Mathware & Soft Computing 5 (1998) 189-199

Graded Sets, Points and Numbers

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Abstract

The basic tool considered in this paper is the so-called "graded set", defined on the analogy of the family of α-cuts of a fuzzy set. It is also considered the corresponding extensions of the concepts of a point and of a real number (again on the analogy of the fuzzy case). These new "graded concepts" avoid the disadvantages pointed out by Gerla (for the fuzzy points) and by Kaleva and Seikkala (for the convergence of sequences of fuzzy numbers).
Keywords: α-cut, fuzzy point, fuzzy number.

1 Introduction

We can find a great variety of definitions and criteria when we work with the different aspects of the Fuzzy Mathematics (as it is pointed out by Kerre [15]). Against this heterogeneity (which is often due to the use of membership functions), the use of the families of α-cuts provides a fruitful and unifying method [19]. Indeed, this method has been used in theoretical as well as in applied fields of the Fuzzy Mathematics, especially in the Decision and Optimization Theory in a Fuzzy Environment [1, 2, 21, 22]... Nevertheless, the families of α-cuts of fuzzy sets are restricted by the conditions stablished in the Negoita and Ralescu's Representation Theorem, which sometimes give rise to certain disadvantages. For example, when operating fuzzy numbers via the Zadeh's Extension Principle [17] or when calculating the limit of a sequence of fuzzy numbers via their α-cuts [13].

In order to make use of the fruitful tool provided by the families of α-cuts, but also avoiding the disadvantages derived from the conditions of the Representation Theorem, the author introduces the so-called "graded sets" [8]. They are defined as non-increasing families of subsets (let us note that the non-increasing condition is weaker than the conditions imposed by the Representation Theorem). This also enables us to define several "graded concepts" on the analogy of the corresponding "fuzzy concepts" [10]. In particular, we consider here the definitions of a "graded point" [8] and a "graded number" [9], because of their interesting properties (as we shall see below).

The non-increasing condition imposed to the graded sets gives rise to a strong relationship with the fuzzy sets. Specifically, the author proves [8] that given any graded set ψ, there exists only one fuzzy set μ_ψ (called the "fuzzy set associated with" ψ) such that ψ is greater or equal to the family of strong α-cuts of μ_ψ and it is lower or equal to the family of [weak] α-cuts of μ_ψ. Conversely, given any fuzzy set μ, it is obvious that any family of subsets fullfilling this condition must be a non-increasing family of subsets (and so these families of subsets will be called the "graded sets associated with" the fuzzy set μ).

The preceding result suggests representing a fuzzy set by any of their associated graded sets (not necessarily equal to its family of α-cuts). This procedure gives rise to a representation method which takes the graded sets as a useful tool in the Fuzzy Mathematics. In this paper, this method is used in order to obtain properties of the fuzzy numbers via the graded numbers. More specifically, relationships between the Zadeh's fuzzy numbers and the Hutton's fuzzy numbers (which are studied separately in the Fuzzy Mathematics) are stablished and it is defined the "graded convergence" for sequences of fuzzy numbers. This new criterion extends the usual convergence in $\mathbb{R}$ (which is not fullfilled by the pointwise convergence of membership functions) and ensures the existence of a limit for any monotonic and bounded sequence of fuzzy numbers (which is not fullfilled by the α-level convergence).

The paper is organized as follows. First (in Section 2), it is fixed the definitions and notations corresponding to the different fuzzy concepts used in the following. The Sections 3, 4 and 5 are devoted to the definitions and fundamental results about graded sets, graded points and graded numbers, respectively. Finally, in the Section 6, it is enumerated some of the results about fuzzy numbers which are deduced from the analogous results corresponding to the graded numbers.

2 Fuzzy sets, fuzzy points and fuzzy numbers

In the following, $\mathbb{I}$ denotes the unit interval $[0,1]$, $\mathbb{R}$ the real line and $\mathbb{N}$ the set of positive integers. Given any set X, $\mathcal{P}(X) = 2^X$ denotes the power set of X and $\mathcal{F}(X) = \mathbb{I}^X := \{\mu : X \to \mathbb{I}\}$ denotes the set of fuzzy parts of X. We assume that $\mathcal{P}(X) \subset \mathcal{F}(X)$, via the identification $A \equiv \chi_A$, where χ_A is the characteristic function of A, crisp subset of X. For each $\alpha \in \mathbb{I}$ and each $\mu \in \mathcal{F}(X)$, we have the [weak] α-cut $\mu^{[\alpha]} := \mu^{-1}[\alpha, 1]$ and the strong α-cut $\mu^{(\alpha)} := \mu^{-1}(\alpha, 1]$.

The extension of arbitrary mappings $f : X \to Y$ (resp. $f : X^2 \to Y$) to $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ (resp. $\mathcal{P}(X)^2 \to \mathcal{P}(Y)$) is made in the usual form, while the extension to $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ (resp. $\mathcal{F}(X)^2 \to \mathcal{F}(Y)$) is performed via the Zadeh's Extension Principle.

The fuzzy set $\lambda_x := \lambda\chi_{\{x\}}$ is called a **fuzzy point** (or **fuzzy singleton**) with the **support** $x \in X$ and **value** $\lambda \in (0,1]$. Let us consider now three different criteria to define where a fuzzy point belongs to a fuzzy set:

- Following Kerre [14], we have:

$$\lambda_x \in \mu :\Leftrightarrow \lambda_x \subseteq \mu$$

(which is equivalent to: $\lambda \leq \mu(x)$).

- According to Wong [23] (who gave the original definition) and to Gottwald [7] (who changed it in order to avoid some drawbacks), we have this other criterion, which does not admit the value $\lambda = 1$ for the fuzzy points:

$$\lambda_x \in \mu :\Leftrightarrow \lambda < \mu(x).$$

- Pu and Liu [18] define the **quasi-coincidence** of the fuzzy sets $\nu, \mu \in \mathcal{F}(X)$ as the existence of an $x \in X$ such that $\nu(x) + \mu(x) > 1$. For the particular case $\nu = \lambda_x$ it results in the following criterion:

$$\lambda_x \in \mu :\Leftrightarrow \lambda + \mu(x) > 1$$

(which is equivalent to: $\lambda_x \notin 1 - \mu$ according to Kerre's definition)

If we use the definition given by Kerre, then the following property does not hold in general (when I is an infinite index set):

$$\lambda_x \in \bigcup_{i \in I} \mu_i \Leftrightarrow \exists i \in I, \lambda_x \in \mu_i.$$

This fact induces Wong's definition. Thus, using it we obtain the fullfillment of the previous property. Nevertheless, Wong's definition does not ensures the fullfillment of this other dual property (which is obtained with Kerre's definition):

$$\lambda_x \in \bigcap_{i \in I} \mu_i \Leftrightarrow \forall i \in I, \lambda_x \in \mu_i.$$

In this respect, Pu and Liu's definition acts similarly to Wong's definition.

Another remarkable fact is that Wong's definition discards the crisp [or ordinary] points as fuzzy points, because it does not admit the value $\lambda = 1$ for the fuzzy points.

In this situation, Gerla [5, 6] considers different properties in order to obtain axiomatic definitions of fuzzy point and of its belonging to a fuzzy set (simultaneously). In particular, he proves the impossibility to define a pair (X', ϵ') such that X' is a set of fuzzy points, ϵ' is a crisp relationship of belonging to fuzzy sets and the following properties are simultaneously verified: $(\forall P, Q \in X', \forall \mu, \mu_i, \nu, \nu_i \in \mathcal{F}(X))$:

- $P\epsilon' \bigcup_{i\in I} \mu_i \Leftrightarrow \exists i \in I, P\epsilon'\mu_i$.
- $P\epsilon' \bigcap_{i\in I} \mu_i \Leftrightarrow \forall i \in I, P\epsilon'\mu_i$.
- $\{P : P\epsilon' 0\chi_X\} = \emptyset, \quad \{P : P\epsilon'\chi_X\} = X'$.
- $\mu = \nu \Leftrightarrow \{P : P\epsilon'\mu\} = \{P : P\epsilon'\nu\}$.
- $P = Q \Leftrightarrow \{\mu : P\epsilon'\mu\} = \{\mu : Q\epsilon'\mu\}$.

The concept of "fuzzy number" has grown, fundamentally, into two different ways. On the one hand, we have several definitions of common use in Artificial Intelligence, consisting basically of membership functions which first increase from 0 to 1 and then decrease to 0 again. All these definitions can be put together under the denomination of "Zadeh's fuzzy numbers" (according to the original definition given in [24]). On the other hand, we have the "Hutton's fuzzy numbers" [12, 4] used in Fuzzy Topology, whose membership function is non-increasing. Similarly to these, we also have the "Höhle's fuzzy numbers" [11], whose membership function is a probability distribution function. Let us remember that the Hutton's fuzzy numbers are, by definition, conditioned by an equivalence relation. Nevertheless, this equivalence can be ignored when we only consider upper-semicontinuous membership functions [16]. In this paper we shall use specifically the following definitions and notations:

Definition 1 *The* **Zadeh's, Hutton's** *and* **Höhle's [fuzzy] numbers** *are, respectively, the elements belonging to the following sets:*

$$\begin{aligned} \mathcal{F}_Z(\mathbb{R}) &:= \{\mu \in \mathcal{F}(\mathbb{R}) : \mu \text{ is convex, normal} \\ &\qquad \text{and u.s.c., and } \mu^{(0)} \text{ is bounded}\}, \\ \mathcal{F}_H(\mathbb{R}) &:= \{\mu \in \mathcal{F}(\mathbb{R}) : \mu \text{ is non-increasing, normal} \\ &\qquad \text{and u.s.c., and } \mu^{(0)} \text{ is bounded above}\}, \\ \mathcal{F}_D(\mathbb{R}) &:= \{\mu \in \mathcal{F}(\mathbb{R}) : \mu \text{ is increasing, normal} \\ &\qquad \text{and u.s.c., and } \mu^{(0)} \text{ is bounded below}\}. \end{aligned}$$

(Where "u.s.c." stands for "upper-semicontinuous ")

We shall say simply **"fuzzy numbers"** *to refer to any of the elements of the set:*

$$\mathcal{FN}(\mathbb{R}) := \mathcal{F}_Z(\mathbb{R}) \cup \mathcal{F}_H(\mathbb{R}) \cup \mathcal{F}_D(\mathbb{R}).$$

3 Graded sets

Definition 2 ([8], Definitions 3.1, 3.2) *Given any set X, by a* **graded subset** *or* **graded part** *of X (or simply* **graded set***) we means any mapping $\psi : \mathbb{I} \to \mathcal{P}(X)$ such that verifies the following condition:*

$$\forall \alpha, \beta \in \mathbb{I}, \ \ [\alpha < \beta \Rightarrow \psi(\alpha) \supseteq \psi(\beta)].$$

We say that ψ is **normal** *when $\psi(\alpha) \neq \emptyset$, $\forall \alpha \in \mathbb{I}$. We denote by $\mathcal{G}(X)$ the set of graded parts of X. The inclusion $\mathcal{P}(X) \subset \mathcal{G}(X)$ is given by: $A \hookrightarrow (A)_{\mathbf{I}}$, where $(A)_{\mathbf{I}}(\alpha) := A$, $\forall \alpha \in \mathbb{I}$.*

Whenever possible, we use the following general criterion in order to extend to graded sets any property or concept "C" known for crisp subsets:

$$\psi \text{ is "C"} \ \ :\Leftrightarrow \ \ \psi(\alpha) \text{ is "C"}, \forall \alpha \in \mathbb{I}.$$

In particular, we define $(\forall \psi_i, \psi, \phi \in \mathcal{G}(X))$:

- $\psi = \bigcap_{i \in I} \psi_i :\Leftrightarrow \psi(\alpha) = \bigcap_{i \in I} \psi_i(\alpha)$, $\forall \alpha \in \mathbb{I}$.

- $\psi = \bigcup_{i\in I} \psi_i :\Leftrightarrow \psi(\alpha) = \bigcup_{i\in I} \psi_i(\alpha)$, $\forall\alpha \in \mathbb{I}$.
- $\psi \subseteq \phi :\Leftrightarrow \psi(\alpha) \subseteq \phi(\alpha)$, $\forall\alpha \in \mathbb{I}$.
- *Any map $f : X \to Y$ is extended to $f : \mathcal{G}(X) \to \mathcal{G}(Y)$ defined by:*

$$f(\psi)(\alpha) := f(\psi(\alpha)), \ \forall\alpha \in \mathbb{I}.$$

- *Any map $f : X^2 \to Y$ is extended to $f : \mathcal{G}(X)^2 \to \mathcal{G}(Y)$ defined by:*

$$f(\psi, \phi)(\alpha) := f(\psi(\alpha), \phi(\alpha)), \ \forall\alpha \in \mathbb{I}.$$

- *In the case $X = \mathbb{R}$, we say that ψ is* **convex** *(resp.* **closed, bounded below** *or* **bounded above***) when, $\forall\alpha \in \mathbb{I}$, $\psi(\alpha)$ is convex (resp. closed, bounded below or above).*

Obviously, ψ is normal iff $\psi(1) \neq \emptyset$ and ψ is bounded (below or above) iff $\psi(0)$ also is.

The main result with regard to the relationship between graded sets and fuzzy sets is the following theorem. From here on, we assume that $\mu^{[0]} = X$ and $\mu^{(1)} = \emptyset$, $\forall\mu \in \mathcal{F}(X)$:

Theorem 1 ([8], Theorem 4.3) *For any set X and any map $\psi : \mathbb{I} \to \mathcal{P}(X)$, the following holds:*

1. *ψ is a graded set if and only if there exists only one fuzzy set μ_ψ verifying: $\mu_\psi^{(\alpha)} \subseteq \psi(\alpha) \subseteq \mu_\psi^{[\alpha]}$, $\forall\alpha \in \mathbb{I}$.*
2. *The fuzzy set μ_ψ is given by:*

$$\mu_\psi(x) = \sup\{\alpha \in \mathbb{I} : x \in \psi(\alpha)\}, \ \forall x \in X,$$

   and its α-cuts are: $\mu_\psi^{[\alpha]} = \bigcap\{\psi(\beta) : 0 \leq \beta < \alpha\}$, $\mu_\psi^{(\alpha)} = \bigcup\{\psi(\beta) : \alpha < \beta \leq 1\}$, $\forall\alpha \in \mathbb{I}$.

When the conditions given in this theorem holds, we say that μ_ψ is **the fuzzy set associated with the graded set** ψ and that ψ is **a graded set associated with** μ_ψ. In particular, the family of α-cuts of any fuzzy set μ (as well as its family of strong α-cuts) is a graded set associated with μ. Taking into consideration this particular case in the previous theorem, they result as corollaries the Negoita and Ralescu's Representation Theorem (which stablishes the conditions that characterize to the α-cuts) and the so-called Resolution Principle (which obtain a fuzzy set from its α-cuts).

In [8] it is proved that the operations and relations previously defined for graded sets are related to the corresponding operations and relations defined by Zadeh for their associated fuzzy sets, in the following sense:

Proposition 1 *For any sets X, Y and any $\psi_i, \psi, \phi \in \mathcal{G}(X)$, we have:*

1. $\mu_{\bigcap_{i\in I} \psi_i} = \bigcap_{i\in I} \mu_{\psi_i}$.

2. $\mu_{\bigcup_{i\in I}\psi_i} = \bigcup_{i\in I}\mu_{\psi_i}$.

3. $\psi \subseteq \phi \Rightarrow \mu_\psi \subseteq \mu_\phi$.

4. *The extensions* $\mathcal{G}(X) \to \mathcal{G}(Y)$ *and* $\mathcal{F}(X) \to \mathcal{F}(Y)$ *obtained from any map* $f : X \to Y$ *verify:*
$$\mu_{f(\psi)} = f(\mu_\psi).$$

5. *The extensions* $\mathcal{G}(X)^2 \to \mathcal{G}(Y)$ *and* $\mathcal{F}(X)^2 \to \mathcal{F}(Y)$ *obtained from any* $f : X^2 \to Y$ *verify:*
$$\mu_{f(\psi,\phi)} = f(\mu_\psi, \mu_\phi).$$

Corollary 1 *For any set* X *and any* $\mu, \nu \in \mathcal{F}(X)$*, we have the inclusion* $\mu \subseteq \nu$ *if and only if there exist two graded sets,* ψ *associated with* μ *and* ϕ *associated with* ν*, such that* $\psi \subseteq \phi$*.*

These properties enable us to extend the representation of fuzzy sets by α-cuts, via the additional use of other associated graded sets. This "extended method" is used in [10] in the study of several aspects about sets, binary relations, topological spaces and some algebraic structures. Here, the method will be used (in Section 6, for the case $X = Y = \mathbb{R}$) to transfer certain properties from the graded numbers to the fuzzy numbers.

4 Graded points

Definition 3 *By a* **graded point** *with* **support** $x \in X$ *and* **value** $\lambda \in (0,1]$*, we means the graded subset* x_λ *of* X *defined by:*

$$\begin{aligned} x_\lambda(\alpha) &:= \{x\}, && \text{if } \alpha \leq \lambda, \\ &:= \emptyset, && \text{if } \alpha > \lambda. \end{aligned}$$

We say that the graded point x_λ **belongs** *to the graded set* $\psi \in \mathcal{G}(X)$ *when we have the inclusion* $x_\lambda \subseteq \psi$ *as graded sets. Obviously, this definition can be expressed in the form:*
$$x_\lambda \in \psi :\Leftrightarrow x \in \psi(\lambda). \tag{1}$$

With this definition, all the conditions considered by Gerla in [5, 6] are satisfied:

Proposition 2 *The set of graded points of* X*,* $X_\mathcal{G} := \{x_\lambda : x \in X, \lambda \in (0,1]\}$ *together with the previous definition of belonging, satisfies the following five conditions:*

1. $x_\lambda \in \bigcup_{i\in I}\psi_i \Leftrightarrow \exists i \in I, x_\lambda \in \psi_i$.

2. $x_\lambda \in \bigcap_{i\in I}\psi_i \Leftrightarrow \forall i \in I, x_\lambda \in \psi_i$.

3. $\{x_\lambda : x_\lambda \in \emptyset\} = \emptyset, \quad \{x_\lambda : x_\lambda \in X\} = X_\mathcal{G}$.

4. $\psi = \phi \Leftrightarrow \{x_\lambda : x_\lambda \in \psi\} = \{x_\lambda : x_\lambda \in \phi\}$.

5. $x_\lambda = y_\kappa \Leftrightarrow \{\psi : x_\lambda \in \psi\} = \{\psi : y_\kappa \in \psi\}$.

Proof: Parts 1 and 2 result immediately from (1) and the corresponding classical properties. Part 3 is trivial, as it is side "⇒" in the two remaining parts. Let us prove both converses:

4. If $\psi \neq \phi$, then there exists $\alpha \in \mathbb{I}$ such that $\psi(\alpha) \neq \phi(\alpha)$. Therefore, $\exists \alpha \in \mathbb{I}$, $\exists x \in X$ such that $x \in \psi(\alpha) \backslash \phi(\alpha)$ (or $x \in \phi(\alpha) \backslash \psi(\alpha)$), and so $\exists x_\alpha \in X_{\mathcal{G}}$ such that $x_\alpha \in \psi$ and $x_\alpha \notin \phi$ (or vice versa). In conclusion, $\{x_\lambda : x_\lambda \in \psi\} \neq \{x_\lambda : x_\lambda \in \phi\}$.

5. Let us assume that $x_\lambda \neq y_\kappa$ and $\lambda \leq \kappa$ (we should have a similar proof in the case $\lambda \geq \kappa$). Next, let us consider the graded set $\psi = x_\lambda$, which verifies obviously that $x_\lambda \in \psi$. Nevertheless $y_\kappa \notin \psi$, because the initial hypothesis implies that $x \neq y$ or $\kappa > \lambda$. Therefore, $\{\psi : x_\lambda \in \psi\} \neq \{\psi : y_\kappa \in \psi\}$.

5 Graded numbers

Definition 4 *Using the following classes of intervals*

$$\begin{aligned} \mathcal{C} &:= \{[a,b] : a, b \in \mathbb{R},\ a \leq b\}, \\ \mathcal{C}_{-\infty} &:= \{(-\infty, a] : a \in \mathbb{R}\}, \\ \mathcal{C}_{+\infty} &:= \{[a, +\infty) : a \in \mathbb{R}\}, \end{aligned}$$

we define the sets $\mathcal{G}_Z(\mathbb{R})$ *(resp.* $\mathcal{G}_H(\mathbb{R})$*,* $\mathcal{G}_D(\mathbb{R})$*) of* **Zadeh's** *(resp.* **Hutton's** *and* **Höhle's***)* **graded numbers** *by:*

$$\begin{aligned} \mathcal{G}_Z(\mathbb{R}) &:= \{\psi \in \mathcal{G}(\mathbb{R}) : \forall \alpha \in \mathbb{I},\ \psi(\alpha) \in \mathcal{C}\}, \\ \mathcal{G}_H(\mathbb{R}) &:= \{\psi \in \mathcal{G}(\mathbb{R}) : \forall \alpha \in \mathbb{I},\ \psi(\alpha) \in \mathcal{C}_{-\infty}\}, \\ \mathcal{G}_D(\mathbb{R}) &:= \{\psi \in \mathcal{G}(\mathbb{R}) : \forall \alpha \in \mathbb{I},\ \psi(\alpha) \in \mathcal{C}_{+\infty}\}. \end{aligned}$$

By **graded numbers** *we refer to any of the elements of the set:*

$$\mathcal{GN}(\mathbb{R}) := \mathcal{G}_Z(\mathbb{R}) \cup \mathcal{G}_H(\mathbb{R}) \cup \mathcal{G}_D(\mathbb{R}).$$

Via the Definition 2 we obtain the usual operations and partial order between the graded numbers. This is done in [9, 10], with the additional obtaining of several properties (and with the consideration of alternative definitions, less restrictive, for the graded and fuzzy numbers). With regard to the unary and binary operations there are stablished conditions which ensure that they are inner operations. In general terms, we obtain the conclusion that the Zadeh's (graded or fuzzy) numbers admit more such operations than the Hutton's or Höhle's numbers. This consideration can be added to others given in [3] with respect to the comparison between those three kinds of fuzzy numbers.

Here, we shall restrict ourselves to the order and the convergence of sequences of numbers. First, we consider the following proposition (which can be easily proved via the equations $[a,b] = [a,+\infty) \cap (-\infty, b]$ and $-[a,+\infty) = (-\infty, -a]$.)

Proposition 3 *The following mappings are bijective:*

1. $\{(\psi_D, \psi_H) \in \mathcal{G}_D(\mathbb{R}) \times \mathcal{G}_H(\mathbb{R}) : \psi_D \cap \psi_H$ *is normal*$\} \to \mathcal{G}_Z(\mathbb{R}) : (\psi_D, \psi_H) \to \psi_D \cap \psi_H$.
2. $\mathcal{G}_H(\mathbb{R}) \to \mathcal{G}_D(\mathbb{R}) : \psi \to -\psi$ *(extension of* $\mathbb{R} \to \mathbb{R} : x \to -x$*).*
3. $\{(\psi_1, \psi_2) \in \mathcal{G}_H(\mathbb{R})^2 : -\psi_1 \cap \psi_2$ *is normal*$\} \to \mathcal{G}_Z(\mathbb{R}) : (\psi_1, \psi_2) \to -\psi_1 \cap \psi_2$.

¿From now on we use the notation given in 3.1, thus expressing each Zadeh's graded number ψ as the intersection of a Hutton's graded number ψ_H with a Höhle's one ψ_D. Moreover, we extend this notation as follows:

$$\begin{array}{llll} \forall \psi \in \mathcal{G}_H(\mathbb{R}), & \psi_H := \psi & \text{and} & \psi_D := (\mathbb{R})_{\mathbf{I}}; \\ \forall \psi \in \mathcal{G}_D(\mathbb{R}), & \psi_D := \psi & \text{and} & \psi_H := (\mathbb{R})_{\mathbf{I}}. \end{array} \tag{2}$$

Definition 5 *In* $\mathcal{GN}(\mathbb{R})$ *we use the notation given above and we define:*

- *The partial order:*

$$\psi \leq \phi :\Leftrightarrow \psi_H \subseteq \phi_H \text{ and } \psi_D \supseteq \phi_D.$$

- *The* **graded convergence** *for sequences in* $\mathcal{GN}(\mathbb{R})$*, written* $\{\psi_n\}_{n\in\mathbb{N}} \xrightarrow{g} \psi$*, which happens if and only if:*

$$\begin{array}{c} \forall \alpha \in \mathbf{I}, \quad \{\inf \psi_{nD}(\alpha)\} \to \inf \psi_D(\alpha) \\ \text{and } \{\sup \psi_{nH}(\alpha)\} \to \sup \psi_H(\alpha), \end{array}$$

where $\to$ *denotes the usual convergence in* $\mathbb{R} \cup \{-\infty, +\infty\}$.

With the last definition we obtain the conditions considered by Kaleva and Seikkala:

Proposition 4 ([9], Proposition 3.13) *For the three sets of graded numbers* $\mathcal{G}_X(\mathbb{R})$*, corresponding to the cases* $X = Z, H, D$*, the following properties hold:*

1. *The graded convergence defined in* $\mathcal{G}_X(\mathbb{R})$ *extends the usual convergence in* $\mathbb{R}$.
2. *If* $\{\psi_n\} \subset \mathcal{G}_X(\mathbb{R})$ *is a monotonic and bounded sequence, then it is convergent in* $\mathcal{G}_X(\mathbb{R})$*, with respect to the graded convergence.*

6 Properties obtained for the fuzzy numbers

There is a great analogy between the graded numbers and the fuzzy numbers. More specifically we have:

Theorem 2 ([9], Theorem 4.3) *For any* $\mu \in \mathcal{F}(\mathbb{R})$ *and any of the three kinds of numbers corresponding to the cases* $X = Z, H, D$*, we have:*

$$\mu \in \mathcal{F}_X(\mathbb{R}) \Leftrightarrow \exists \psi \in \mathcal{G}_X(\mathbb{R}) \text{ associated with } \mu.$$

The usual operations are performed in $\mathcal{F}_Z(\mathbb{R})$ via the Zadeh's Extension Principle. Lowen [16] proves that this can also be done in $\mathcal{F}_H(\mathbb{R})$, for the operations previously defined by Rodabaugh [20] in the fuzzy real line. As a consequence of the Theorem 2 we obtain the following bijection:

$$\{(\mu_D, \mu_H) \in \mathcal{F}_D(\mathbb{R}) \times \mathcal{F}_H(\mathbb{R}) : \mu_D \cap \mu_H \text{ is normal}\}$$
$$\to \mathcal{F}_Z(\mathbb{R}) : \quad (\mu_D, \mu_H) \to \mu_D \cap \mu_H.$$

This fact gives rise to the notation $\mu = \mu_D \cap \mu_H$, extended by:

$$\forall \mu \in \mathcal{F}_H(\mathbb{R}), \quad \mu_H := \mu \quad \text{and} \quad \mu_D := \chi_{\mathbf{R}};$$
$$\forall \mu \in \mathcal{F}_D(\mathbb{R}), \quad \mu_D := \mu \quad \text{and} \quad \mu_H := \chi_{\mathbf{R}}.$$

The partial orders considered by Zadeh [24] and by Hutton [12] are then subsumed into the following definition ($\forall \mu, \nu \in \mathcal{FN}(\mathbb{R})$):

$$\mu \leq \nu :\Leftrightarrow \mu_D \supseteq \nu_D \text{ and } \mu_H \subseteq \nu_H.$$

Moreover, this order for the fuzzy numbers is related with the order for the graded numbers as follows:

Proposition 5 *For any of the three cases $X = Z, H, D$, we have ($\forall \mu, \nu \in \mathcal{F}_X(\mathbb{R})$):*

$$\mu \leq \nu \quad \Leftrightarrow \quad \exists \psi \in \mathcal{G}_X(\mathbb{R}) \text{ associated with } \mu \text{ and } \exists \phi$$
$$\in \mathcal{G}_X(\mathbb{R}) \text{ associated with } \nu, \text{ such that} \psi \leq \phi.$$

All these considerations enable us to transfer to the fuzzy numbers the properties obtained for the graded numbers in [9, 10], with regard to the unary and binary operations as well as with regard to the partial order defined. Finally, we obtain a new criterion for the convergence of sequences of fuzzy numbers such that extends the usual convergence in $\mathbb{R}$ (this does not happen with the pointwise convergence of membership functions) and such that ensures the existence of a limit for any monotonic and bounded sequence (this does not happen with the α-level convergence). That is, the two drawbacks considered by Kaleva and Seikkala [13] are avoided with this new criterion:

Definition 6 *In $\mathcal{FN}(\mathbb{R})$, we define the* **graded convergence** *as follows:*

$$\{\mu_n\} \xrightarrow{g} \mu \quad :\Leftrightarrow \quad \forall n, \exists \psi_n \text{ associated with } \mu_n$$
$$\text{and } \exists \psi \text{ associated with } \mu \quad \text{such that } \{\psi_n\} \xrightarrow{g} \psi.$$

References

[1] M. Delgado, J. Kacprzyk, S.A. Orlovski, J.L. Verdegay and M.A. Vila (1990), A survey of fuzzy optimization and fuzzy mathematical programming, in: M. Fedrizzi, J. Kacprzyk and M. Roubens (eds.), *Interactive Fuzzy Optimization and Mathematical Programming* (Springer Verlag, Berlin).

[2] M. Delgado, J.L. Verdegay and M.A. Vila (1989), A General Model for Fuzzy Linear Programming, *Fuzzy Sets and Systems* **29**: 21-29.

[3] M. Delgado, J.L. Verdegay and M.A. Vila (1994), Fuzzy Numbers, Definitions and Properties, *Mathware & Soft Computing* **1**: 31-43.

[4] T.E. Gantner, R.C. Steinlage and R.H. Warren (1978), Compactness in fuzzy topological spaces, *J. Math. Anal. Appl.* **62**: 547-562.

[5] G. Gerla (1986), On the concept of fuzzy point, *Fuzzy Sets and Systems* **18**: 159-172.

[6] G. Gerla (1986), Generalized Fuzzy Points, *J. Math. Anal. Appl.* **120**: 761-768.

[7] S. Gottwald (1981), Fuzzy points and local properties of fuzzy topological spaces, *Fuzzy Sets and Systems* **5**: 199-201.

[8] J.A. Herencia (1996), Graded Sets and Points: A stratified approach to Fuzzy Sets and Points, *Fuzzy Sets and Systems* **77**: 191-202.

[9] J.A. Herencia (1996), Graded numbers and graded convergence of fuzzy numbers, *Fuzzy Sets and Systems* **88**: nr. 2.

[10] J.A. Herencia (1996), *Origen y uso de los conjuntos graduales en la Matemática Difusa*, Tesis Doctoral (Universidad de Granada).

[11] U. Höhle (1978), Probabilistische Topologien, *Manuscripta Math.* **26**: 223-245.

[12] B. Hutton (1975), Normality in fuzzy topological spaces, *J. Math. Anal. Appl.* **50**: 74-79.

[13] O. Kaleva and S. Seikkala (1984), On fuzzy metric spaces, *Fuzzy Sets and Systems* **12**: 215-229.

[14] E.E. Kerre (1980), Fuzzy Sierpinski space and its generalizations, *J. Math. Anal. Appl.* **74**: 318-324.

[15] E.E. Kerre (1989), A call for crispness in Fuzzy Set Theory, *Fuzzy Sets and Systems* **29**: 57-65.

[16] R. Lowen (1983), On $(\mathbb{R}(L), \oplus)$, *Fuzzy Sets and Systems* **10**: 203-209.

[17] H.T. Nguyen (1978), A Note on the Extension Principle for Fuzzy Sets, *J. Math. Anal. Appl.* **64**: 369-380.

[18] Pu PaoMing and Liu YingMing (1980), Fuzzy topology. Part 1: Neighborhood structure of fuzzy point and Moore–Smidt convergence, *J. Math. Anal. Appl.* **76**: 571-599.

[19] D. Ralescu (1979), A survey of the representation of fuzzy concepts and its applications, in: M.M. Gupta, R.K. Ragade and R.R. Yager, Eds., *Advances in Fuzzy Sets Theory and Applications* (North-Holland, Amsterdam) 77-91.

[20] S.E. Rodabaugh (1982), Fuzzy addition in the L-fuzzy real line, *Fuzzy Sets and Systems* **8**: 39-52.

[21] J.L. Verdegay (1980), *Problemas de Decisión en ambiente difuso*, Tesis Doctoral (Universidad de Granada).

[22] J.L. Verdegay (1982), Fuzzy Mathematical Programming, in: M.M. Gupta and E. Sanchez (eds.), *Fuzzy Information and Decision Processes* (North-Holland, Amsterdam) 231-237.

[23] C.K. Wong (1974), Fuzzy points and local properties of fuzzy topology, *J. Math. Anal. Appl.* **46**: 316-328.

[24] L.A. Zadeh (1975), The concept of a linguistic variable and its applications to Approximate Reasoning, *Inform. Sci.* **8**: 199-249 (part I), 301-357 (part II), **9** (1976): 43-80 (part III).