Vagueness and Its Representations:
A Unifying Look

Maciej Wygralak
Adam Mickiewicz University
Faculty of Mathematics and Computer Science
Matejki 48/49, 60-769 Poznan, Poland

Abstract

Using the notion of a vaguely defined object, we systematize and unify different existing approaches to vagueness and its mathematical representations, including fuzzy sets and derived concepts. Moreover, a new, approximative approach to vaguely defined objects will be introduced and investigated.

1 Properties, vaguely defined objects, sets and subdefinite sets

Let us consider the class $\mathbf{P}$ composed of all properties which: (i) can be formulated in a natural language, (ii) make sense for the elements of an infinite universe $\mathbf{M}$, and (iii) do not lead to antinomy. Although the definition of $\mathbf{P}$ is not very formalized, it seems to be intuitively clear and sufficient for the purpose of this paper. The assumption (i) implies that, generally, $\mathbf{P}$ contains properties which are more or less *vague*, e.g. 'to be a *tall* man', 'to be a number *approximately equal* to 1', 'to have a *high* salary'. Therefore, the elements of $\mathbf{P}$ will be called *vague properties*. Consequently, even infinitely many intermediate states between the states of fulfilment and nonfulfilment of a property $\mathbf{p} \in \mathbf{P}$ are generally possible, and the transition from one of these extreme states to the other is generally gradual.

In particular, the number of the intermediate states can be equal to zero and, then, $\mathbf{p}$ will be called a *sharp* or *crisp property* (e.g. 'to be a prime number' or 'to be married'; indeed, either a natural number is prime or not, and any intermediate states are not possible, etc.). The transition between the states of fulfilment and nonfulfilment of $\mathbf{p}$ is now abrupt. Thus, by definition, each sharp property is also a vague property, however its "vagueness" is reduced "to zero".

We shall assume that each (vague) property from $\mathbf{P}$ separates in $\mathbf{M}$ an object, generally distinct in each case, which will be called *a vaguely defined object(VD-object*, in short). Clearly, a VD-object is a set if the property separating it is a sharp property. VD-objects which are not sets will be called *proper VD-objects*, and can be imagined as nebular objects in $\mathbf{M}$. As one knows, sets can be identified

with (sharp) properties separating them. Analogously, arbitrary VD-objects can be identified with vague properties separating them. Generally, if the properties differ, VD-objects differ, too. However, we remind that some sharp properties are "isomorphic" in that sense that they separate the same set in **M**, e.g. the properties 'to be a solution of the inequality $x^2 - 1 < 0$' and 'to be a real number between -1 and 1'. The same can happen for arbitrary vague properties from **P**.

As concerns sets, the problem of practical settlement "Does $x \in$ **M** have a (sharp) property **p** or not?" is certainly not in the domain of set theory, naive or axiomatized. However, that problem is often essential and nontrivial in solving. It happens that a set is incompletely known because, practically speaking, we are not able to indicate or to recognize all elements of that set. The difficulty lies in unknown or uncertain status of some x's with respect to **p**, caused by a lack of sufficient information/knowledge about the x's. For instance, we are practically unable to check (at least, in a reasonably short time) if an arbitrarily large integer belongs to the set of all prime numbers or not; the reader can easily give a lot of similar examples from outside mathematics. Such incompletely known sets will be called *subdefinite sets*. Obviously, the treatment of a set as a subdefinite set is generally a question of one's personal choice.

2 Mathematical representations of vaguely defined objects

In this section, we like to recall some existing approaches to the problem of mathematical representation of (proper) VD-objects and subdefinite sets.

As concerns proper VD-objects, most of the approaches make use of an intuition which incites us to apply a "generalization" of characteristic functions of sets. More precisely, a VD-object is then represented or described by a *generalized characteristic function or membership function* **M** $\rightarrow$ **V**, where **V** denotes a set differing from $\{0, 1\}$. In some approaches, one assumes that membership functions are precisely determined, at least theoretically speaking. In the others, membership functions are generally treated as imprecisely determined functions. VD-objects can be identified with their membership functions. Consequently, the classical two-valued logic, which supports the classical set theory, has to be replaced by another *supporting logic* (*sl*, in short), namely by a many-valued logic (see [5]). Throughout the paper, membership functions will be denoted by capital letters in italic *A, B, C, ...* , whereas bold capitals **A, B, C, ...** symbolize sets. The sum $A \cup B$, intersection $A \cap B$, and cartesian product $A \times B$, respectively, represent the sum, intersection, and cartesian product of the VD-objects represented by A and B, respectively, where

$$(A \cup B)(x) := A(x) \lor B(x), \ (A \cap B)(x) := A(x) \land B(x), \ (A \times B)(x, y) := A(x) \land B(y).$$

The symbols $\lor$ and $\land$, respectively, denote binary operations which are numerical interpretations (in the *sl* one uses) of inclusive disjunction and conjunction, respectively, whereas := stands for 'equals by definition' or 'denotes'. Further-

more, inclusion between VD-objects is defined in the classical way with implication interpreted by means of an implication operator $\rightarrow$ specific for the *sl* (see [5]).

On the other hand, the idea of semisets can be viewed as an attempt at representing VD-objects without the use of any generalization of characteristic functions of sets. Finally, subdefinite sets can be represented in a probabilistic or possibilistic way. The following list contains a more detailed characterization of the approaches.

I. *Representations of proper VD-objects*

   I.1. *The membership functions are used*

   I.1.1. *The membership functions are precisely determined, at least theoretically speaking*

(a) $\mathbf{V} := [0,1]$, *sl*:= infinite-valued Lukasiewicz logic L_∞.

   Consequently, $\vee:=$ max, $\wedge:=$ min and $p \rightarrow q := 1 \wedge 1 - p + q$ for $p, q \in [0,1]$.

   VD-objects are then called *fuzzy sets* (see [20], [3]). Worth mentioning here is the case of $\mathbf{V}:=$ residuated lattice and so-called *bold fuzzy sets* (see e.g. [10]).

(b) $\mathbf{V} := [0,1]$, *sl*:= intuitionistic logic.

   So, $\vee:=$ max and $\wedge:=$ min, whereas $p \rightarrow q := (1$ if $p \leq q$, else $q)$.

   VD-objects are called *intuitionistic fuzzy sets* (see [14]).

(c) $\mathbf{V} :=$ complete Heyting algebra, *sl*:= intuitionistic logic. Consequently,.

$$p \vee q := \bigvee\{p,q\}, \ p \wedge q := \bigwedge\{p,q\}, \ p \rightarrow q := \bigvee\{r : p \wedge r \leq q\} \text{ for } p, q \in \mathbf{V}.$$

   VD-objects are then called *L-fuzzy sets* or *Heyting algebra valued sets* (see [4], [6], [9], [13]).

(d) $\mathbf{V} := [0,1]$, *sl*:= intuitionistic type logic.

   Then $\wedge:=$ residual t-norm $\mathbf{t}$, $\vee:=$ related t-conorm, $p \rightarrow q := \bigvee\{r : p\mathbf{t}r \leq q\}$
.

   VD-objects collapse to *fuzzy sets with triangular norms* (see [5]).

   I.1.2. *The membership functions are generally imprecise*

(a) $\mathbf{V} := [0,1]^{[0,1]}$; VD-objects are called *type 2 fuzzy sets* (see [8]).

(b) $\mathbf{V} := \{[p,q] : p, q \in [0,1]\}$; VD-objects are called *ultrafuzzy sets or interval-valued fuzzy sets* (see [12]).

(c) Approximative approach to VD-objects in which membership functions are approximated by some other membership functions which fulfil some simple postulates. It will be presented in detail in Section 3.

I.2. *No membership functions are used*

It is evident that any assumption about preciseness of the membership functions of VD-objects is practically difficult to defend. Moreover, type 2 fuzzy sets and ultrafuzzy sets do not solve the problem of impreciseness, but they only move it a bit farther. On the other hand, one can try to find a representation of VD-objects without the use of membership functions. In [15], a new notion has been added to the Gödel-Bernays set theory, namely that of a semiset which is defined as a proper class being a subclass of a set, whereas a class is defined as a property being understood as an object. Proper VD-objects in **M** can be treated as semisets. Unfortunately, semisets do not seem to be useful as a basis for applicational theories or models. The reason is that the membership functions (in spite of their immanent imperfectness) are convenient and "handy". They make fuzzy sets and derived concepts a more constructive tool than semisets are.

II. *Representations of subdefinite sets*

II.1. *Probabilistic representations*

We shall disregard them in this paper. Probabilistic representations (e.g. *probabilistic sets* or *random sets*) deserve an exhausting treatment which goes beyond the scope of this paper. We do focus our attentions on nonclassical, possibilistic representations which seem to be more universal and flexible.

II.2. *Possibilistic representations*

(a) A subdefinite set **A** in **M** is represented by a pair (**D**, **E**) called *a partial* (or *flou*) *set* (see [2], [7], [9]). One assumes that $\mathbf{D} \subset \mathbf{E} \subset \mathbf{M}$. The set **D** contains sure elements of **A**, whereas **E** is composed of sure and possible elements of **A**. Basic operations on partial sets are defined via operations on their components. Finally, partial sets can be identified with membership functions $\mathbf{M} \to \{0, 1/2, 1\}$ and this representation via 3-valued membership functions can be extended to sums and intersections of partial sets. Worth mentioning here is also the idea of *rough sets* which is connected with the concept of knowledge bases (see e.g. [11]). Quite formally, rough sets are pairs (**D**, **E**) with $\mathbf{D} \subset \mathbf{E} \subset \mathbf{M}$, too. However, **D** and **E**, respectively, are now the interior and the closure, respectively, of a set in a topological space generated from **M** by an equivalence relation. This topological feature of rough sets and (hence) operations on them causes that the representation of rough sets via 3-valued membership functions cannot be extended to sums and intersections of rough sets.

(b) A subdefinite set **A** in **M** is represented by *a twofold fuzzy set* which is a pair (*C, P*) of generalized characteristic functions belonging to GP:= $[0,1]^{\mathbf{M}}$ and such that
$$C \subset 1_{\ker(P)},$$
where
$$\ker(Y) := \{x : Y(x) = 1\} \text{ for } Y \in \mathrm{GP},$$
$$Y \subset Z \Leftrightarrow \forall x : Y(x) \leq Z(x)$$

and $1_{\mathbf{H}} :=$ the characteristic function of $\mathbf{H} \subset \mathbf{M}$ (see [1]). The fuzzy sets characterized by C and P, respectively, are composed of elements which, respectively, more or less certainly and more or less possibly are in $\mathbf{A}$. More precisely, $C(x)$ expresses a minimal degree of certainty that x is in $\mathbf{A}$, whereas $P(x)$ expresses a maximal degree of possibility that x is in $\mathbf{A}$. The condition $C \subset 1_{\ker(P)}$ says that the more or less certain elements of $\mathbf{A}$ have to be considered to be 'totally' possible elements of $\mathbf{A}$.

3   Approximative approach to vaguely defined objects

In this section, we like to develop the idea of approximative approach to VD-objects which was initiated in two variants in [17, 18] and [19] as a starting point for a nonclassical cardinality theory. We shall investigate that approach in a more detailed way. Generally speaking, we shall represent VD-objects by means of generalized characteristic functions rejecting, however, the fiction of their preciseness. Throughout, $\mathbf{V} := [0, 1]$ and $\mathbf{L}_{\infty}$ will be used (cf. [16]). Again, two variants will be considered. The first one seems to be especially suitable for proper VD-objects, whereas the second one is useful also for subdefinite sets.

Variant 1.

We shall assume that a VD-object in $\mathbf{M}$ is described by a function $A \in$ GP which is maybe imprecisely determined. Therefore, A will be approximated by two other membership functions $f(A)$ and $g(A)$, where $f, g : \mathrm{GP} \to \mathrm{GP}$ are defined by means of the following system of postulates ($x, y \in \mathbf{M}$, $B \in$ GP, PS$:=\{0, 1\}^{\mathbf{M}}$, $\&:=$ conjunction, $\perp:=$ inclusive disjunction, id$:=$ the identity function):

(A1) $f(A) \subset A \subset g(A)$,

(A2) $A \in$ PS $\Rightarrow f(A), g(A) \in$ PS,

(A3) $A(x) \leq B(y) \Rightarrow f(A)(x) \leq f(B)(y)$ & $g(A)(x) \leq g(B)(y)$,

(A4) $(f, g) \neq$ (id,id) $\Rightarrow f(\mathrm{GP}) \subset \mathrm{PS} \perp g(\mathrm{GP}) \subset \mathrm{PS}$.

Obviously, $(f(A), g(A))$ is an ultrafuzzy set, but the lower approximation $f(A)$ and the upper approximation $g(A)$ of A are not so arbitrary. They must be constructed by means of monotonic transformations of the membership grades $A(x)$ (see (A3)). Both the approximations of a set have to be sets, too (see (A2)). Finally, we see that the pair $(f, g) =$ (id,id) fulfils (A1)-(A3) and corresponds to the case of precise A. If A is imprecise (subjective), we do approximate it using a pair differing from (id , id). However, we do not like to 'proliferate' that imprecision by constructing imprecise lower and upper bounds for A. We accept that the imprecision of each $A(x)$ is more or less total: the only unquestionable lower evaluation of each $A(x)$ is $A(x) \geq 0$, possibly excluding the $A(x)$'s being equal to 1

if they are assumed to be precise, or/and the only unquestionable upper evaluation of each $A(x)$ is $A(x) \leq 1$, possibly excluding the $A(x)$'s being equal to 0 if they are assumed to be precise. This means that at least one of the approximations of a VD-object represented by A has to be a set, i.e. has to be a 'simpler' object, which is described in (A4). The family of all pairs (f, g) fulfilling (A1)-(A4) will be denoted by **F**, excluding the trivial pair $(f, g) = (T, M)$ with

$$T := 1_\emptyset \text{ and } M := 1_{\mathbf{M}}.$$

Clearly, the choice of a suitable pair (f, g) from **F** should be correlated with the type of imprecision of A. For the moment, we keep from giving any examples of pairs from **F**. The reason is that they will be easier to construct knowing a bit more about consequences of the postulates (A1)-(A4) (see Theorem 3.1 and Corollary 3.2).

Let f1, gs:GP $\rightarrow$ GP be defined in the following way:

$$\text{f1}(Y) := 1_{\ker(Y)} \text{ and } \text{gs}(Y) := 1_{\text{supp}(Y)}$$

with

$$\text{supp}(Y) := \{x : Y(x) > 0\} \text{ for } Y \in \text{GP}.$$

Theorem 3.1. *Let* $(f, g) \in \mathbf{F}$ *and* $A, B \in$ GP. *The following implications and equalities are satisfied:*

(a) $A \subset B \Rightarrow f(A) \subset f(B) \ \& \ g(A) \subset g(B)$.

(b) $f(A * B) = f(A) * f(B)$ *and* $g(A * B) = g(A) * g(B)$ *for each* $* \in \{\cap, \cup, \times\}$.

(c) $A \in \text{PS} \Rightarrow f(A) = g(A) = A$, *unless* $f \equiv T$ *or* $g \equiv M$.

(d) $(f, g) \neq (\text{id,id}) \Rightarrow (f \equiv T \perp f = \text{f1}) \perp (g \equiv M \perp g = \text{gs})$.

Proof. Part (a) follows from (A3) by putting $y := x$. Further, we see that (A3) implies $A(x) = B(y) \Rightarrow f(A)(x) = f(B)(y) \ \& \ g(A)(x) = g(B)(y)$, which leads to (b). (c) is a consequence of (A2) and (A3). Finally, (d) follows from (A1) and (A4). ∎

Corollary 3.2. *For each* $(f, g) \in \mathbf{F}$ *and* $A, B \in$ GP *the following properties hold true:*

(a) $A = B \Rightarrow f(A) = f(B) \ \& \ g(A) = g(B)$.

(b) $\text{f1}(A) \subset f(A)$ *and* $g(A) \subset \text{gs}(A)$, *unless* $f \equiv T$ *or* $g \equiv M$.

(c) $\ker(f(A)) = \ker(A)$ *and* $\text{supp}(g(A)) = \text{supp}(A)$, *unless* $f \equiv T$ *or* $g \equiv M$.

(d) $f(A) \subset 1_{\ker(g(A))}$, *unless* $(f, g) = (\text{id,id})$.

Proof. Immediate consequences of Theorem 3.1 and (A1)-(A4). ∎

Worth noticing is that Theorem 3.1(d) implies a decomposition of **F** into five subfamilies, namely:

$$\mathbf{F} := \{(\mathrm{id},\, \mathrm{id})\} \cup \{(f,g) : f \equiv T\} \cup \{(f,g) := \mathrm{f1}\} \cup \{(f,g) : g \equiv M\} \cup \{(f,g) : g = \mathrm{gs}\}.$$

Using this decomposition, one can easily give many examples of pairs from **F**. The following eight pairs seem to be particularly significant and useful:

(id,id),
(T,id), (f1,id),
(id,gs), (id,M),
(T,gs), (f1,M), (f1,gs).

So, each VD-object in **M** with maybe imprecisely determined membership function $A \in \mathrm{GP}$ can be viewed as a pair $(f(A), g(A))$ with $(f, g) \in \mathbf{F}$. The following particular cases of such pairs should be mentioned:

fuzzy sets $((f, g) = (\mathrm{id}, \mathrm{id}))$,
twofold fuzzy sets $((f, g) \neq (\mathrm{id}, \mathrm{id})$; see Corollary 3.2(d)),
partial sets $((f, g) = (T, gs), (f1, M), (f1, gs))$.

Basic relations and operations are defined as follows (cf. [1], [7]):

$$(f(A), g(A)) \subset (f(B), g(B)) \Leftrightarrow f(A) \subset f(B) \,\&\, g(A) \subset g(B), \qquad \text{(inclusion)}$$

$$(f(A), g(A)) = (f(B), g(B)) \Leftrightarrow f(A) = f(B) \,\&\, g(A) = g(B), \qquad \text{(equality)}$$

$$(f(A), g(A)) * (f(B), g(B)) := (f(A) * f(B), g(A) * g(B)) \text{ with } x \in \{\cap, \cup, \times\},$$
$$\text{(intersection, sum, cartesian product)}$$

$$(f(A), g(A))' := (g(A)', f(A)') \text{ with } Y'(x) := 1 - Y(x). \qquad \text{(complement)}$$

In virtue of Theorem 3.1(a, b), we have

$(f(A), g(A)) \subset (f(B), g(B))$ for $A \subset B$,
$(f(A), g(A)) * (f(B), g(B)) = (f(A * B), g(A * B))$ for $* \in \{\cap, \cup, \times\}$.

Contrary to ultrafuzzy sets, the approximative approach to VD-objects with imprecise membership functions allows one to treat VD-objects as fuzzy sets. Namely, if a pair $(f, g) \in \mathbf{F}$ is fixed, the fuzzy set-like notation $\mathrm{obj}(A)$ for a VD-object described by an imprecise $A \in \mathrm{GP}$ can be used, as if A would be precise, and the following natural definitions can be then formulated:

$$[\![\, x \in_m \mathrm{obj}(A) \,]\!] := A(x), \qquad \text{(membership)}$$

$$\mathrm{obj}(A) \subset_m \mathrm{obj}(B) := \forall_m x \in \mathbf{M}:\ x \in_m \mathrm{obj}(A) \Rightarrow_m x \in_m \mathrm{obj}(B), \text{ (many-valued inclusion)}$$

$$\mathrm{obj}(A) =_m \mathrm{obj}(B) := \mathrm{obj}(A) \subset_m \mathrm{obj}(B) \,\&_m\, \mathrm{obj}(B) \subset_m \mathrm{obj}(A), \text{ (many-valued equality)}$$

$$\mathrm{obj}(A) \subset \mathrm{obj}(B) \Leftrightarrow A \subset B, \qquad\qquad\qquad \text{(inclusion)}$$

$$\mathrm{obj}(A) = \mathrm{obj}(B) \Leftrightarrow A = B, \qquad\qquad\qquad \text{(equality)}$$

$$\mathrm{obj}(A) * \mathrm{obj}(B) := \mathrm{obj}(A * B) \text{ for } * \in \{\cap, \cup, \times\}, \qquad \text{(intersection, sum,}$$
$$\text{cartesian product)}$$

where $[\![\,s\,]\!] :=$ truth value of a sentence s, $\in_m :=$ many-valued membership predicate, $\forall_m :=$ many-valued general quantifier, $\Rightarrow_m :=$ implication and $\&_m :=$ conjunction in L_∞. Obviously, $\mathrm{obj}(A)$ is a set iff $A \in \mathrm{PS}$; informally, $\mathbf{D} = \mathrm{obj}(1_{\mathbf{D}})$ for each set $\mathbf{D} \subset \mathbf{M}$. VD-objects with the above defined sum $\cup$, intersection $\cap$ and neutral elements $\mathrm{obj}(T)$ and $\mathrm{obj}(M)$ are isomorphic to $(\mathrm{GP}, \cup, \cap, T, M)$, and form a bounded distributive lattice. In virtue of the above mentioned consequences of Theorem 3.1(a, b), the definitions of $(f(A), g(A)) * (f(B), g(B))$ and $\mathrm{obj}(A) * \mathrm{obj}(B)$ are coincident for each $(f, g) \in \mathbf{F}$, whereas the definitions of $\mathrm{obj}(A) \subset \mathrm{obj}(B)$ and $\mathrm{obj}(A) = \mathrm{obj}(B)$ are generally stronger than those of $(f(A), g(A)) \subset (f(B), g(B))$ and $(f(A), g(A)) = (f(B), g(B))$. In this case, full coincidence holds if $(f, g) \in \mathbf{F}$ is such that $f = \mathrm{id}$ or $g = \mathrm{id}$. Indeed, the implication connective in Theorem 3.1(a) and Corollary 3.2(a) can be then replaced by the equivalence connective and (hence) we get $(f(A), g(A)) \subset (f(B), g(B))$ iff $A \subset B$ as well as $(f(A), g(A)) = (f(B), g(B))$ iff $A = B$.

We should ask if the fuzzy set-like treatment of VD-objects can be extended to complementation and generalized operations on VD-objects. We easily see that the fuzzy set-like definition of complementation

$$\mathrm{obj}(A)' := \mathrm{obj}(A')$$

is coincident with that of $(f(A), g(A))'$ only if $(g(A)', f(A)') = (f(A'), g(A'))$ for each $A \in \mathrm{GP}$.

Theorem 3.3. *Let* $(f, g) \in \mathbf{F}$. *We have* $f(A) = g(A')'$ *and* $g(A) = f(A')'$ *for each* $A \in \mathrm{GP}$ *iff* (f, g) *is equal to* (id,id) *or* (f1, gs) .

Proof. By routine transformations.   ∎

So, the above defined complementation of a VD-object coincides with $(f(A), g(A))'$ only if (f, g) is equal to (id,id) or (f1,gs).

Finally, we like to formulate a few remarks about generalized operations on VD-objects with respect to the approximative approach. Let $\mathbf{J}$ denote a nonempty set of indices, $A_e \in \mathrm{GP}$ for each $e \in \mathbf{J}$, and $(f, g) \in \mathbf{F}$. As usual, we define

$$\left(\bigcap_{e \in \mathbf{J}} A_e \right)(x) \quad := \quad \bigwedge_{e \in \mathbf{J}} A_e(x),$$

$$\left(\bigcup_{e \in \mathbf{J}} A_e \right)(x) \quad := \quad \bigvee_{e \in \mathbf{J}} A_e(x),$$

$$\left(\times_{e \in \mathbf{J}} A_e \right)(y) \quad := \quad \bigwedge_{e \in \mathbf{J}} A_e(y(e)),$$

where $x \in \mathbf{M}$ and $y : \mathbf{J} \to \mathbf{M}$. Let

$$*_{e \in \mathbf{J}} \mathrm{obj}(A_e) := \mathrm{obj}(*_{e \in \mathbf{J}} A_e)$$

with $* \in \{\bigcap, \bigcup, \times\}$ provided that

$$*_{e \in \mathbf{J}} h(A_e) := h(*_{e \in \mathbf{J}} A_e)$$

(h symbolizes any element of a pair $(f, g) \in \mathbf{F}$; cf. Theorem 3.1(b)). One easily notices that the last condition is satisfied, for instance, by $(f, g) = (\mathrm{id},\mathrm{id}),(\mathrm{id},M),(T,\mathrm{id})$.

Variant 2.

In this variant, we shall assume that a VD-object in $\mathbf{M}$ is described by a pair (F, G) such that $F, G \in \mathrm{GP}$ and

$$F \subset 1_{\ker(G)}.$$

The pair (F, G) will be called *a free representing pair*; the family of all free representing pairs will be denoted by $\mathbf{K}$. So,

$$F \subset G \text{ and } F(x) > 0 \Rightarrow G(x) = 1 \text{ for each } x \in \mathbf{M}.$$

The VD-object represented by (F, G) will be denoted by $\mathrm{obj}(F, G)$. The following two interpretations of (F, G) can be used.

Possibilistic interpretation. (F, G) can be understood as a twofold fuzzy set with F and G being interpreted in the language of necessity and possibility degrees (cf. II.2(b) in Section 2). $\mathrm{obj}(F, G)$ is then a subdefinite set in $\mathbf{M}$.

Approximative interpretation. One can assume that $F = f(A)$ and $G = g(A)$ for some $A \in \mathrm{GP}$ and for some transformations $f, g : \mathrm{GP} \to \mathrm{GP}$. In other words, F and G are lower and upper approximations of a membership function A. We easily see that $F = G$ implies $F, G \in \mathrm{PS}$. So, if $F = G$, $\mathrm{obj}(F, G)$ is a set and its characteristic function is precisely known, else $\mathrm{obj}(F, G)$ is a proper VD-object in $\mathbf{M}$ and its membership function A is assumed to be always imprecise. Again, we assume that the imprecision of A is more or less total: either no nontrivial bounds for $A(x)$ can be given ($0 \leq A(x) \leq 1$) or only one of them can be established ($0 \leq A(x) \leq a$ or $b \leq A(x) \leq 1$ with $a, b \in [0, 1]$ depending on x); cf. Variant 1. This means that $F(x) = 0$ or/and $G(x) = 1$ for each $x \in \mathbf{M}$, which is equivalent to the condition $F \subset 1_{\ker(G)}$. So, this inclusion, usually connected only with twofold fuzzy sets, appears to be more universal than one could expected it to be.

Let $(F, G), (H, S) \in \mathbf{K}$. The following definitions of basic relations and operations for free representing pairs can be introduced (cf. [1] and Variant 1):

$$(F, G) \subset (H, S) \Leftrightarrow F \subset H \,\&\, G \subset S,$$
$$(F, G) = (H, S) \Leftrightarrow F = H \,\&\, G = S,$$
$$(F, G) * (H, S) := (F * H, G * S) \text{ with } * \in \{\cap, \cup, \times\},$$
$$(F, G)' := (G', F').$$

Generally,

$$*_{e \in \mathbf{J}}(F_e, G_e) := (*_{e \in \mathbf{J}}F_e, *_{e \in \mathbf{J}}G_e)$$

for $* \in \{\bigcap, \bigcup, \times\}$; as previously, $\mathbf{J}$-a set of indices, $(F_e, G_e) \in \mathbf{K}$ for each $e \in \mathbf{J}$. We define

$$\mathrm{obj}(F, G) \subset \mathrm{obj}(H, S) \Leftrightarrow (F, G) \subset (H, S),$$
$$\mathrm{obj}(F, G) = \mathrm{obj}(H, S) \Leftrightarrow (F, G) = (H, S),$$
$$*_{e \in \mathbf{J}}\mathrm{obj}(F_e, G_e) := \mathrm{obj}(*_{e \in \mathbf{J}}F_e, *_{e \in \mathbf{J}}G_e) \text{ with } * \in \{\bigcap, \bigcup, \times\},$$
$$\mathrm{obj}(F, G)' := \mathrm{obj}(G', F').$$

One can easily check that $(\mathbf{K}, \cup, \cap, ', (T, T), (M, M))$ forms an infinitely distributive de Morgan algebra.

The approximative approach to VD-objects seems to be a compromise solution of the problem of mathematical representation of VD-objects under immanent impreciseness of any kind of generalized characteristic functions. Moreover, that approach can be a convenient and realistic starting point for developing different mathematical theories for VD-objects. For example, in [17-19], a general nonclassical cardinality theory is constructed in this way.

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