Fuzzy Approximation Relations, Modal Structures and Possibilistic Logic*

F. Esteva[1], P. Garcia [1], L. Godo [1] and R.O. Rodriguez[2]

[1] Institut d'Intel.ligència Artificial – CSIC
Campus Univ. Autònoma Barcelona, s/n. 08193 Bellaterra, Spain
[2] Dept. de Ciencias de la Computación
Universidad Nacional de Buenos Aires, Argentina
e-mail: {*esteva,garcia,godo,ricardo*}*@iiia.csic.es*

Abstract

The paper introduces a general axiomatic notion of approximation mapping, a mapping that associates to each crisp proposition p a fuzzy set representing "approximately p". It is shown how it can be obtained through fuzzy relations, which are at least reflexive. We study the corresponding multi-modal systems depending on the properties satisfied by the approximate relation. Finally, we show some equivalences between possibilistic logical consequences and global/local logical consequences in the multi-modal systems.

Keywords: approximation relation, similarity, approximate reasoning, multi-modal system, possibilistic logic.

1   Introduction

Recently, similarity-based reasoning has been investigated from different perspectives [1, 2, 3, 7, 9, 12, 16, 17, 18, 22, 23]. The kind of statements which are in the scope of similarity-based reasoning are of the form *if p is true then q is close to be true*, in the sense that, although it may be false, *q* is semantically *close or similar* to some other proposition which is true. The matter is to take into account that some situations resemble more than anothers. Apart from some qualitative approaches like [13] or [23], the formalism which has been considered in most of the above papers is related somehow to the approach initiated by Rupini in [18], which is based on the introduction of a fuzzy binary relation

$$S : W \times W \to [0,1]$$

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called similarity relation, that maps pairs of possible worlds into numbers between 0 and 1. This function captures a notion of semantical proximity between possible worlds, with a value of 1 corresponding to the identity of possible worlds and a value 0 indicating that knowledge of propositions that are true in one possible world does not provide any indication about propositions that are true in the other. Some requirements must be satisfied by similarity relations. Usually, the similarity degree of any world with itself has the highest value ($S(\omega, \omega) = 1$, i.e. it is *reflexive*), it is *symmetric* ($S(\omega, \omega') = S(\omega', \omega)$) and satisfies a relaxed form of *transitivity*. This is expressed as : $S(\omega, \omega") \geq (S(\omega, \omega') \otimes S(\omega', -\omega"))$, where, in general, $\otimes$ is considered to be a triangular norm (or t-norm).

However, it can be argued that a notion of semantical proximity or similarity may be also captured by fuzzy relations which are not necessarily neither symmetric nor transitive. Reflexivity is much harder to be dropped out. In this paper we focus on weaker relations than similarity relations. Namely we introduce an axiomatic definition of what we call *upper approximation mappings* which are proved to be obtained through the so-called approximation relations, i.e, fuzzy reflexive relations on the set of interpretations. In section 2, axioms characterising approximation mappings and symmetric, t- norm transitive and separating approximations mappings are also provided. Following Ruspini we generalise implication and consistency measures for approximation relations. Section 3 is devoted to present semantical definitions, axiomatic systems and some completness results of the multi-modal systems which arise from these generalised implication measures. Kripke structures are defined as triples $(W, R, \models)$ where W is the set of possible worlds and R is a fuzzy relation $R : W \times W \to [0, 1]$. Local and global logical consequence relations with respect to different classes of models are studied. Finally, section 4 is devoted to the study of relationships between consequence relations in the multi-modal system with respect to different classes of models and the possibilistic consequence relation. Although possibilitic logic [4] is a logic of uncertainty and similarity (or approximation) logic is rather a logic of graded truth (see [6] for a discussion), it turns out that they share some features from a formal point of view. Previous results in that direction already appear in [7] and [14, 15, 16]. In this paper we extend some results appearing in [16] to classes of models defined by different types of approximation mappings.

2   Upper Approximation mappings

In this section we give an axiomatic approach to upper approximation mappings of classical propositions and prove to be in direct relation with a kind of fuzzy relations for which we generalise the notions of implication and consistency measures.

Let **L** be a classical propositional language obtained from a *finite* set of propositional symbols and connectives $\vee, \wedge, \to, \neg$ and let Ω be the (finite) set of boolean interpretations of **L**. We will denote by $[p]$ the subset of interpretations w that make the proposition p true, written $w \models p$, and by $\delta_p : \Omega \to \{0, 1\}$ its characteristic function. It is well known that the mapping between propositions and subsets of interpretations given by $p \to [p]$ is one-to-one and so we will sometimes

identify, if no confusion is possible, an interpretation w with the proposition p_w such that $[p_w] = \{w\}$. Besides, we will denote by $PF(\Omega)$ the set of fuzzy sets on Ω with values on [0,1], by μ_F the characteristic function of a fuzzy set $F \in PF(\Omega)$ and by $[F]_\alpha$ the a-cut of a fuzzy set F, that is, $[F]_\alpha = \{\omega \in W \mid \mu_F(\omega) \geq \alpha$. Finally we will use the symbol $\otimes$ to denote an arbitrary t-norm on the unit interval [0,1].

In previous papers [3, 9], the starting point to model similarity- based reasoning was to assume to have a fuzzy similarity relation, i.e. a t-norm transitive, symmetric and reflexive fuzzy relation [25, 20], defined on the set of interpretations Ω. A fuzzy relation $R : \Omega \to [0, 1]$ is:

- reflexive if $R(\omega, \omega) = 1$, $\forall \omega \in \Omega$,

- symmetric if $R(\omega, \omega') = R(\omega', \omega)$, $\forall \omega, \omega' \in \Omega$

- $\otimes$-transitive if $R(\omega, \omega') \otimes R(\omega', \omega") \leq R(\omega, \omega")$, $\forall \omega, \omega', \omega" \in \Omega$.

Usually a similarity relation is also required to be separating:

- R is separating if $R(\omega, \omega') = 1$ iff $\omega = \omega'$, $\forall \omega, \omega' \in \Omega$.

However, in this paper our main interest is in the weaker notion of approximation relations, modelling statements of the type *p is approximately described by q*, and for which we only require to be reflexive fuzzy relations.

Let $R : \Omega \to [0, 1]$ be a reflexive fuzzy relation. Then, for each proposition $p \in \mathbf{L}$, we define the fuzzy set p^* of interpretations which approximately describe p by:

$$\mu_{P^*}(\omega) = \max \{R(\omega, \omega') | \omega' \in [p]\} \quad (1)$$

Actually, p^* is an (fuzzy) upper approximation of p in the sense that p^* includes $[p]$, i.e. it holds,

A1: $\mu_{p^*} \geq \delta_p$.

Moreover, it can be easily checked that the kind of upper approximations defined by (1) is functional w.r.t the disjunction: the upper approximation of a disjunction is the union of the upper approximations, that is, it holds

A2: $\left(\vee_{i \in I} p_i\right)^* = \cup_{i \in I} (p_i)^*$,

where $\cup$ denotes the fuzzy set union defined by the maximum, i.e. $\mu_{\cup_{i \in I}(p_i)^*}(\omega) = \max_{i \in I} \mu_{(p_i)^*}(\omega)$. What it is interesting is that **A1** and **A2** completely characterize fuzzy upper approximations which are definable from reflexive fuzzy relations by (1), and therefore they provide an axiomatic definition of (fuzzy) upper approximations.

Proposition 1 *A mapping* $* : \mathbf{L} \to PF(\Omega)$, *satisfies* **A1** *and* **A2** *if and only if there exists a reflexive fuzzy relation* $R : \Omega \to [0,1]$ *such that*

$$\mu_{p^*}(\omega) = max\{R(\omega,\omega') \mid \omega' \in [p]\}$$

for any proposition $p \in \mathbf{L}$.

Proof: If $* : \mathbf{L} \to PF(\Omega)$ satisfies **A1** and **A2** then the relation defined as $R(\omega',\omega) = \mu_{\omega *}(\omega')$, for all $\omega, \omega' \in \Omega$ does the job. □

From now on, mappings satisfying **A1** and **A2** will be simply referred as upper approximation mappings. Proposition 1 shows that upper approximations are determined by the upper approximations of the interpretations. Moreover, it is not difficult to see which are the required properties needed to characterize upper approximations defined by fuzzy relations stronger than reflexive.

Proposition 2 *Given an upper approximation mapping* $* : \mathbf{L} \to PF(\Omega)$, *consider the fuzzy relation* R *on* Ω *defined by* $R(\omega',\omega) = \mu_{\omega *}(\omega')$. *Then the following conditions hold:*

1. *R is symmetric iff, for every $p,q \in \mathbf{L}$, $[p^*]_\alpha \cap [q] \neq \emptyset$ iff $[q*]_\alpha \cap [p] \neq \emptyset$ for every $\alpha \in [0,1]$.*

2. *R is $\otimes$-transitive iff, for every $p,q,r \in \mathbf{L}$,*

$$[p] \subseteq [q*]_\alpha \; and \; [q] \subseteq [r*]_\beta \; imply \; [p] \subseteq [r*]_{\alpha \otimes \beta}$$

   for every $\alpha, \beta \in [0,1]$.

3. *$R*$ is separating iff $[p^*]_1 = [p]$, for every $p \in \mathbf{L}$.*

Proof: In one direction, take p,q and r as propositions such that $[p], [q]$ and $[r]$ are singletons. Then the corresponding conditions for R are easily verified. The other direction is obvious for cases (1) and (3), so we will only prove case (2). Let p,q and r be propositions and let R be $\otimes$- transitive. Suppose $[p] \subseteq [q^*]_\alpha$ and $[q] \subseteq [r^*]_\beta$. Recall that $[p] \subseteq [q^*]_\alpha$ means that for all $\omega \models p$ there exists ω' such that $\omega' \models q$ and $R(\omega,\omega') \geq \alpha$. For this ω', since $[q] \subseteq [r*]_\beta$, there exists ω'' such that $\omega'' \models r$ and $R(\omega',\omega'') \geq \beta$. Therefore, by the $\otimes$-transitivity of R we have $R(\omega,\omega'') \geq R(\omega,\omega') \otimes R(\omega',\omega'') \geq \alpha \otimes \beta$. Thus, $[p] \subseteq [r*]_{\alpha \otimes \beta}$. □

3   Multi-modal Systems based on Approximation Relations

Generalizing Ruspini's definition for similarity relations [18], given an appproximation relation R, we can define its corresponding implication measure on pairs of propositions as:

$$I_R(p \mid q) = \inf_{\omega \models q} \sup_{\omega' \models p} R(\omega,\omega') = \inf_{\omega \models q} \mu_{p^*}(\omega).$$

It is obvious that if q is a proposition such that $[q]$ is a singleton $\{\omega\}$, then, identifying q and ω, we have that $I_R(p \mid \omega) = \mu_{p^*}(\omega)$. Moreover, $I_R(p \mid q)$ provides the degree with which p can be considered an upper approximation of q, in the sense that the following relation holds:

$$
\begin{aligned}
I_R(p \mid q) \geq \alpha \quad &\text{iff} \quad [q] \subseteq [p^*]_\alpha \\
&\text{iff} \quad \text{for each } \omega \in [q], \text{ there exists } \omega' \in [p] \text{ and } R(\omega, \omega') \geq \alpha.
\end{aligned}
$$

By interpreting the fuzzy relation R as a graded accesibility relation, this last condition makes clear the suitability of a modal framework to capture some form of reasoning based on upper approximations, in an analogous way the modal systems introduced in [9] model similarity-based reasoning. In the rest of this section we provide the main semantical notions of multi-modal systems based on fuzzy approximation relations which will be used in next section.

- Modal Language: Formulas of the new language $\mathcal{L}$ are built over the formulas of **L** by adding modal operators $\Diamond_\alpha^c$ and $\Diamond_\alpha^o$ for every rational $\alpha \in [0, 1]$.

- Approximation Kripke Models: An approximation Kripke model is a structure $M = (W, R, \models_M)$ where W is the set of possible worlds, R is a reflexive fuzzy relation on W, and $\models_M : W \times Prop \to \{0, 1\}$ gives, for each world a truth-value assignment of propositional variables ($Prop$) of **L**. We shall write $(M, \omega) \models p$ for $\models_M (\omega, p) = 1$.

- Satisfiability: Let $M = (W, R, \models_M)$, $\omega \in W$ and A be a formula of $\mathcal{L}$. Then, we define:
  a) $(M, \omega) \models \Diamond_\alpha^c A$ if $I_R^M(A \mid \omega) \geq \alpha$,
  b) $(M, \omega) \models \Diamond_\alpha^o A$ if $I_R^M(A \mid \omega) > \alpha$.

The rest of conditions are the usual ones.

Note that this notion of satisfiability needs a definition of implication measure for modal formulas since the definition given above is only valid for non modal formulas. Nevertheless, the implication measure for modal formulas A is defined as a natural extension in the following way,

$$
I_R^M(A \mid \omega) = \sup\{R(\omega, \omega') \mid (M, \omega') \models A\}.
$$

We can also introduce the corresponding family of dual modal operators $\Box_\alpha^c$ and $\Box_\alpha^o$ as $\neg \Diamond_\alpha^c \neg$ and $\neg \Diamond_\alpha^o \neg$ respectively, and whose satisfiability conditions are:
c) $(M, \omega) \models \Box_\alpha^c A$ if $I_R^M(\neg A \mid \omega) < \alpha$,
d) $(M, \omega) \models \Box_\alpha^o A$ if $I_R^M(\neg A \mid \omega) \leq \alpha$.

In the case when W is finite, $\Diamond_\alpha^c$ and $\Box_\alpha^c$ have usual Kripke semantics with respect to the accesibility relation R_α^c defined as

$$
\omega R_\alpha^c \omega' \quad \text{iff} \quad R(\omega, \omega') \geq \alpha.
$$

In contrast, the *strict* cuts R_α^o of R, i.e. $\omega R_\alpha^o \omega'$ iff $R(\omega, \omega') > \alpha$, always provide the modal operators $\Diamond_\alpha^o$ and $\Box_\alpha^o$ with usual Kripke semantics, even when W is not

finite. As usual, the above definitions of satisfiablity can be extended to the notion of validity of a formula A with respect to a model M, written by $\models_M A$, and with respect to a class of models Σ, written by $\Sigma \models A$.

- Logical consequence: We will consider three different definitions in this paper. Given a set of formulas Γ and a formula A, then we define:

  1. *Logical consequence* inside a model M (written $\models_M$): $\Gamma \models_M A$ iff $(\forall \omega \in W)[(M, \omega) \models B$ for every $B \in \Gamma$ implies $(M, \omega) \models A]$.

  2. *Local* logical consequence in a class of models Σ (written $\models_{l\Sigma}$): $\Gamma \models_{l\Sigma} A$ iff $(\forall M \in \Sigma)(\Gamma \models_M A)$

  3. *Global* logical consequence in a class of models Σ (written $\models_{g\Sigma}$): $\Gamma \models_{g\Sigma} A$ iff $(\forall M \in \Sigma)(\models_M \Gamma$ implies $\models_M A)$.

  Notice that the notion of local consequence is stronger than the notion of global consequence, i.e. $\Gamma \models_{l\Sigma} A$ always implies $\Gamma \models_{g\Sigma} A$, but the converse is not true in general.

In [9] the authors studied the corresponding multi-modal system for the case of Kripke models corresponding to separating similarity relations. In [14, 15] Liau and Lin define a multi-modal system like the one presented here. One goal of that paper is the relationship of their modal system with *possibilistic logic* and therefore they consider models such that the relation R only satisfies the so-called *serial* property, i.e. for all $\omega \in W$, $\sup_{\omega' \in W} R(\omega, \omega') = 1$. Obviously this property is weaker than reflexivity, but to model approximation mappings it does not seem meaningful to consider serial relations which are not reflexive, since in that case the corresponding mapping might be such that the approximation p^* of a proposition p could not contain the set $[p]$ of interpretations of p. Nevertheless, for the sake of a global perspective about the already known results of axiomatization of different multi-modal systems, we shall consider all the following classes of models:

$\Sigma_0 \quad = \quad \{(W, R, \models) \mid R$ is a fuzzy relation $\}$,

$\Sigma_1 \quad = \quad \{(W, R, \models) \mid R$ is a serial fuzzy relation $\}$,

$\Sigma_2 \quad = \quad \{(W, R, \models) \mid R$ is a reflexive fuzzy relation $\}$,

$\Sigma_3 \quad = \quad \{(W, R, \models) \mid R$ is a reflexive and symmetric fuzzy relation $\}$,

$\Sigma_\otimes \quad = \quad \{(W, R, \models) \mid R$ is a $\otimes$- similarity relation $\}$.

Moreover, we will use the notation Σ_i^* to denote the subclass of Σ_i $(i \in \{1, 2, 3, \otimes\})$ where the fuzzy relation is separating as well. As it is obvious, we have that $\Sigma_0 \supset \Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset \Sigma_\otimes$, and therefore, their corresponding sets of valid formulas fullfil the inverse inclusionships. As for the axiomatic characterization of the different multi-modal systems, let us consider the following schemes, where G denotes the rang of the fuzzy relations and it is assumed to be of the form $\{0, 1\} \subseteq G \subseteq [0, 1]$ and closed with respect to the operation $\otimes$:

K^c:　　$\Box_\alpha^c (A \to B) \to (\Box_\alpha^c A \to \Box_\alpha^c B), \forall \alpha \in G$

K^o:　　$\Box_\alpha^o (A \to B) \to (\Box_\alpha^o A \to \Box_\alpha^o B), \forall \alpha \in G$

D:　　　$\Box_1^c A \to \Diamond_1^c A$

T^c:　　$\Box_\alpha^c A \to A, \forall \alpha \in G$

T^o:　　$\Box_\alpha^o A \to A,$ for $\alpha < 1$

C^c:　　$A \to \Box_1^c A$

B^c:　　$A \to \Box_\alpha^c \Diamond_\alpha^c A,$ for $\alpha > 0$

B^o:　　$A \to \Box_\alpha^o \Diamond_\alpha^o A, \forall \alpha \in G$

4^c:　　$\Box_{\alpha \otimes \beta}^c A \to \Box_\beta^c \Box_\alpha^c A, \forall \alpha, \beta \in G$

4°:　　$\Box_{\alpha \otimes \beta}^o A \to \Box_\beta^o \Box_\alpha^o A, \forall \alpha, \beta \in G$

N^c:　　$\Box_\alpha^c A \to \Box_\beta^c A,$ for $\beta \geq \alpha,$

N^o:　　$\Box_\alpha^o A \to \Box_\beta^o A,$ for $\beta \geq \alpha$

EX^c:　$\Diamond_0^c A,$

EX^o:　$\neg \Diamond_1^o A,$

CO:　　$\Box_\alpha^c A \to \Box_\alpha^o A, \forall \alpha \in G$

OC:　　$\Box_\alpha^o A \to \Box_\beta^c A,$ for $\alpha < \beta,$

and the following inference rules

MP:　　From A and $A \to B$ infer B

RN^c:　From A infer $\Box_\alpha^c A,$ for $\alpha > 0$

RN^o:　From A infer $\Box_\alpha^o A, \forall \alpha \in G$

In [14, 15] Liau and Lin propose a Quantitative modal logic (QML) with $G = [0, 1]$ and prove the following completeness results (in the following, PL stands for propositional tautologies):

- The axiom system **SK** consisting of PL, K^c, K^o, CO, OC, EX^c, EX^o, together with the MP and RN^o inference rules is complete with respect the class of models Σ_0.

- The axiom system **SKD** = SK + D is complete with respect to the class of models Σ_1.

- The axiom system **SKT** = SK + T^c is complete with respect to the class of models Σ_2.

In [9] the authors prove further completeness results:

- The axiom system $\mathbf{MS5^{++}(G, min)}$ = SKT + B^o + B^c + 4° + 4^c + C^c plus MP and RN^c is complete with respect to the subclass of finite models of $\Sigma_\otimes^*$ when G is a dense and denumerable and $\otimes = \min$.

- If G is finite, then the axiom system $\mathbf{MS5^+(G, \otimes)}$ consisting of PL, K^c, T^c, B^c, 4^c, C^c, N^c, EX^c, plus MP and RN^c is complete with respect to the class of models $\Sigma_\otimes^*$, for any t-norm $\otimes$. Notice that in this case the open and closed modalities are interdefinable, and the resulting modal system can be simplified to :

PL: Tautologies of propositional logic,
K: $\Box_\alpha(A \to B) \to (\Box_\alpha A \to \Box_\alpha B)$,
T: $\Box_\alpha A \to A$,
B: $A \to \Box_\alpha \Diamond_\alpha A$,
4: $\Box_{\alpha \otimes \beta} A \to \Box_\beta \Box_\alpha A$,
C: $A \to \Box_1 A$,
N: $\Box_\alpha A \to \Box_\beta A$, with $\beta \geq \alpha$,
EX: $\Diamond_0 A$,

where $\Box_\alpha$ stands for $\Box_\alpha^c$.

Moreover, it can be easily checked the following:

- If we remove axiom C^c from the system **MS5^{++}(G, min)** we get a complete system with respect to the subclass of finite models of $\Sigma_\otimes$ when G is dense and denumerable and the $\otimes = \min$.

- If G is finite and we remove axiom C^c from the system **MS5$^+$(G, min)** we get a complete system with respect to $\Sigma_\otimes$.

- If G is finite and we remove axiom 4^c ($+ C^c$) from the system **MS5$^+$(G, min)** we get a complete system with respect to Σ_2^* (w.r.t. Σ_2 respectively).

4 Relationship to Possibilisitic Logic

In [16] the relationships between similarity-based multi-modal systems and possibilistic logic (PL) were explored. In this section we will complete and extend those results by taking into account the multi-modal systems corresponding to different classes of approximation relations. The basic features of Possibilistic logic[1] [4] are the following ones. Possibilistic formulas are pairs of classical propositions and lower bounds of necessity or possibility measures, so the language of possibilistic logic over a propositional language **L** is $L_{PL} = \{(p, \Pi_\alpha), (p, N_\alpha) \mid p \in \mathbf{L}, \alpha \in [0,1]\}$. Models are normalized possibility distributions $\pi : \Omega \to [0,1]$, where Ω is the set of classical interpretations of the propositional language **L**. Possibilistic satisfiability relation is defined by:

$\pi \models_{PL} (p, \Pi_\alpha)$ iff $Poss_\pi(p) = \sup\{\pi(\omega) \mid \omega \models p\} \geq \alpha$,
$\pi \models_{PL} (p, N_\alpha)$ iff $Nec_\pi(p) = 1 - Poss_\pi(\neg p) \geq \alpha$,

The notion of logical consequence in PL is defined as usual: a possibilistic formula φ is a logical consequence of a set of possibilistic formulas Γ, written $\Gamma \models_{PL} \varphi$, if for every normalised possibility distribution π such that $\pi \models_{PL} \psi$, for all $\psi \in \Gamma$, it is the case that $\pi \models_{PL} \varphi$. Let us recall now some results relating similarity relations and possibility distributions given in [7]. First two definitions:

1. Given a similarity relation S and a subset A of Ω, we may define the following possibiliy distribution:

$$\pi_{S,A}(\omega) = \sup\{S(\omega, \omega') \mid \omega' \in A\}.$$

---

2. Given a possibility distribution π, we may build the similarity relation S_π generated by π according to Valverde's representation theorem [21] as follows:

$$S_\pi(\omega, \omega') = \min(\pi(\omega) \otimes \to \pi(\omega'), \pi(\omega') \otimes \to \pi(\omega))$$

being $\otimes \to$ the residuated implication defined by $\otimes$.

Lemma 1 ([7]) *The following relations hold:*
   (i)   *If $S = S_\pi$ and $A \subseteq Core(\pi) = \{\omega \mid \pi(\omega) = 1\}$, then $\pi_{S,A} = \pi$.*
   (ii)   *If S' is the similarity relation defined from $\pi_{S,A}$, A being a subset of Ω, then $S' \geq S$ and $S'(\omega, \omega') = 1$ for every $\omega, \omega' \in A$.*

In order to relate the modal systems and possibilistic logic we need some preliminar settings. Possibilistic models are defined on the set Ω of classical interpretations of **L**. Therefore in all this section we shall restrict ourselves to approximation Kripke models where the set of worlds is just Ω, that is, models of the type $M_R = (\Omega, R, \models)$, where R is an approximation relation on Ω, and thus fulfil $(M_R, \omega) \models p$ iff $\omega(p) = true$, for every propositional variable p of **L**. We will also need a way to translate possibilistic into multi-modal formulas. The following proposition shows how to do this.

Proposition 3 *Given a similarity Kripke model $M_S = (\Omega, S, \models)$ and a non-modal formula $p \in$ **L**, then, for all $\omega \in \Omega$, the following equivalences hold:*
   (i) $(\omega, M_S) \models \Diamond_\alpha^c p$ *iff* $\pi_{S,\omega} \models_{PL} (p, \Pi_\alpha)$,
   (ii) $(\omega, M_S) \models \Box_\alpha^o p$ *iff* $\pi_{S,\omega} \models_{PL} (p, N_{1-\alpha})$.

Proof: One can show that $Poss_{\pi_{S,\omega}}(p) = I_S(p \mid \omega)$. From that, the proposition easily follows. $\square$

Therefore, the natural transformation **T** from the language L_{PL} of possibilistic logic to the language $\mathcal{L}$ of multi-modal logic we propose is defined as:

$$\mathbf{T}(p, \Pi_\alpha) = \Diamond_\alpha^c p \quad \text{and} \quad \mathbf{T}(p, N_{1-\alpha}) = \Box_\alpha^o p.$$

It is interesting to remark that **T** is one-to-one but the image set

$$\mathcal{L}' = \mathbf{T}(L_{PL}) = \{\Diamond_\alpha^c p, \Box_\alpha^o p \mid p \in \mathbf{L}, \alpha \in [0,1]\}$$

is only an strict sublanguage of $\mathcal{L}$. Next theorem proves that the possibilistic consequence is equivalent to the local consequence in the multi-modal system with respect to the class of models $\Sigma_\otimes$ corresponding to $\otimes$-similarity relations.

Theorem 1 *For any proposition $p \in$ **L**, any subset $\Gamma \subseteq L_{PL}$ and any $\alpha \in [0,1]$, the following equivalences hold:*
   (i) $\Gamma \models_{PL} (p, \Pi_\alpha)$ *iff* $\mathbf{T}(\Gamma) \models_{l\Sigma_\otimes} \Diamond_\alpha^c p$,
   (ii) $\Gamma \models_{PL} (p, N_{1-\alpha})$ *iff* $\mathbf{T}(\Gamma) \models_{l\Sigma_\otimes} \Box_\alpha^o p$,
where $\mathbf{T}(\Gamma) = \{\mathbf{T}(\varphi) \mid \varphi \in \Gamma\}$.

Proof: We will prove only the first equivalence. Suppose that $\mathbf{T}(\Gamma) \models_{l\Sigma_\otimes} \Diamond_\alpha^c p$, this is, for every ω of Ω, if $(M_S, \omega) \models \mathbf{T}(\varphi)$, for every φ of Γ, then $(M_S, \omega) \models \Diamond_\alpha^c p$. Now let π such that $\pi \models_{PL} \Gamma$ (i.e. $\pi \models_{PL} \psi$, for every $\psi \in \Gamma$) . By Lemma 1, there exist $S = S_\pi$ and $\omega \in Core(\pi)$ such that $\pi = \pi_{S,\omega}$. Thus applying (i) of Proposition 3, $(M_S, \omega) \models \mathbf{T}(\psi)$ for every $\psi \in \Gamma$, and by our asumption, $(M_S, \omega) \models \Diamond_\alpha^c p$, which is equivalent, by (i) of Proposition 3, $\pi \models_{PL} (p, \Pi_\alpha)$. Therefore we have proved that $\Gamma \models_{PL} (p, \Pi_\alpha)$. The proof of the converse is analogous taking into account the second equivalence of Proposition 3.   □

These equivalences do not hold for the global consequence relation, as the following counterexample shows.

Counterexample 1. Let p and q be two different *maximal elementary conjunctions*, and let $[p] = \{\omega'\}$ and $[q] = \{\omega''\}$ with $\omega' \neq \omega''$. Let $\alpha \in (0, 1]$ such that $\alpha \otimes \alpha \neq 0$. Then, the following global logical consequence holds:

$$\Diamond_\alpha^c p \models_{g\Sigma_\otimes} \Diamond_{\alpha\otimes\alpha}^c q.$$

The proof is as follows. If a similarity relation S satisfies $I_S(p \mid \omega) = S(\omega, \omega') \geq \alpha$ for every $\omega \in \Omega$, then $I_S(q \mid \omega) = S(\omega, \omega'') \geq S(\omega, \omega') \otimes S(\omega', \omega'') \geq \alpha \otimes \alpha$. But that consequence relation is not true in the local sense. Namely, if S is the minimal classical equivalence relation ($S(\omega, \omega') = 1$ if $\omega = \omega'$ and $S(\omega, \omega') = 0$ otherwise), it is obvious that

$$(M_S, w') \models \Diamond_\alpha^c p \quad \text{and} \quad (M_S, w') \not\models \Diamond_{\alpha\otimes\alpha}^c q.$$

Perhaps it is even more evident that the "translated" consequence relation is not true in possibilistic logic, that is,

$$(p, \Pi_\alpha) \not\models_{LP} (q, \Pi_{\alpha\otimes\alpha}).$$

An easy calculation shows that the possibility distribution π defined by $\pi(\omega) = 1$ if $\omega = \omega'$ and $\pi(\omega) = 0$ otherwise, satisfies the left hand side but not the right hand side of the consequence relation.   □

On the other hand, Liau and Lin prove in [16] that the possibilistic logical consequence coincides with the global multi-modal consequence relation with respect to the class of serial models Σ_0. Of course, the multi-modal language has to be restricted also to possibilistic formulas, i.e. formulas of the type $\Diamond_\alpha^c p$ and $\Box_\alpha^o p$, p being a classical proposition. We present next an extension of this result based on the following propositions.

Proposition 4 *Given a serial relation R on Ω, and for every $\omega \in \Omega$ let π_ω be the possibility distribution defined by $\pi_\omega(\omega') = R(\omega', \omega)$, for all $\omega' \in \Omega$. Then, for any $p \in \mathbf{L}$ and $\alpha \in [0, 1]$, the following equivalences hold:*
  (i) $(\omega, M_R) \models \Diamond_\alpha^c p$ *iff* $\pi_\omega \models_{PL} (p, \Pi_\alpha)$,
  (ii) $(\omega, M_R) \models \Box_\alpha^o p$ *iff* $\pi_\omega \models_{PL} (p, N_{1-\alpha})$.

Proof: The proposition follows from the following equalities: $I_R(p \mid \omega) = \sup\{R(\omega', \omega) \mid \omega' \models p\} = \sup\{\pi_\omega(\omega') \mid \omega' \models p\} = Poss_{\pi_\omega}(p)$.   □

Proposition 5 *Given a possibility distribution π, let R_π be the serial relation given by $R_\pi(\omega, \omega') = \pi(\omega)$. Then for any $p \in \mathbf{L}$ and $\alpha \in [0, 1]$, the following equivalences hold:*

(i) $\pi \models_{PL} (p, \Pi_\alpha)$ iff,   for all ω, $(\omega, M_{R_\pi}) \models \Diamond_\alpha^c p$,

(ii) $\pi \models_{PL} (p, N_{1-\alpha})$ iff,   for all ω, $(\omega, M_{R_\pi}) \models \Box_\alpha^o p$.

The proof is analogous to the one of Proposition 4. The main result is the following one which extends Liau and Liu's results in [16] which are given only for global consequence.

Theorem 2 *Let Σ_0 be the class of serial models. Then for any $p \in \mathbf{L}$ and $\alpha \in [0, 1]$, the following equivalences hold:*

(i) $\Gamma \models_{PL} (p, \Pi_\alpha)$ iff $\mathbf{T}(\Gamma) \models_{l\Sigma_0} \Diamond_\alpha^c p$ iff $\mathbf{T}(\Gamma) \models_{g\Sigma_0} \Diamond_\alpha^c p$,

(ii) $\Gamma \models_{PL} (p, N_{1-\alpha})$ iff $\mathbf{T}(\Gamma) \models_{l\Sigma_0} \Box_\alpha^o p$ iff $\mathbf{T}(\Gamma) \models_{g\Sigma_0} \Box_\alpha^o p$.

Proof: The proof is an easy consequence of propositions 3 and 4, taking into account that $R_\pi(\omega, \omega') = \pi(\omega)$, i.e. R_π is not dependent on the second variable. As a consequence, a formula is satisfiable in a world ω of the model (ω, M_{R_π}) iff it is satisfiable in the model M_{R_π}. This implies that the local and global consequence for formulas of $\mathbf{T}(L_{PL})$ with respect to serial models coincide. $\square$

Remark. From Proposition 4, and as it is noted in [16], it is obvious that serial models can be determined by giving, for every $\omega \in W$, a possibility distribution π_ω, i.e. R and the family of π_ω's are related by the equality $R(\omega', \omega) = \pi_\omega(\omega')$. In some sense we can identify the consequence relation with respect to serial models with some kind of local possibilistic logic in the sense that for each world we have a different possibility distribution π_ω. But surprisingly, the consequence relation with respect to the class of serial models coincides with possibility consequence relations. This result seems to indicate that this type of "local possibilistic logic" has no sense because it coincides with usual possibilistic logic.

Finally we turn our attention to logical consequences with respect to the classes Σ_1 and Σ_2 corresponding to approximation and symmetric approximation mappings respectively.

Theorem 3 *For any $p \in \mathbf{L}$ and $\alpha \in [0, 1]$, the following equivalences hold:*

(i) $\Gamma \models_{PL} (p, \Pi_\alpha)$ iff $\mathbf{T}(\Gamma) \models_{l\Sigma_1} \Diamond_\alpha^c p$ iff $\mathbf{T}(\Gamma) \models_{l\Sigma_2} \Diamond_\alpha^c p$,

(ii) $\Gamma \models_{PL} (p,, N_{1-\alpha})$ iff $\mathbf{T}(\Gamma) \models_{l\Sigma_1} \Box_\alpha^o p$ iff $\mathbf{T}(\Gamma) \models_{l\Sigma_2} \Box_\alpha^o p$.

Proof: The equivalences follow from the fact that they hold in the class Σ_0, which contains Σ_1 and Σ_2, and in the class $\Sigma_\otimes$, which is contained in Σ_1 and Σ_2. $\square$

Analogously to Theorem 1, (i) and (ii) of Theorem 3 do not hold for the global consequence relation. This can checked in the following counterexample.

Counterexample 2. Let $p \in \mathbf{L}$ such that p and $\neg p$ are satisfiable and let $\alpha \in (0, 1)$. Then there exists at least one world ω such that $\omega \models \neg p$ and so, for every reflexive

R, we have $I_R(\neg p \mid \omega) = 1$. Thus, for any approximation relation R, $(\Omega, R) \not\models \Box^o_\alpha p$. Therefore, for any proposition q, the following global consequence trivially holds:

$$\Box^o_\alpha p \models_{g\Sigma} \Box^o_\alpha (p \wedge q),$$

where Σ stands either for Σ_1, Σ_2 or for $\Sigma_\otimes$. But this logical consequence is obviously not true in the local sense. It is easy to see that the corresponding translated possibilistic entailment

$$(p, N_{1-\alpha}) \models_{LP} (p \wedge q, N_{1-\alpha})$$

is not true in general. $\qquad\qquad\square$

Finally, the following counterexample proves that Theorem 3 does not hold either for none of the classes of models Σ^*_1, Σ^*_2, Σ^*_3 and $\Sigma^*_\otimes$ corresponding to separating relations that are besides reflexive, proximity (reflexive and symmetric) and similarity relations respectively.

Counterexample 3. Let $p, q, r \in \mathbf{L}$ be three different propositions such that $[p] = \{\omega\}$, $[q] = \{\omega'\}$ and $[r] = \{\omega"\}$. An easy computation shows that the following local logical consequence holds:

$$\{\Diamond^c_1 p, \Diamond^c_1 q\} \models_{l\Sigma^*} \Diamond^c_1 r,$$

where Σ^* stands for any of the classes Σ^*_1, Σ^*_2, Σ^*_3 or $\Sigma^*_\otimes$. However this is not the case if the class of models is any of the classes Σ_1, Σ_2, Σ_3 or $\Sigma_\otimes$. The reason why is the separating property. No model in Σ^* satisfies the left hand side and so the logical consequence is trivially valid. But one can easily find models with non-separating relations satisfying the left hand side and not the right hand side. It suffices to take a model (ω, M_S) defined by a similarity S such that $S(\omega, \omega') = 1$ and $S(\omega, \omega") = S(\omega', \omega") \neq 1$. $\qquad\qquad\square$

The following schema summarizes the relationships which have been considered in this section. As usual, Γ and φ denote a family of possibilistic formulas and a possibilistic formula respectively and $\mathbf{T}(\Gamma)$ and $\mathbf{T}(\varphi)$ denote the corresponding formulas of $\mathcal{L}'$.

$$\mathbf{T}(\Gamma) \models_{\Sigma_0} \mathbf{T}(\varphi) \quad \Leftrightarrow \quad \mathbf{T}(\Gamma) \models_{l\Sigma} \mathbf{T}(\varphi) \quad \Rightarrow_1 \quad \mathbf{T}(\Gamma) \models_{l\Sigma^*} \mathbf{T}(\varphi)$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Downarrow_2 \qquad\qquad\qquad\qquad \Downarrow_3$$

$$\Gamma \models_{PL} \varphi \qquad \Rightarrow \quad \mathbf{T}(\Gamma) \models_{g\Sigma} \mathbf{T}(\varphi) \quad \Rightarrow \quad \mathbf{T}(\Gamma) \models_{g\Sigma^*} \mathbf{T}(\varphi)$$

where Σ stands for any class Σ_1, Σ_1 or $\Sigma_\otimes$, and Σ^* stands for the corresponding class Σ^*_1, Σ^*_2 or $\Sigma^*_\otimes$, respectively. Moreover $\mathbf{T}(\Gamma) \models_{\Sigma_0} \mathbf{T}(\varphi)$ means that the consequence is true both in the local and in the global sense. On the other hand subscripted arrows mean that the converse does not hold. Counterexample 3 proves that the converse of $\Rightarrow_1$ does not hold, and counterexamples 1 and 2 prove that the converse of $\Rightarrow_2$ does not hold. To prove that the converse of $\Rightarrow_3$ does not

hold is easy and it is left to the reader. Moreover, the following relationships hold: $\models_{g\Sigma_0}$ is *stronger* than $\models_{g\Sigma_1}$, $\models_{g\Sigma_1}$ is *stronger* than $\models_{g\Sigma_2}$, and $\models_{g\Sigma_2}$ is *stronger* than $\models_{g\Sigma_\otimes}$. These relations hold in the strict sense, i.e. it is easy to show that the converse relationships do not hold. Namely, take the axioms characterising serial, reflexive, symetric and transitive relations in the multi-modal system and prove that these axioms are valid only in the classes of models defined by relations satisfying the corresponding property. Of course, in this schema it is necessary to remark that the multi-modal system is restricted to the formulas of $\mathcal{L}'$ and so, the general result is that possibilistic logical consequence is only a restriction of logical consequence respect to the class Σ_0 and also with local logical consequence respect to the classes Σ_1, Σ_2 and $\Sigma_\otimes$. Obviously, the same arrows are valid when considering the corresponding "separating" classes Σ_i^* for $i \in \{1, 2, 3, \otimes\}$.

5   Conclusions and future work

In this paper we have proposed an axiomatic definition of so-called approximation mappings which provide, for each crisp proposition p, an upper fuzzy approximation representing the fuzzy set of interpretations which are (semantically) *close to p*. The basic type of approximation mappings have been proved to be definable through fuzzy reflexive relations. Then, we have shown some multi-modal systems accounting for the notion of approximation and finally we have explored some relationships between logical consequences in these modal systems and the possibilistic entailment. As for a natural extension of the work presented here, one could consider the problem of defining upper approximations of fuzzy propositions. For instance, a possible definition of what an upper approximation (with respect to a $\otimes$-similarity relation S on X) $\phi_S(h)$ of a fuzzy set $h : X \to [0,1]$ is has been proposed in [10] as

$$\phi_S(h)(x) = sup_{y \in X} \left(S(x,y) \otimes h(y) \right). \quad (3)$$

This kind of upper approximation also appears in [5]. Of course, this definition agrees with our approximation mapping when restricted to crisp subsets (taking $X = \Omega$). The mapping ϕ_S provides, for every fuzzy set h , the least extensional fuzzy set (w.r.t. S) containing h, and it is proved [11] to satisfy the conditions of a fuzzy closure operator in the lattice $(FP(X), \leq, \vee, \wedge)$ of fuzzy subsets of X with pointwise max and min, in the sense that it fulfils the following propeties:

- **C1**: $h \leq \phi_S(h)$,

- **C2**: $\phi_S(h1 \vee h2) = \phi_S(h_1) \vee \phi_S(h_2)$

- **C3**: $\phi_S \circ \phi_S = \phi_S$

- **C4**: $\phi_S(k) = k$, for all constant fuzzy set k (i.e. $k(x) = k$, for all $x \in X$).

Notice that **C1** and **C2** correspond to the axioms **A1** and **A2** of approximation mappings (see Section 2). Moreover, the mapping ϕ_S satisfies this further property:

- **C5$^\otimes$**: $\phi_S(\{x\} \wedge k) = \phi_S(\{x\}) \otimes k$,

where $\{x\}$ denotes a singleton, i.e. $\{x\}(x) = 1$ and $\{x\}(z) = 0$ for all $z \neq x$, and k a constant fuzzy set. **C5$^\otimes$** basically states that ϕ_S is determined by the approximations of the (crisp) singletons of X. Then, we can extend the characterization of approximation mappings given in Proposition 1 to fuzzy approximations defined by (3).

Proposition 6 *A mapping ϕ satisfies* **C1**,**C2***, and* **C5$^\otimes$** *if, and only if, there exist a fuzzy reflexive relation R on X such that $\phi = \phi_R$.*

Proof: It is easy to show that, for any fuzzy reflexive relation, ϕ_R satisfies **C1**, **C2**, and **C5$^\otimes$**. The proof of the converse is as follows. For all $x, y \in X$, define $R(x, y) = f(\{x\})(y)$. Notice that R so defined is reflexive due to **C1**. Any fuzzy set h can be written as $h = \vee_{x \in X} \{x\} \wedge k_{h(x)}$, where $k_{h(x)}$ is the constant fuzzy set $k_{h(x)}(y) = h(x)$, for all $y \in X$. Therefore, by **C2**, $\phi(h) = \vee_{x \in X} \phi(\{x\} \wedge k_{h(x)})$, and by **C5$^\otimes$**, $\phi(h) = \vee_{x \in X} \phi(\{x\}) \otimes h(x)$, and thus $\phi(h)(y) = sup_{x \in X} \phi(x)(y) \otimes h(x) = sup_{x \in X} R(y, x) \otimes h(x) = \phi_R(h)(y)$. □

So, **C1**, **C2**, and **C5$^\otimes$** are the natural counterparts of **A1** and **A2** when considering upper approximations of fuzzy sets. Moreover, **C4** is a consequence of **C1**, **C2**, and **C5$^\otimes$**. If we further require to the approximation mapping to be a closure operator (**C3**), then we are led to fuzzy $\otimes$-transitive relations.

Proposition 7 *A mapping ϕ satisfies* **C1***,* **C2***, and* **C5$^\otimes$** *and* **C3** *if, and only if, there exist a fuzzy reflexive and $\otimes$- transitive relation R on X such that $\phi = \phi_R$.*

Proof: A simple computation shows that, for any fuzzy reflexive, $\otimes$- transitive relation R, then ϕ_R satisfies **C1**, **C2**, **C3** and **C5$^\otimes$**. On the contrary if ϕ satisfies conditions **C1**, **C2**, **C3** and **C5$^\otimes$** and we define R as in Proposition 6, then one can show that the relation R is $\otimes$-transitive. It is sufficient to apply condition **C2** to a singleton, i.e. $(\phi \circ \phi)(\{x\}) = \phi(\{x\})$, and take into account condition **C5$^\otimes$**. □

Notice that the symmetry of the relation does not play any role in the axiomatic definition of closure operators. This agrees with the fact that closure operators are related to fuzzy preorderings rather than to fuzzy similarity relations (see [19] and the references there). Of course, if needed, we could add a condition to guarantee the symmetry of the relation. One obvious way is by requiring the following axiom:

- **C6**: $\phi(\{x\})(y) = \phi(\{y\}(x)$, for all $x, y \in X$.

Analogously to the approximation mappings considered in Section 2, a natural question is to look for the corresponding modal systems to these more general fuzzy approximation mappings. Obviously, in this case, the initial logical system will be many-valued rather than two-valued as in Section 3. This subject is clearly an interesting matter of future work.

Acknowledgments

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