially $\mathbf{1} \in {}^{\perp}A$. Assume $x, x \to y \in {}^{\perp}A$. Let $a \in A$. Then $x \to y \leq x \to (y \vee a)$, $a \leq y \vee a = \mathbf{1} \to (y \vee a) = (x \vee a) \to (y \vee a)$. Therefore

$$\begin{aligned} \mathbf{1} &= (x \to y) \vee a \\ &\leq [x \to (y \vee a)] \vee [(x \vee a) \to (y \vee a)] \\ &\leq [x \wedge (x \vee a)] \to (y \vee a)] \\ &= x \to (y \vee a). \end{aligned}$$

Thus $x \leq y \vee a$, so $\mathbf{1} = x \vee a \leq (y \vee a) \vee a = y \vee a$. Hence $y \in {}^{\perp}A$, whence $^{\perp}A$ is a ds. If $A \neq \{\mathbf{1}\}$ and as A is non-void, there is an element $a \in A$ such that $a \neq \mathbf{1}$ and $\mathbf{0} \vee a = a \neq \mathbf{1}$. Therefore $^{\perp}A$ is proper.□

By Proposition 3, $\langle x \rangle = \{y \in L \mid x^n \leq y \text{ for some } n \in \mathcal{N}\}$ is a ds of an BL-algebra L. It is easy to see that $ord(x) < \infty$ iff $\langle x \rangle = L$ and $\langle x \rangle$ is proper iff $ord(x) = \infty$. Given $a \in L$, define

$$D^a = \{x \in L \mid x \to a = a, a \to x = x\}.$$

Then trivially $x \in D^a$ iff $a \in D^x$.

Proposition 17 *For any $a \in L$, D^a is a ds of L and $D^a = ^{\perp}\{a\}$.*

Proof. If $a \vee x = \mathbf{1}$ then, by (15), $(a \to x) \to x = \mathbf{1}$ and $(x \to a) \to a = \mathbf{1}$, i.e. $a \to x = x$ and $x \to a = a$. Therefore $^{\perp}\{a\} = \{x \in L \mid a \vee x = \mathbf{1}\} = \{x \in L \mid x \to a = a, a \to x = x\} = D^a$.□

Proposition 18 *If $ord(a) < \infty$ then $D^a = \{\mathbf{1}\}$.*

Proof. Assume $ord(a) = m < \infty$, $x \in D^a$. Then $a \in D^x$ which is a ds. Therefore $a^m \in D^x$, hence $\mathbf{0} \in D^x$ and so $\mathbf{0} \to x = x$. This means $x = \mathbf{1}$.□

Proposition 19 *For any $\emptyset \neq X \subseteq L$, $^{\perp}X = \bigcap_{x \in X} D^x = \bigcap_{x \in X} {}^{\perp}\{x\}$.*

Proof. $a \in {}^{\perp}X$ iff $\forall x \in X$: $a \vee x = \mathbf{1}$ iff $\forall x \in X$: $a \in {}^{\perp}\{x\}$ iff $a \in \bigcap_{x \in X} {}^{\perp}\{x\} = \bigcap_{x \in X} D^x$.□

Proposition 20 *For any non-void set $X \subseteq L$, $\langle X\rangle \cap {}^{\perp}X = \{\mathbf{1}\}$, in particular, for all $x \in L$, $D^x \cap \langle x\rangle = \{\mathbf{1}\}$ and, if D is a ds, then $D \cap {}^{\perp}D = \{\mathbf{1}\}$.*

Proof. If $a \in \langle X\rangle \cap {}^{\perp}X$, then $a \in {}^{\perp}X$, hence $x \to a = a$ for all $x \in X$. Since $a \in \langle X\rangle$, $x_1 \odot \ldots \odot x_n \leq a$ for some $x_1, \ldots, x_n \in X$, thus

$$\mathbf{1} = x_1 \to (\ldots x_{n-1} \to (x_n \to a)\ldots) = a.$$

The second claim follows by the fact $D^x = {}^{\perp}\{x\}$. If, in particular, D is a *ds* and $a \in \langle D\rangle$, then $a \in D$ as $x_1 \odot \ldots \odot x_n \leq a$ for some $x_1, \ldots, x_n \in D$ and $x_1 \odot \ldots \odot x_n \in D$, hence $\langle D\rangle = D$, whence $D \cap {}^{\perp}D = \langle D\rangle \cap {}^{\perp}D = \{\mathbf{1}\}$.□

Proposition 21 *If ${}^{\perp}A$ is a prime ds and $a, b \in A$ then either for all $x \in A$ holds $x \in D^{a\to b}$ or for all $x \in A$ holds $x \in D^{b\to a}$.*

Proof. Since $(a \to b) \vee (b \to a) = \mathbf{1}$ and ${}^{\perp}A$ is a prime *ds*, either $a \to b \in {}^{\perp}A = \bigcap_{x\in A} D^x$ or $b \to a \in \bigcap_{x\in A} D^x$, thus either for all $x \in A$ holds $x \in D^{a\to b}$ or for all $x \in A$ holds $x \in D^{b\to a}$.□

Proposition 22 *Let $A \subseteq L$ be a ds. Then ${}^{\perp}A$ is a prime ds if, and only if A is linear and $A \neq \{\mathbf{1}\}$.*

Proof. Assume A is linear and $A \neq \{\mathbf{1}\}$. Let $a \vee b \in {}^{\perp}A$, but $a \notin {}^{\perp}A$, $b \notin {}^{\perp}A$. Then there exists $x', x'' \in A$ such that $a \vee x' \neq \mathbf{1}$, $b \vee x'' \neq \mathbf{1}$. Set $x = x' \wedge x''$. Then $x \in A$ as A is a *ds* and, clearly, $a \vee x \neq \mathbf{1}$, $b \vee x \neq \mathbf{1}$. Since $x \leq a \vee x$, $b \vee x$, we conclude $a \vee x, b \vee x \in A$ and, as A is linear, we may assume $b \vee x \leq a \vee x$. Now

$$\mathbf{1} = (a \vee b) \vee x = a \vee (b \vee x) \leq a \vee (a \vee x) = a \vee x,$$

which contradicts the fact $a \vee x \neq \mathbf{1}$. Therefore $a \in {}^{\perp}A$ or $b \in {}^{\perp}A$, hence ${}^{\perp}A$ is prime. Conversely, assume ${}^{\perp}A$ is prime. Then $A \neq \{\mathbf{1}\}$ as otherwise we would have ${}^{\perp}A = L$. Let $a, b \in A$. Since $b \leq a \to b$, $a \leq b \to a$ and A is a *ds*, we have $a \to b, b \to a \in A$. By Proposition 21, either $a \to b, b \to a \in D^{a\to b}$ or $a \to b, b \to a \in D^{b\to a}$. In the first case $\mathbf{1} = (a \to b) \vee (a \to b) = (a \to b)$, thus $a \leq b$, in the second case $\mathbf{1} = (b \to a) \vee (b \to a) = (b \to a)$, hence $b \leq a$. Therefore A is linear.□

Proposition 23 *If $X \subseteq Y$, then ${}^{\perp}Y \subseteq {}^{\perp}X$.*

Proof. If $z \in \bigcap_{y\in Y} {}^{\perp}\{y\}$ then, for any $x \in X \subseteq Y$, $x \to z = z$, $z \to x = x$, thus $z \in \bigcap_{x\in X} {}^{\perp}\{x\}$. Therefore ${}^{\perp}Y = \bigcap_{y\in Y} {}^{\perp}\{y\} \subseteq \bigcap_{x\in X} {}^{\perp}\{x\} = {}^{\perp}X$.□

Theorem 3 *If $\emptyset \neq X \subseteq L$, then*

$$X \subseteq {}^{\perp\perp}X, \tag{28}$$
$${}^{\perp}X = {}^{\perp\perp\perp}X, \tag{29}$$
$${}^{\perp}X = {}^{\perp}\langle X\rangle. \tag{30}$$

Proof. ${}^{\perp\perp}X = \{a \in L \mid a \vee x = \mathbf{1} \text{ for all } x \in {}^{\perp}X\}$. If now $b \in X$, then $b \vee x = \mathbf{1}$ for all $x \in {}^{\perp}X$, hence $b \in {}^{\perp\perp}X$ and (28) holds. By (28), ${}^{\perp}X \subseteq {}^{\perp\perp\perp}X$ and, by Proposition 23, ${}^{\perp\perp\perp}X \subseteq {}^{\perp}X$. Therefore (29) holds. To verify (30), we first reason that a fact $X \subseteq \langle X \rangle$ implies ${}^{\perp}\langle X \rangle \subseteq {}^{\perp}X$. To see that also the converse inclusion holds, assume $y \in {}^{\perp}X$. Then, for any $x_i \in X$, $i = 1, \ldots n$, $x_i \vee y = \mathbf{1}$. We demonstrate, by induction on n, that also $(x_1 \odot \ldots \odot x_n) \vee y = \mathbf{1}$. If $n = 1$, then the claim is clearly true, so assume it is true for $n = k$. Let $n = k + 1$ and set $x = (x_1 \odot \ldots \odot x_k)$. By induction hypothesis,

$$\begin{aligned} \mathbf{1} &= (y \vee x) \odot (y \vee x_{k+1}) \\ &= [y \odot (y \vee x_{k+1})] \vee [x \odot (y \vee x_{k+1})] \\ &= y \vee [(x \odot y) \vee (x \odot x_{k+1})] \\ &\leq y \vee [y \vee (x_1 \odot \ldots \odot x_{k+1})] \\ &= y \vee (x_1 \odot \ldots \odot x_{k+1}). \end{aligned}$$

Thus, the claim holds for all natural numbers n. If now $z \in \langle X \rangle$ then, for some $x_1, \ldots, x_n \in X$, $x_1 \odot \ldots \odot x_n \leq z$. Therefore $\mathbf{1} = y \vee (x_1 \odot \ldots \odot x_n) \leq y \vee z$. We conclude that $y \vee z = \mathbf{1}$ for any $z \in \langle X \rangle$ and so $y \in {}^{\perp}\langle X \rangle$. This proves ${}^{\perp}X \subseteq {}^{\perp}\langle X \rangle$ and the proof is complete.□

Proposition 24 *If a linear ds D contains an element $x \neq \mathbf{1}$ and $x \vee x^* = \mathbf{1}$, then x is the least element of D.*

Proof. Since $x \vee x^* = \mathbf{1}$ we have, by (13), that $x \wedge x^* = \mathbf{0}$. Let $a \in D$. Then $a = a \vee \mathbf{0} = a \vee (x \wedge x^*) = (a \vee x) \wedge (a \vee x^*)$, where the last equation follows by the distributivity of L. By Proposition 22, ${}^{\perp}D$ is a prime *ds*. Since $x \vee x^* = \mathbf{1}$, either $x \in {}^{\perp}D$ or $x^* \in {}^{\perp}D$ and, as $x \vee x = x \neq \mathbf{1}$, we necassarily have $x^* \in {}^{\perp}D$. Now $a \in D$, hence $a \vee x^* = \mathbf{1}$, whence $a = a \vee x$, thus $x \leq a$ and the proof is complete.□

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