Mathware & Soft Computing 5 (1998) 99-114

The Distribution of Mathematical Expectations of a Randomized Fuzzy Variable

V.B. Kuz'min and S.I. Travkin
Institute System Analysis RAN,
117312 Moscow, pr. 60 let Octiabria, 9, Russia
Submitted by S. Ovchinnikov

Abstract

The Shaffer's definition of the upper and lower expectations of fuzzy variables is considered with respect to randomized fuzzy sets. The notion of randomized fuzzy sets is introduced in order to evaluate fuzzy statistical indices for an arbitrary chosen fuzzy variable. Provided the distribution of the mathematical expectation of a randomized fuzzy variable is known, it is possible to adopt the traditional methods of testing statistical hypotheses for fuzzy variables.

We show that this distribution has a specific analytical structure and may be represented by means of Wan-der-Mond determinant derivatives. The relation between the notions of expectations of fuzzy variables and the Pareto optimality is demonstrated.

The mathematical expectation of the upper and lower expected values of a randomized fuzzy variable and their asymptotics are calculated.

1 The expected value of a fuzzy variable

By the Dempster definition [1] of the fuzzy variable γ is a mapping $\gamma : S \to R$ of the finite support set S of the fuzzy set $\tilde{A}$ into the set of real numbers R. Two Shafer's expectation of a fuzzy variable are defined as:

1) the *lower expectation* $E_*\gamma$ with respect to the measure $\{PL(A),\, A \subseteq S\}$ of *plausibililty* of the crips set A to be a subset of the fuzzy set $\tilde{A}$ is

$$E_*\gamma = \int vdPl(\{s : \gamma(s) < v\}); \tag{1}$$

2) the *upper expectation* $E^*\gamma$ with respect to the measure $\{CR(A),\, A \subseteq S\}$ of *credibility* of te crisp set A to be a subset of the fuzzy set $\tilde{A}$ is

$$E^*\gamma = \int vdCr(s : \gamma(s) < v\}). \tag{2}$$

According to the Shafert theory of evidence [4] these two measures are seen as *upper* and *lower* probability measures, i.e. the probability space (Ω, σ, P) and the mapping $\Sigma : \Omega \to 2^S$ of measurable sets form Ω onto the power set of S are assumed to be specified. Then by the definition:

$$Pl(A) = P^*(\omega : \gamma(\omega) \cap A \neq 0) \quad (3)$$
$$Cr(A) = P_*(\omega : \gamma(\omega) \subset A) \quad (4)$$

If the mapping γ is not a function then the right parts of the equations (3) and (4) should be normalized (divided by

$$P(\omega : \gamma(\omega) \cap S \neq \emptyset \quad \text{or} \quad P(\omega : \gamma(\omega) \subset A)$$

correspondingly). Thus, $Pl(A) = P^*(A)$ and $Cr(A) = P_*(A)$.

The membership function of a fuzzy set $\tilde{A}$ is the projection of the plausibility measure over the set of the singletons of the set $S : \mu(s) = Pl(\{s\})$.

If the family $\boldsymbol{F} = \{F : P(\omega : \gamma(\omega) = F) > 0\}$ of *focal elements* [3] is the collection of imbedded subsets of the set S then the plausibility measure $Pl(.)$ is the meausre of *possibility - Poss(.)* introduced by L. Zadeh [3]. Only in this case the plausibility measure may be inferred from the membership function $\mu(s)$ of the fuzzy set $\tilde{A}$.

Let's consider the fuzzy variable γ on the finite set S and arrange the values $\{\gamma(s), s \in S\}$, in non-decreasing order:

$$v_1 \leq v_2 \leq \ldots \leq v_n, \quad \text{where} \quad v_j = \gamma(s_i);\ i = 1, \ldots, n;\ |S| = n.$$

The definitions of the fuzzy variable expected values (1) and (2) imply

$$E^*\gamma = \sum_i v_i \circ \left(\operatorname{Max}_{k \geq i} \mu_k - \operatorname{Max}_{k > i} \mu_k \right) \quad (5)$$

$$E_*\gamma = \sum v_i \circ \left(\operatorname{Max}_{k \leq i} \mu_k - \operatorname{Max}_{k < i} \mu_k \right), \quad (6)$$

where $\mu_k = \mu(s_k) = Poss(\{s_k\})$.

Definition 1 *The upper (lower) expectation of a fuzzy set $\tilde{A}$ (defined on $S \subseteq R^1$) is the Shafer expectation of the Dempster identical fuzzy variable $\gamma(s) \equiv s$.*

In case the mapping γ is a function we may use the *extension principle* of L. Zadeh [2] and change the support set S by the set $V = \{v : v = \gamma(s),\ s \in S\}$ with the associated membership function $\nu(v) = \mu(\gamma^{-1}(v))$, and hence $\nu_j \equiv \nu(v_j) = \mu(s_i) = \mu_i$.

In general case

$$\nu(v) = \sup_{s : \gamma(s) = v} \mu(s).$$

Let's define the partial order $\succ$ on the graph

$$G = \{(v, \nu(v)) : v \in V\} :$$

$$\succ : (v, \nu) \succ (v', \nu') \text{ iff } v > v' \text{ and } \nu > \nu'. \quad (7)$$

2 The Pareto set of a fuzzy variable

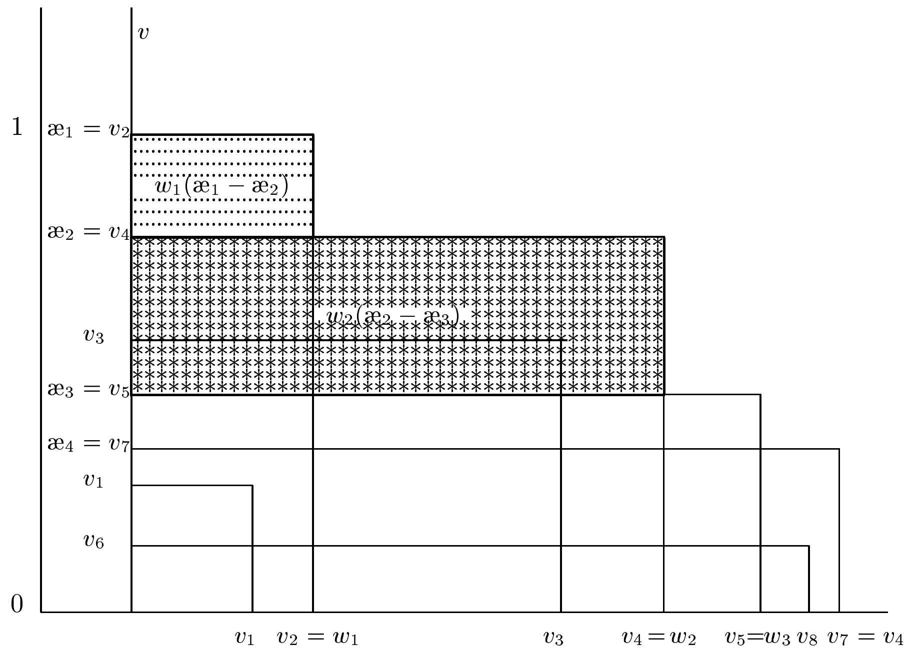
A non-zero contribution to the value of the upper expectation $E^*\gamma$ is only of those variables v_i for which points (v_i, μ_i) are in the Pareto set Π (the set-theoretical maximum with respect to the partial order (7). Let's renumber the Pareto points

$$\Pi = \{(w_j, æ_j)\}_{j=1}^{m}$$

in the increasing order of the values of memberships $\nu(v)$. Then the equation (5) for the upper expectation of the fuzzy set $\tilde{A}$ implies

$$E(\gamma^*) = \sum_{i=1}^{m} w_i(æ_i - æ_{i+1}). \tag{8}$$

That means the value of the upper expectation of the fuzzy set $\tilde{A}$ is equal by the Euclidean measure of the area (in the first quadrant of the plane $V \times [0,1]$. Pic.1) dominated by the points of the Pareto set Π- the union of the partial order (7) main filters.



Pic. 1. The expectation as the mesure of the union of the Pareto filters

Proposition 1 *When the mapping γ is not a bijection but a function then the equation (7) holds true. The expectation of the fuzzy variable is the same as the expectation of the fuzzy set $\tilde{\gamma}$ generated by the extension principle over the set of different images of the mapping γ:*

$$V = \{v\}_{k=1}^{m} = \{v : v = \gamma(s), s \in S\}$$

while corresponding membership function is

$$\nu(v) = \underset{s:\gamma(s)=v}{Max}\ \mu(s), \quad S = \bigcup_{k=1}^{m} S_k,$$

To prove the Proposition let's partition the set S by level-sets with levels $\nu_k = \nu(v_k)$, $S = \bigcup_{k=1}^{m} S_k$, $S_k = \arg \underset{s:\gamma(s)=v_k}{\text{Max}} |\mu(s)|$ and accordingly decompose the definition (5) of the upper expectation

$$E^*(\gamma) = \sum_{k=1}^{m} v_k \sum_{i \in S} \left(\underset{j \geq i}{\text{Max}}\, \mu_j - \underset{j > i}{\text{Max}}\, \mu_j \right).$$

The set S is ordered non-increasingly with respect to $\gamma(s)$. Thus,

$$S_k = \{s_i\}_{i=i_k}^{i_k+m_k-1}$$

where $m_k = |S_k|$ is the number of the elements of the level k and $i_k + m_k = i_{k+1}$. Since

$$\underset{k \geq i_j}{\text{Max}}\, \mu = \underset{k \geq i}{\text{Max}}\, \underset{l \in S_k}{\text{Max}}\, \mu_l = \underset{k \geq j}{\text{Max}}\, \nu_k,$$

then

$$E^*\gamma = \sum_{k} v_k \left(\underset{j \geq i_k}{\text{Max}}\, \mu_j - \underset{j \geq i_{k+1}}{\text{Max}}\, \mu_j \right) 1 = \sum_{k+1}^{m} v_k \cdot (\underset{j \geq k}{\text{Max}}\, \nu_j - \underset{j \geq k+1}{\text{Max}}\, \nu_j) = E^*\tilde{\gamma}.$$

3 Randomized fuzzy sets

The set of all fuzzy sets $\mathfrak{F}$ on the finite universum of the cardinality m is isomorphic to the llattice $[0,1]^m$, i.e., the unit cube R^m. Let's consider the borelian subsets of the set $\mathfrak{F}$ which are the inverse imges of the borellian sets in $[0,1]^m$ under the isomorphism $\mathfrak{F} \to [0,1]^m$ and provide the measurable space $< \mathfrak{F}, \mathfrak{B} >$ with the probability measure P. The ellement of $\mathfrak{F}$ may be treated as the realization of the randomized fuzzy set $\tilde{\Lambda}$ which is identified with the random vector

$$\Lambda = (\Lambda_1, \Lambda_2, ..., \Lambda_m),$$

$$P(\tilde{\Lambda} \subset V = \sum_{i=1}^{m} \nu_i | s_i) = \int_{(\lambda:\lambda_i < \nu_i)} dP(\lambda).$$

Note: The term of the randomize fuzzy set $\tilde{\wedge}$ is used to clarify the difference from the well-known notion of the random fuzzy set $(\tilde{F})$, [3]. L. Zadeh assigned the probability measure not to the measurable space $\mathfrak{F}$ but to the measurable space $\mathfrak{G} : (S, \mathfrak{U}, P)$ defined on the universal set S. The class of the random fuzzy sets is a class of measurable mappings $\{\mu : S \to [0,1]\}$, i.e. membership functions over the set V. Thus, the membership function μ of the random fuzzy set $\tilde{F}$ is considered as a random variable and the probability $P(\tilde{F} = V)$ is a mathematical expectation of this variable $\mu(.)$.

The mapping $E^* : \mathfrak{F} \to R^1$ which maps each fuzzy set V to its expected value $E^*(V)$ is measurable. The expected value of the randomized fuzzy set $\tilde{\wedge}$ is a random variable $E^*\wedge$. Now we are ready to calculate the distribution

$$F(u) = P(E^*\tilde{\wedge} < u) = \int_{\left\{v : \sum v_k \left(\operatorname{Max}\limits_{j \geq k} y - \operatorname{Max}\limits_{j \geq k+1} y\right) \leq u\right\}} dP(v) \tag{9}$$

4 The sample ranking and the probability of the identical permutation

Our aim is to calculate the distribution (9) with the uniform measure over V : $dP(v) = dv$. In this case all the variables $\{\nu_i\}$ of the membership function ν are independent and uniformly distributed over $[0,1]$ random variables.

Let's introduce the set of hypothesis about the ordering of these variables using the notion of a *rank*.

The *rank* $r(i)$ of the membership function value ν_i is a serial number of the element ν_i in the non-increasingly ordered array of all values ν_i, $r(i) =$RANK ν_i.

The definition of ranks of membership function values is correct for the events with non-zero probability of occurrence (and only such events are considered further on). Every realization of the randomized fuzzy set $\tilde{\wedge}$ corresponds the permutation $r : \boldsymbol{n} \to \boldsymbol{n}$, an element of symmetrical group S_n such that $r(i) =$RANK ν_i, ($\boldsymbol{n}$ denotes an ordinal $\{1, 2, \ldots, n\}$).

Let's consider $n!$ hypothesis $\{H_r; r \in S_n\}$ about the ranking of the membership function values $H_r = \{\text{RANK } \nu_i = r(i),\ i = n\}$. Then

$$F(u) = \sum_{\rho \in S_n} P\left((E^*\tilde{\wedge} < u) \& H_\rho\right).$$

Let's calculate the probability $F_e(u) = P((E^*\tilde{\wedge} < u)\&H_e)$ for the identical permutation $r = e$, $(e : e(i) =\text{RANK } \nu_i = i)$. $F_e(u)$ is equal to the volume of the intersection of the unique cube $[0,1]^n$ with two simplexes:

1) $\{x : \sum\limits_{i=1}^{n} (v_i - v_{i-1}) x_i < u;\ \forall i \in n : x_i > 0;\ v_0 = 0\}$

2) $\{x : x_i > x_{i+1};\ \forall i \in \boldsymbol{n} : x_i > 0;\ (x_1 \leq 1,\ x_{n+1} = 0)\}$.

Changing the variables $\eta_i = v_i(\nu_i - \nu_{i+1})$, $i \in \boldsymbol{n}$ we reduce the problem to the calculation of the volume of intersection of regions σ_u and σ_v cut from the first quadrant of $R^n = \prod_{i=1}^n Y_i$, by two planes:

$$\begin{aligned} L_u(y) &= \sum_{i=1}^n y_i - u = 0, \\ L_v(y) &= \sum_{i=1}^n \frac{y_i}{v_i} - 1. \end{aligned}$$

Since the volume element $\wedge_{i=1}^n dx_i$ is equal to $\wedge_{i=1}^n dy_i / \prod_{i=1}^n v_i$ then

$$F_e(u) = \text{Measure}\,(\sigma_u \cap \sigma_v) \,/\, \prod_{i=1}^n v_i$$

The sought probability may be written in the form of the polynomial with respect to u of the order n, the type of which is determined by the interval (v_i, v_{i+1}) covering the value u.

If $u < \sigma_1$ then $\sigma_u \subset \sigma_v$ and Measure $(\sigma_u \cap \sigma_v)$ =Measure $\sigma_u = u^n/n!$.

If $v_1 < u < v_2$ then among vertices of the simplex σ_u only the vertex $u \cdot e_1$ (where $e_1 = (1, 0, ..., 0)$ and in general e_j denotes jth basic vector in R^n) turns out to be outside of the simplex σ_v. The simplex σ_v cuts off from the simplex σ_u the simplex σ^1 with the vertices $v_1 e_1$, $u_1 e_1$ and with $n-1$ vertices located in the plane $L = 0_v$:

$$\left\{ v_1 e_1 \frac{u - v_i}{v_1 - v_i} + v_i e_i \frac{u_i - v_1}{v_i - v_1} \right\}, \quad i = \{2, 3, ..., n\}.$$

The volume of the simplex σ^1 which is equal to

$$\frac{1}{n!} \prod_{i \neq 1} \frac{v_i}{v_1 - v_i} (u - v_i)^n$$

has to be subtracted from the volume of the simplex σ_u,

$$\text{Measure}\,(\sigma_u \cap \sigma_u) = \frac{1}{n!} \left(u^n - \prod_{i \neq 1} \frac{v_i}{v_i - v_1} (u - v_1)^n \right).$$

If $v_2 < u < v_3$ then the vertex ue_2 of the simplex σ_u is not in σ_v. The volume of the simplex σ^2 (with the vertices $v_2 e_2$, ue_2 and $n-1$ vertices located in the plane $L_v = 0$:

$$\left\{ v_2 \frac{u - v_i}{v_2 - v_i} e_2 + v_i \frac{u - v_2}{v_i - v_2} e_2, \quad i \neq 1 \right\}$$

has to be subtracted from the volume of the simplex σ^1. Hence

$$\text{Measure}\,(\sigma_u \cap \sigma_u) = \frac{1}{n!} \left[u^n - \sum_{j=1}^2 \prod_{i \neq j} \frac{v_i}{v_i - v_j} (u - v_j)^n \right].$$

This result is a hint on a more general answer:

$$F_e(u) = \frac{1}{n! \prod_{i=1}^n v_i} \left[u^n - \sum_{j=1}^n \prod_{i \neq j} \frac{v_i}{v_i - v_j} \chi_j(u)(u - v_j)^n \right] \tag{10}$$

where $\chi_j(u)$ is the characteristic function of the interval $\{u : u \geq v_j\}$. It may be proved by the method of inclusion-exclusiion and we have already made the first two steps in this direction.

5 The coinciding variables

A general denominator of the summonds in the above written sum (10) is equal to the value $\prod_{i>j}(v_i - v_j)$ of the Wan-der-Mond's determinant Det W of the matrix

$$W = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_n \\ v_1^2 & v_2^2 & \cdots & v_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ v_1^{n-1} & v_2^{n-1} & \cdots & v_n^{n-1} \end{bmatrix}$$

Let's substitute the row of units (the first row in W) by the row

$$\chi_1(u)(u - v_1)^n, \ \chi_2(u)(u - v_2)^n, \ \ldots, \ \chi_n(u)(u - v_n)^n$$

and denote the result as $W_u = W_u(v)$, then rewrite the expression (10) for the probability $F_e(u)$ using the matrix $W_u(v)$:

$$F_e(u) = \frac{1}{n! \prod_{i=1}^n v_i} \left(u^n - \frac{\text{Det } W_u}{\text{Det } W} \right) \tag{11}$$

If $u > v_n$ then all characteristic functions for the sets $\{u : u > v_i\}$ are units, $\chi_i(u) = 1$, and the simplest σ_v is included in σ_u. Hence, $F_e(u) = 1/n!$ and the following identity holds true:

$$\prod_{i=1}^n v_i = u^n - \text{Det } \frac{\text{Det} W_u(v)}{\text{Det } W}.$$

By the statement of the problem all the variables v_i are different but to calculate the probabilities $F_\rho(u)$ under hypotheses H_ρ for non-identical permutations $\rho \in S_n$ it will be useful to known the distribution in the case $v_i = v_{i+1}$ and even in more general cases

$$v_i = v_{i+1} = \ldots = v_{i+k}, \quad i \geq 1, \quad i + k = m < n..$$

The inclusion-exclusion method is inapplicable now: for not all regions needed for inference are the simplexes. We have to use the other idea.

The distribution $F_e(u) = F_e(u,v)$ is continuous with respect to u, v and the problem is solved by taking the limit

$$\lim_{v_{i+1}\to v_i} F_e(u).$$

By the eliminating the first order uncertainty (of the type 0/0) in the ratio of the determinants $\frac{\mathrm{Det} W(u,v)}{\mathrm{Det}\ W}$ we get

$$F_e(u,v)\Big|_{v_i=v_{i+1}} = \frac{1}{n!\prod_{i=1}^n v_i}\left(u^n - \frac{\partial}{\partial v_{i+1}}\mathrm{Det}\ W_u / \frac{\partial}{\partial v_{i+1}}\ \mathrm{Det}\ W\right)\Big|_{v_i=v_{i+1}}$$

where $v = (v_1, v_2, ..., v_n)$. Since the variable v_{i+1} appears only in the $(i+1)$th column of the matrix W' (any of the matrices W or $W(u,v)$) then

$$\frac{\partial}{\partial v_{i+1}}\ \mathrm{Det}\ W' = \ \mathrm{Det}\ \frac{\partial}{\partial v_{i+1}} W'$$

The limit as v_{i+2} tends to v_i is equal to $F_e(u,v)$ with $v_i = v_{i+1} = v_{i+2}$. But now the ratio of the determinants $\frac{\mathrm{Det} W(u,v)}{\mathrm{Det}\ W}$ has the uncertainty of the second order. The matrices

$$\frac{\partial}{\partial v_{i+2}}\left[\frac{\partial}{\partial v_{i+1}} W_{\cdot}\Big|_{v_{i+1}=v_i}\right]\Big|_{v_{i+2}=v_i}$$

have two identical columns: $(i+1)$th and $(i+2)$th.

In order to get the general formula we introduce the strictly increasing step function v over $\boldsymbol{n} = \{1, 2, \ldots, n\}$,

$$\begin{aligned} v &: \boldsymbol{n} \to R^1 \\ v_i &= \sum_{j=1}^k v_{i,j}\chi_{\Delta_j}(i), \end{aligned} \tag{12}$$

where χ_{Δ_j} is the characteristic function of the set Δ_j, $\Delta_j = \{i\}_{i=i_{j-1}+1}^{i_j}$ and $0 = i_0 < i_1 < ... < i_k = n$. Hence $\boldsymbol{n} = \bigcup_{j=1}^k \Delta_j$. Denote as $\partial\Delta_j$ the operator of taking mixed derivative in the points $v_i = i \in \Delta_j$ (if Δ_j is not singleton, i.e. $\Delta_j \neq \{i\}$):

$$\partial\Delta_j = \frac{\partial^{\Delta_{j-1}}}{\partial v_{i_j}\partial v_{i_j-1}...\partial v_i}$$

where $\Delta_j = |\Delta_j| = i_j - i_{j-1}$ is the number of the elements in Δ_j. If Δ_j is a singleton $\{i_j\}$ then $\Delta_j = 1$. In this case $\partial\Delta_j$ is the identity operator $\partial\Delta_j W_0 = W$. Let's introduce an another differential operator ∂ defined as the superposition of k operators $(j \in k)$

$$\partial = *_{k=1}^k \partial\Delta_j = \partial_{\Delta_1}\left(\partial_{\Delta_2}(...\partial_{\Delta_k})\right).$$

Thus, for the case of the coinciding points of the support set for the fuzzy set $\tilde{\Lambda}$ (when this coincidence is determined by the function v, (12)) the distribution $F_e(u)$ is

$$F_e(u)\Big|_v = \frac{1}{n!\prod_{j=1}^k v_{i_j}}\left(u^n - \frac{\mathrm{Det}\ \partial W_u}{\mathrm{Det}\ \partial W}\right). \tag{13}$$

In particular when $v_j = v$ for all j the determinant Det ∂W_u in the numerator of (13) is equal to $\chi_1(u)/D_n$ where

$$D_n = \begin{bmatrix} (u-v)^n & -n(u-v)^{n-1} & n(n-1)(u-v)^{n-2} & \cdots & (-1)^{n-1}n!(u-v) \\ v & 1! & 0 & \cdots & 0 \\ v^2 & 2v & 2! & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ v^{n-1} & (n-1)v^{n-2} & (n-1)(n-2)v^{n-3} & \cdots & (n-1)! \end{bmatrix}$$

The recurrence of its principal minors is

$$D_{j+1} = j!D_j + \binom{n}{j-1}(u-v)^{n-j+1}v^{j-1}(j-1)!(j-2)!,$$

where $\binom{n}{j}$ is the binomial coefficient.

Solving the inferred recurrence we arrive at

$$D_n = \prod_{j=1}^{n-1} j! \sum_{m=0} \binom{n}{m}(u-v)^{n-m}v^m = \prod_{j=1}^{n-1} j!(u^n - v^n),$$

The factor $\prod j!$ is equal to the multiple derivative of Wan-der-Mond's determinant which appears in the denominator in the left part of the equation (13), (with the parameters $\Delta_j = 1$ and $v_j = v$)

$$\det \partial W = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ v & 1! & 0 & \cdots & 0 \\ v^2 & 2v & 2! & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v^{n-1} & (n-1)v^{n-2} & (n-1)(n-2)v^{n-3} & \cdots & (n-1)! \end{vmatrix}$$

Hence, for the $v_i \equiv v$ (constant) we get

$$F_e(u)\Big|_{v_i \equiv v} = \begin{cases} \left\{ {u \atop v} \right\}/n! & \text{if } u < v. \\ 1/n! & \text{else} \end{cases}$$

6 The conditional probabilities under the hypothesis of non-identical permutations. The absolute distribution

To calculate the conditional distribution $F\rho(u)$ under the hypothesis H_ρ we need a proper replica of the function 12) $v : \boldsymbol{n} \to \boldsymbol{n}$. For this purpose let's partition the set $\boldsymbol{n} = \{1, 2, ..., n\}$ to "segments" $\Delta_j = \Delta_j(\rho)$, wich right points $i_j = i_j(\rho)$, $j \in k$ (the maxima of the segments) are defined recursively:

$$\begin{aligned} i_0 &= 0, \\ i_j &= \rho^{-1}(\min_{m > i_{j-1}} \rho(m)), \end{aligned}$$

where ρ^{-1} is inverse permutation of ρ. Then $\Delta_j = \{i\}_{i=i_{j-1}+1}^{i_j}$ and $\boldsymbol{n} = \bigcup_j \Delta_j$ are in full accordance with the recent notations.

The number k of segments is equal to the cardinality of the Pareto set $\prod = \{v_{i_j}, \nu(v_{i_j})\}$ in any realization of the membership function of the randomized fuzzy set $\tilde{\Lambda}$ under the hypothesis H_ρ.

The conditional distribution $F_\rho(u)$ may be rewritten by means of the equation (13), but for the step function $v_\rho : \boldsymbol{n} \to \boldsymbol{n}$,

$$v_\rho(i) = \sum_{j=1}^{n} v_{i_j}(\rho)\chi_{\Delta_j(\rho)}(i).$$

By gathering the hypothesis we prove the following theorem.

Theorem. *The absolute probability distribution of the randomized fuzzy set* $\tilde{\Lambda}$ *is*

$$F(u) = \frac{1}{n!}\sum_{\rho\in S_n}\left(u^n - \frac{\det \partial_{v_\rho} W_k}{\det \partial_{v_\rho} W}\right) / \prod_{j=1}^{k(\rho)} v_{i_{(\rho)}}^{\Delta_j(\rho)}.$$

7 The mathematical expectation of the upper expectation of a normal fuzzy set

The randomization of the fuzzy set defined on the finite support $V = \{v\}_{i=1}^n$ (as it have been done above) implies all realizations of the randomized fuzzy set $\tilde{\Lambda}$ to be subnormal fuzzy sets almost certainly (with the probability 1). And to be precise we shell call it a *subnormal randomized fuzzy set.* If we are to consider the normal case then we have to classify all normal fuzzy sets $\mathfrak{R}\mathfrak{F}(V)$ as factor-sets $\mathfrak{F}_k(V)$, $0 \le k \le n\}$ with fixed cardinality Card $V_1 = k$ of their subsets $V_1 \subseteq V$ of the level 1.

Definition 2.

1) Any fuzzy set of the class $\mathfrak{F}_k(V)$ *is called a k-normal fuzzy set.*

2) $\tilde{\gamma}_{n,k}$ *is k-normal uniformly randomized fuzzy set over the finite set* V *iff*

- *only strict subset of cardinality* k *may be a subset of level 1 in the realization of the randomized fuzzy set.*
- *any strict subset (of* V*) of cardinality* k *may be equiprobably found as a subset of the level 1 among the realizations of the randomized fuzzy set.*
- *for any element of the complement* $V \backslash V_1$ *any grade of the membership function is an independent uniformly distributed over [0,1] random variable.*

The definition 2 is correct if we are not paying attention to the events of probability 0. We still assume the linear order over V is $v_1 < v_2 ... < v_n$.

Theorem 2. *The mathematical expectation of the upper expectation of k-normal uniformly randomized fuzzy set is equal to*

$$\mathcal{E}^*\tilde{\gamma}_{n,k} = \sum_{m=k}^{n} \frac{n+1-(1-1/k)m}{(n-m+1)(n-m+2)} \frac{\binom{m-1}{k-1}}{\binom{n}{k}} v_m.$$

To prove the Theorem 2 we are in need of the next Lemma which is interesting by itself..

Lemma 1.*The mathematical expectation of the upper expectation of the uniformly randomized subnormal fuzzy set γ_n is equal to*

$$\mathcal{E}^*\tilde{\gamma}_n = \sum_{m=1}^{n} \frac{v_m}{(n-m+1)(n-m+2)}.$$

Proof. Note if the set V_1 is a singleton then

$$\mathcal{E}^*\tilde{\gamma}_{n,1} = (1+1/n) \sum_{m=1}^{n} \frac{v_m}{(n-m+1)(n-m+2)} = (1+\frac{1}{n})\mathcal{E}^*\tilde{\gamma}_n.$$

Every fuzzy set over V which has no elements with the same grade of the membership function has been mapped to some permutation $\rho \in S_n$. Let's define the function $æ_\rho : \boldsymbol{n} \to \boldsymbol{n}$, $æ_\rho(i) = \min_{j \geq i} \rho(j)$ and use it for writing the expression of the upper expectation

$$E^*_\rho = \sum_{i=1}^{n} y_{æ_\rho(i)} w_i = y_0 w,$$

where

$$y_\rho = \left(y_{æ_\rho(1)}, y_{æ_\rho(2)}, ..., y_{æ_\rho(n)}\right), \quad w = (w_1, w_2, ..., w_n),$$

$$w_i = v_i - v_{i-1}, \quad v_0 = 0.$$

The variables y_i of the uniformly distributed subnormal fuzzy set $\tilde{\gamma}$ are rank-test statistics. Their mathematical expectations are equal to $1 - \frac{1}{n+1}$. The random permutation ρ is an element of measured space S_n (the uniform measure of Haar: $P(\rho = r) = \frac{1}{n!}$, $r \in S_n$). Taking all this into consideration we get

$$\begin{aligned} \mathcal{E}^* &= \mathcal{E}E^*\tilde{\gamma} = \frac{1}{n!} \sum_{r \in S_n} \mathcal{E}(E^*_\rho \tilde{\gamma} | \rho = r) = \\ &= \frac{1}{n!} \sum_{r \in S_n} \sum_{i=1}^{n} \left(1 - \frac{æ_r(i)}{n+1}\right) w_i = v_n - \frac{1}{(n+1)!} \left(\sum_{r \in S_n} æ_r\right) w. \end{aligned}$$

where $æ_r = (æ_{r(1)}, ..., æ_{r(n)})$.

Let's note that $\mathcal{E}^*$ is the function of the elements of the support $\mathcal{E}^* = \mathcal{E}^*(v) = v$, $v = (v_1, ..., v_n)$. Introducing the notation

$$\lambda_n = \sum_{r \in S_n} æ_r$$

we may write

$$\mathcal{E}^* = v_n - \frac{1}{(n+1)!} \lambda_n w.$$

Let's show that

$$\lambda_n(i) = (n+1)!/(n+2-i),$$

by induction over n:

- The case $n = 1$ is trivial: $\lambda_1(1) = 1$.
- Assume the equation $\lambda_{n-1} = \frac{n!}{n+1-i}$ holds true.

Let's consider some permutation $r \in S_n$ with which we relate n permutations $s_i(r) \in S_n$, $i \in n$ defined as

$$s_i(r) = (r(1)+1, r(2)+1, ..., r(i-1)+1, 1, r(i)+1, ..., r(n-1)+1),$$

i.e. every element of r is increased by 1 and unite ith element is added. The mapping is a covering of $S_n = \bigcup_{r \in S_{n-1}} S_r$. Denote

$$\lambda'_r(i) = \sum_{m=1}^{n} æ_{s_m(r)}(i),$$

then

$$\lambda_n(i) = \sum_{s \in S_n}^{n} æ_s(i) = \sum_{r \in S_{n-1}} \lambda'_r(i).$$

The function $æ_{s(r_m)}(i)$ with respect to $s_m(r) \in S_n$ is

$$æ_{s_m(r)}(i) = \begin{cases} 1, & \text{if } i \leq j, \\ æ_r(i-1)+1, & \text{if } i > j. \end{cases}$$

Then

$$\lambda'_r(1) = \sum_{m=1}^{m} æ_{s_m(r)}(1) \equiv n$$

and

$$\lambda_n(1) = n! = \frac{(n+1)!}{n+1}.$$

In general case

$$\lambda'_r(i) = (i-1)(æ_r(i-1)+1) + n - (i-1) = (i-1)æ_n(i-1) + n$$

and we get the recurrence

$$\lambda_r(i) = (i-1)\lambda_{n-1}(i-1) + n!.$$

Now ise the inductive assumption about the values of $\lambda_{n-1}(i)$ to get

$$\lambda_n(i) = (i-1)\frac{n!}{n+2-i} + n! = (n+1)!/(n+2-i).$$

Hence

$$\mathcal{E}^* = v_n - \sum_{i-1}^{n} \frac{w_i}{n+2-i} = \sum_{i-1}^{n} \frac{n+1-i}{n+2-i}$$

and after substitution of $v_i - v_{i-1}$ for $w(i)$ and some simplification of the resulting expression the Lemma is proved.

The proof of the Theorem 1. Let's introduce the complete set of hypotheses $\{H_m\}$, $m \in \{k, k+1, ..., n\}$ about the maximum of the 1-level set V_1 of the uniformly randomized k-normal fuzzy set $\tilde{\lambda}_{n,k}$:

$$H_m : \operatorname*{Max}_{v_i \in V_1} v_i = m.$$

The probability of this event is equal to

$$P(H_m) = \binom{m-1}{k-1} / \binom{n}{k}.$$

Since the hypothesis H_m are disjunctive and $\bigcup_{m=k}^{n} H_m$ is a certain event then

$$\mathcal{E}^*_{n,k} = \mathcal{E}E^*\tilde{\gamma}_{n,k} = \frac{1}{\binom{n}{k}} \sum_{m=1}^{k} \binom{m-1}{k-1} \mathcal{E}(E^*\gamma_{n,k}|H_m).$$

Then $m = n$, (i.e. $\mu(v_n) = 1$) then

$$\mathcal{E}(E^*\gamma_{n,k}|H_n) = v_n.$$

If $m < n$ then the conditional mathematical expectation $\mathcal{E}(E^*\gamma_{n,k}|H_n)$ is expressed by the mathematical expectation of the subnormal randomized fuzzy set over the support

$$V_m = \{v_{m+1} - v_m, v_{m+2} - v_m, ..., v_n - v_m\},$$

i.e.

$$\mathcal{E}^*\left(E^*\gamma_{n,k}|H_n\right) = v_n - \mathcal{E}^*\{v_{m+1} - v_m, ..., v_n - v_m\}.$$

Hence

$$\mathcal{E}^*_{n,k} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} v_n - \sum_{m=k}^{n-1} \binom{m-1}{k-1} \left[m + \sum_{j=m+1}^{n} (v_j - v_m)/2\binom{n-j+2}{2} \right] / \binom{n}{k}.$$

The expression in the square brackets might be simplified

$$v_m + \sum_{j=m+1}^{n} (v_j - v_m)/(n+1-j)(n+2-j) =$$

$$= \left[1 - \sum_{j=m+1}^{n} \left(\frac{1}{n+1-j} - \frac{1}{n+2-j}\right)\right] v_m - \sum_{j=m+1}^{n} \frac{v_j}{(n+1-j)(n+2-j)}$$

$$= \frac{v_m}{n+1-m} - \sum_{j=m+1}^{n} \frac{v_j}{(n+1-j)(n+2-j)}.$$

Thus

$$\mathcal{E}^*_{n,k} = \left[\sum_{m=k}^{n} \binom{m-1}{k-1} \frac{v_m}{n+1-j} - \sum_{m=k}^{n-1} \binom{m-1}{k-1} \sum_{j=m+1}^{n} \frac{v_j}{(n+1-j)(n+2-j)}\right] / \binom{n}{k}. \tag{14}$$

Changing the order of summation of the multiple $\sum$ we get

$$\sum_{j=k+1}^{n} \frac{v_j}{2\binom{n-j+2}{2}} \sum_{m=k}^{j-1} \binom{m-1}{k-1} = \sum_{j=k+1}^{n} \binom{j-1}{k} \frac{v_j}{(n+1-j)(n+2-j)}.$$

After changing j to m and substituting this expression into (14) we get

$$\mathcal{E}^*_{n,k} = \frac{v_k}{n+1-k} + \sum_{m=k+1}^{n} v_m \left[\frac{\binom{m-1}{k-1}}{n+1-m} + \frac{\binom{m-1}{k}}{(n-m+1)(n-m+2)}\right].$$

The expression in the square brackets may be rewritten as

$$\frac{n+1-(1-1/k)m}{(n-m+1)(n-m+2)} \binom{m-1}{k-1}$$

and if $m = k$ then it is equal to $1/(n-k+1)$. Thus

$$\mathcal{E}^*_{n,k} = \sum_{m=k}^{n} \frac{n+1-(1-1/k)/m}{(n-m+1)(n-m+2)} \binom{m-1}{k-1} v_m.$$

The Theorem 2 is proved.

8 The mathematical expectation of the lower expectation of a subnormal fuzzy set

Lemma 1. *The mathematical expectation of the lower expectation of subnormal uniformly randomized fuzzy set may be found by substituting* v_{n+1-i} *instead of* v_i*,*

$i \in n$, in Lemma 1, and is equal to

$$\mathcal{E}E_*\tilde{\gamma}_n = \sum_{i=1}^{n} \frac{v_i}{i(i+1)}$$

Proof. According to the definition of the lower expectation for the realization of the randomized fuzzy set $\tilde{\Lambda}$ under the hypothesis $H_r : \rho = r \in S_n$ we have

$$E_*(r) = \sum_{i=1}^{n} v_i \left(\operatorname*{Max}_{k \le i} v_k - \operatorname*{Max}_{k \le i-1} v_k \right)$$

(where it is assumed that $\operatorname*{Max}_{k \le 0} \nu_k := 0$ and Rank $\nu_k = r_k$, $k \in n$.

Using the Abel formula of summation we get

$$E_*(r) = \sum_{i=1}^{n} w_i \left(h - \operatorname*{Max}_{k \le i-1} v_k \right),$$

where $h = \max v_k$, $k \in n$, is the height of the randomized fuzzy set $\tilde{\Lambda}$. The expected value of the stochastic variable ν is

$$\mathcal{E}\nu_j \equiv 1 - \frac{r(j)}{n+1},$$

and thus

$$\begin{aligned} \mathcal{E}E_*(\rho)|\rho = r) &= \sum_{i=1}^{n} w_i \left(1 - \frac{1}{n+1} - \operatorname*{Max}_{k \le i-1} \left(1 - \frac{r(k)}{n+1} \right) \right) = \\ &= \frac{1}{n+1} \left[n w_1 + \sum_{i=2}^{n} w_i (\min_{k \le i-1} r(k) - 1) \right]. \end{aligned}$$

Let $\min_{k \le -1} r_k = n + 1$. Then using the function $æ_r : \boldsymbol{n} \to \boldsymbol{n}$,

$$æ_r(i) = \min_{k \le i-1} r(k) - 1,$$

we get

$$\mathcal{E}_*(r) = \mathcal{E}(E_*(\rho) \& \rho = r) = \frac{1}{(n+1)!} \sum_{i=1}^{n} w_i æ_r(i)$$

and the unconditional mathematical expectation is equal to

$$\mathcal{E}E_*\tilde{\gamma}_n = \sum_{r \in S_n} \mathcal{E}_*(r) = \frac{1}{(n+1)!} w \lambda_n,$$

where i-th coordinate of the vector λ_n is defined by the relation

$$\lambda_n(i) = \sum_{r \in S_n} æ_r(i).$$

Let's proof that

$$\lambda_n(i) = n!(n+1-i)/i. \tag{15}$$

To this end as it have been done previously let's consider the permutation $r \in S_{n-1}$ and inducing by it n permutations from S_n, $S_i(r)$, $i \in \boldsymbol{n}$. Let's express the coordinates of the vector

$$æ_{s_j(r)} = (æ_{s_j(r)}(1), ..., æ_{s_j(r)}(n))$$

by means of the coordinates $æ_r(i)$, $i \in n-1$,

$$æ_{s_j(r)}(i) = \begin{cases} æ_r(i)+1 & \text{if } i \leq j, \\ 0 & \text{if } i > j. \end{cases}$$

Since $S_n = \bigcup_{r \in S_{n-1}} \{S_i(r), i \in n\}$, then

$$\begin{aligned} \lambda_n(i) &= \sum_{s \in S_n} æ_s(i) = \sum_{r \in S_{n-1}} \sum_{j=1} æ_{j(r)}(i) = \\ &= \sum_{r \in S_{n-1}} (n+1-i)(æ_r(i)+1) = (n+1-i)(\lambda_{n-1}(i)-(n-1)!). \end{aligned}$$

It is easily to test whether (15) is the resolution of the obtained equation

$$\lambda_n(1) = (n+1-i)(\lambda_{n-1}(i)-(n-1)!).$$

Thus

$$\mathcal{E}E_*\tilde{\Lambda} = \frac{1}{n+1}\sum_{i=1}^{n} w_i(n+1-i)/i = \sum_{i=1}^{n} w_i\left(\frac{1}{i}-\frac{1}{n+1}\right),$$

and adding on the Abel's formula we get the Lemma 2.

Let's note that if $w_i = 1/n$ then

$$\mathcal{E}_* = \mathcal{E}^* = \frac{\mathcal{E}|\pi_n|}{n} - \frac{1}{n(n+1)} \sim \frac{\ln n}{n}.$$

The expected value is evaluated by the number of the Pareto points in the random substitution.

References

[1] Dempster, A.P., Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Statist.* 1967, v. 38.

[2] Zadeh, L.A., Fuzzy sets. *Inform. Control*, 1965, v. 8.

[3] Zadeh, L.A., Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1978, v. 1, N. 3.

[4] Shafer, G., *A Mathematical Theory of Evidance.* Princeton, University Press, 1976.