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On Some Geometric Transformation of t-norms*

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Abstract

Given a triangular norm T, its t-reverse T^*, introduced by C. Kimberling (*Publ. Math. Debrecen* 20, 21-39, 1973) under the name invert, is studied. The question under which conditions we have $T^{**} = T$ is completely solved. The t-reverses of ordinal sums of t-norms are investigated and a complete description of continuous, self-reverse t-norms is given, leading to a new characterization of the continuous t-norms T such that the function $G(x, y) = x + y - T(x, y)$ is a t-conorm, a problem originally studied by M.J. Frank (*Aequationes Math.* 19, 194-226, 1979). Finally, some open problems are formulated.

1 Introduction

Triangular norms (t-norms) and the corresponding t-conorms play a fundamental role in several branches of mathematics, e.g., in probabilistic metric spaces [6], in the theory of generalized measures and games [1] and in fuzzy logic [5]. In [3], the t-reverse T^* of a t-norm T was introduced (under the name invert). We somewhat extend and complete the study of t-reverses done there.

A *triangular norm* (t-*norm* for short) is a function $T : [0, 1]^2 \to [0, 1]$ which is commutative, associative, non-decreasing in both components, and satisfies the boundary condition $T(x, 1) = x$ for each $x \in [0, 1]$. Given a t-norm T, its dual t-*conorm* S_T is defined by

$$S_T(x, y) = 1 - T(1 - x, 1 - y).$$

The most important t-norms, together with their dual t-conorms are

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$T_{\mathbf{M}}(x,y) = \min(x,y),$ $S_{\mathbf{M}}(x,y) = \max(x,y);$

$T_{\mathbf{P}}(x,y) = xy,$ $S_{\mathbf{P}}(x,y) = x + y - xy;$

$T_{\mathbf{L}}(x,y) = \max(0, x + y - 1),$ $S_{\mathbf{L}}(x,y) = \min(1, x + y);$

$$T_{\mathbf{W}}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad S_{\mathbf{W}}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

It is obvious that these t-norms satisfy the inequality $T_{\mathbf{W}} \leq T_{\mathbf{L}} \leq T_{\mathbf{P}} \leq T_{\mathbf{M}}$. Moreover, for each t-norm T we have $T_{\mathbf{W}} \leq T \leq T_{\mathbf{M}}$. A continuous t-norm is called *Archimedean* if for each $x \in]0,1[$ we have $T(x,x) < x$

An interesting family of t-norms $\{T_s^{\mathbf{F}}\}_{s\in[0,+\infty]}$ was studied in [2]:

$$T_s^{\mathbf{F}}(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \text{if } s = 0, \\ T_{\mathbf{P}}(x,y) & \text{if } s = 1, \\ T_{\mathbf{L}}(x,y) & \text{if } s = \infty, \\ \log_s \left[1 + \frac{(s^x-1)(s^y-1)}{s-1}\right] & \text{otherwise.} \end{cases}$$

These t-norms will be referred to as the *Frank t-norms*, the family of the dual *Frank t-conorms* will be denoted $\{S_s^{\mathbf{F}}\}_{s\in[0,+\infty]}$. The family $\{T_s^{\mathbf{F}}\}_{s\in[0,+\infty]}$ of Frank t-norms is decreasing (see [1] and [4]) and continuous in the sense that we have

$$(s_n)_{n\in\mathbb{N}} \uparrow t \Rightarrow (T_{s_n}^{\mathbf{F}})_{n\in\mathbb{N}} \downarrow T_t^{\mathbf{F}}.$$

2 Definition of the t-reverse

Let T be a t-norm. Then the function $T^* : [0,1]^2 \to [0,1]$ defined by

$$T^*(x,y) = \max(0, x + y - 1 + T(1-x, 1-y)) \tag{1}$$

is called the *t-reverse* of T. This definition goes back to [3] where the name *invert* was used for T^*.

Using the dual t-conorm S_T of T, this definition can be rewritten as

$$T^*(x,y) = \max(0, x + y - S_T(x,y)). \tag{2}$$

The construction of T^* can be conceived geometrically as follows (it is visualized in Figure 1):

(i) The graph of T is rotated 180^o around the vertical symmetry axis of the unit cube

(ii) The plane $z = x + y - 1$ is added to the rotated graph (this implies that the boundary conditions $T^*(x,1) = x$ and $T^*(x,0) = 0$ are satisfied).

(iii) Any negative values are replaced by zero.

Figure 1: Visualization of the reversion: a t-norm (top left), rotating it around the vertical symmetry axis (top right), adding the plane $x+y-1$ (bottom left), cutting off negative values (bottom right).

It is clear that T^* satisfies the symmetry and boundary conditions required for t-norms. The monotonicity and associativity, however, may not hold for T^*:

Example 2.1. (i) $T_{\mathbf{W}}^* = T_{\mathbf{L}}$.

(ii) $T_{\mathbf{L}}^* = T_{\mathbf{L}}$.

(iii) If T is the t-norm given by

$$T(x,y) = \begin{cases} \frac{xy}{x+y-xy}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

then T^* is not associative, since, e.g., $T^*(T^*(0.2, 0.9), 0.9) \approx 0.1952$ and $T^*(0.2, T^*(0.9, 0.9)) \approx 0.1948$.

(iv) Let T be the ordinal sum $\{\langle 0, 0.5, T_{\mathbf{W}} \rangle, \langle 0.5, 1, T_{\mathbf{L}} \rangle\}$ (for the general definition of ordinal sums see Section 4). Then T^* is not non-decreasing, since, e.g., $T^*(0.4, 0.6) = 0.4 > 0.2 = T^*(0.6, 0.6)$.

Examples 2.1 (iii) and (iv) both show that the t-reverse T^* of a t-norm T not necessarily is a t-norm. We shall say that a t-norm T is *t-reversible* if its t-reverse T^* is also a t-norm, and we shall denote the family of all t-reversible t-norms by $\mathcal{R}$.

3 General properties

In [3] it was conjectured that a t-norm T is t-reversible only if T equals one of the basis t-norms $T_{\mathbf{M}}, T_{\mathbf{P}}, T_{\mathbf{L}}, T_{\mathbf{W}}$ or a specific ordinal sum (for the general definition of ordinal sums see again Section 4) thereof. However, this conjecture turns out to be incorrect, as a consequence of the following result.

Theorem 3.1. *For all t-norms T with $T \leq T_{\mathbf{L}}$ we have $T^* = T_{\mathbf{L}}$.*

Proof. If $x+y \leq 1$ then $x+y = S_{\mathbf{L}}(x,y) \leq S_T(x,y)$, where S_T is the dual t-conorm of T, in which case we have $x + y - S_T(x,y) \leq 0$ and, therefore, $T^*(x,y) = 0$. If $x + y > 1$ then $1 = S_{\mathbf{L}}(x,y) \leq S_T(x,y)$, implying $S_T(x,y) = 1$ and, consequently, $T^*(x,y) = x + y - 1$. ■

Theorem 13 in [3] claims that for a t-norm T we always have $T^{**} = T$.This is not true since T may not be t-reversible, in which case $T^{**} = (T^*)^*$ is not properly defined. Even if T is t-reversible, this claim is wrong: from Example 2.1 (i) and (ii) we have $T_{\mathbf{W}}^* = T_{\mathbf{L}}$ and $T_{\mathbf{L}}^* = T_{\mathbf{L}}$, showing that $T_{\mathbf{W}}^{**} \neq T_{\mathbf{W}}$. However, we get the following result:

Theorem 3.2. *Let T be a t-reversible t-norm. Then $T^{**} = T$ if and only if $T \geq T_{\mathbf{L}}$.*

Proof. By definition we have

$$T^{**}(x,y) = \max[0, x+y-S_{T^*}(x,y)],$$

where S_{T^*} is the dual of the t-norm T^*, for which we get

$$\begin{aligned} S_{T^*}(x,y) &= 1-T^*(1-x,1-y) \\ &= 1-\max[0,1-x+1-y-S_T(1-x,1-y)] \\ &= 1-\max[0,T(x,y)+1-x-y] \\ &= \min[1,x+y-T(x,y)]. \end{aligned}$$

This implies

$$\begin{aligned} T^{**}(x,y) &= \max[0,x+y-\min(1,x+y-T(x,y))] \\ &= \max[0,\max(x+y-1,T(x,y))] \\ &= \max(T_{\mathbf{L}}(x,y),T(x,y)). \end{aligned}$$

Now it is clear that $T^{**} = T$ if and only if $T \geq T_{\mathbf{L}}$. ∎

Corollary 3.3. *Suppose that both T and T^* are t-reversible t-norms. Then we have $T^{***} = T^*$.*

Proof. This is obvious since we always here

$$T^*(x,y) = \max(0,x+y-S_T(x,y)) \geq \max(0,x+y-1) = T_{\mathbf{L}}(x,y).$$

∎

Theorem 3.4. *Let T be a continuous Archimedean, t-reversible t-norm. Then T^* is a continuous Archimedean t-norm.*

Proof. Continuity follows from the definition. That T^* is Archimedean is a consequence of the fact that for all $x \in]0,1[$

$$T^*(x,x) = \max(0,x+x-S_T(x,x)) < x,$$

since the dual t-conorm S_T of T satisfies $S_T(x,x) > x$ for all $x \in]0,1[$. ∎

4 t-reverses of ordinals sums

An important way to construct new t-norms from given ones is that of an ordinal sum: let $\{]\alpha_k,\beta_k[\}_{k\in K}$ be a non-empty countable family of pairwise disjoint open subintervals of [0,1] and let $\{T_k\}_{k\in K}$ be a family of corresponding t-norms. Then the *ordinal sum* $\{\langle\alpha_k,\beta_k,T_k\rangle\}_{k\in K}$ is the function $T:[0,1]^2 \to [0,1]$ defined by

$$T(x,y) = \begin{cases} \alpha_k + (\beta_k-\alpha_k)\cdot T_k\left(\frac{x-\alpha_k}{\beta_k-\alpha_k}, \frac{y-\alpha_k}{\beta_k-\alpha_k}\right) & \text{if } x,y \in [\alpha_k,\beta_k], \\ \min(x,y) & \text{otherwise,} \end{cases}$$

which is always a t-norm. In order to keep the notation short, we also consider here the trivial ordinal sum $T = \{\langle 0, 1, T_1 \rangle\}$, i.e., where $K = \{1\}$ is a one point set and $\alpha_1 = 0$ and $\beta_1 = 1$, in which case we have $T = T_1$.

Ordinal sums of t-conorms are defined in the same way as ordinal sums of t-norms, only replacing min by max. Observe, however, that the dual t-conorm of an ordinal sum $\{\langle \alpha_k, \beta_k, T_k \rangle\}_{k \in K}$ of t-norms is the ordinal sum $\{\langle 1 - \beta_k, 1 - \alpha_k, S_{T_k} \rangle\}_{k \in K}$ of t-conorms which, in general, is different from the ordinal sum $\{\langle \alpha_k, \beta_k, S_{T_k} \rangle\}_{k \in K}$.

Each continuous t-norm can be written as an ordinal sum $\{\langle \alpha_k, \beta_k, T_k \rangle\}_{k \in K}$ such that all T_k are continuous Archimedean t-norms.

Denote by $\mathcal{F}$ the family of t-norms T such that the function $G : [0,1]^2 \to [0,1]$ given by

$$G(x, y) = x + y - T(x, y) \tag{3}$$

is associative, i.e., a t-conorm.

Each element of $\mathcal{F}$ can be written as an ordinal sum $\{\langle \alpha_k, \beta_k, T_k \rangle\}_{k \in K}$ such that all T_k are Frank t-norms (see [2]). For more details about ordinal sums, see, e.g., [6].

In [3] the class of all t-norms satisfying the condition

$$x \le u \text{ and } y \le v \Rightarrow u + v - T(u, v) \ge x + y - T(x, y) \tag{4}$$

was denoted by $\mathcal{M}$ (in the language of [3], these t-norms are said to be *of moderate growth*). In [3, Theorem 12] it is shown that, given $T \in \mathcal{M}$, then T^* is necessarily non-decreasing in each component, so only the associativity of the t-reverse can be a problem. Finally, Theorem 16 in [3] proves that if $T \in \mathcal{M}$ is an ordinal sum of t-reversible t-norms, i.e., $T = \{\langle \alpha_k, \beta_k, T_k \rangle\}_{k \in K}$, with T_k reversible, then T itself is t-reversible, and T^* equals the ordinal sum $\{\langle 1 - \beta_k, 1 - \alpha_k, T_k^* \rangle\}_{k \in K}$.

An interesting question is now the relation between the three families $\mathcal{R}, \mathcal{M}$, and $\mathcal{F}$, i.e., of the families of t-norms which are t-reversible, of moderate growth, and which are solutions of the problem of Frank [2], respectively. Here are some simple observations concerning this problem.

Example 4.1. (i) The monotonicity of t-conorms implies that all elements of $\mathcal{F}$ belong to $\mathcal{M}$, i.e., $\mathcal{F}$ is a subfamily of $\mathcal{M}$.
(ii) Conversely, an element of $\mathcal{M}$ need not be an element of $\mathcal{F}$: the t-norm T mentioned in Example 2.1 (iii) is an example for this, showing that $\mathcal{F}$ is a proper subfamily of $\mathcal{M}$.
(iii) Not each t-reversible t-norm belongs to $\mathcal{M}$: $T_{\mathbf{W}}$ is an example for this. Hence, $\mathcal{R}$ is not a subfamily of $\mathcal{M}$.

The exact relationship relation between the three families $\mathcal{R}, \mathcal{M}$ and $\mathcal{F}$ is given as follows.

Theorem 4.2. *A t-norm T is both t-reversible and an element of $\mathcal{M}$ if and only if T is an element of $\mathcal{F}$ (this means that we have $\mathcal{F} = \mathcal{R} \cap \mathcal{M}$).*

Proof. Assume first that $T = \{\langle \alpha_k, \beta_k, T^{\mathbf{F}}_{s_k}\rangle\}_{k\in K}$ is an element of $\mathcal{F}$ and, consequently, of $\mathcal{M}$. Let S_T be the dual t-conorm of T, i.e., S_T is the ordinal sum $\{\langle 1-\beta_k, 1-\alpha_k, S^{\mathbf{F}}_{s_k}\rangle\}_{k\in K}$. Then from [2] we know that the expression

$$x + y - S_T(x, y)$$

is always nonnegative and defines a t-norm. Taking into account

$$\begin{aligned} T^*(x,y) &= \max(0, x+y-S_T(x,y)) \\ &= x+y-S_T(x,y), \end{aligned}$$

it is clear that T is t-reversible.

If, conversely, $T \in \mathcal{R} \cap \mathcal{M}$, observe first that (4) implies the inequality

$$1 = 1 + 1 - T(1,1) \geq 1 - x + 1 - y - T(1-x, 1-y),$$

from which we get

$$0 \leq x + y - 1 + T(1-x, 1-y) = x + y - S_T(x, y).$$

Now, using $T \in \mathcal{R}$ and (2), we get

$$T^*(x, y) = x + y - S_T(x, y)$$

or, equivalently,

$$S_T(x, y) = x + y - T^*(x, y),$$

which, as a consequence of the results in [2], means that S_T can be written as an ordinal sum $\{\langle \alpha_k, \beta_k, S^{\mathbf{F}}_{s_k}\rangle\}_{k\in K}$, implying that we have $T = \{\langle 1-\beta_k, 1-\alpha_k, T^{\mathbf{F}}_{s_k}\rangle\}_{k\in K}$, i.e., $T \in \mathcal{F}$. ■

Remark 4.3. (i) Note that from the proof of Theorem 4.2 we can conclude that for $T \in \mathcal{F}$ we have

$$T^*(x, y) = 1 - S(1-x, 1-y),$$

where S is the t-conorm defined by $S(x, y) = x + y - T(x, y)$.
(ii) Let T be an ordinal sum of Frank t-norms, i.e., $T = \{\langle \alpha_k, \beta_k, T^{\mathbf{F}}_{s_k}\rangle\}_{k\in K}$. Using the fact that for each pair $(T^{\mathbf{F}}_{s_k}, S^{\mathbf{F}}_{s_k})$ we have

$$T^{\mathbf{F}}_{s_k}(x, y) + S^{\mathbf{F}}_{s_k}(x, y) = x + y$$

(see again [2]), we see that T^* equals the ordinal sum $\{\langle 1-\beta_k, 1-\alpha_k, T^{\mathbf{F}}_{s_k}\rangle\}_{k\in K}$, the dual t-conorm S_{T^*} of which is just given by $S_{T^*}(x, y) = x + y - T(x, y)$.
(iii) This means that all Frank t-norms are self-reverse, i.e., we have $(T^{\mathbf{F}}_s)^* = T^{\mathbf{F}}_s$ for all $s \in [0, +\infty]$ (for a more detailed discussion see Section 5).

Example 2.1 (iv) and Theorem 3.1 show that ordinal sums of t-reversible t-norms, in general, need not be t-reversible (this fact is visualized in Figure 2). The following proves that a t-reversible ordinal sum can have at most one summand which is smaller than $T_{\mathbf{L}}$.

Figure 2: Ordinal sum $\{\langle 0.3, 0.9, T\rangle\}$ with $T(x,y) = 1-\min[1-(\sqrt{1-x}+\sqrt{1-y})^2]$, i.e., $T < T_{\mathbf{L}}$ (top left) whose t-reverse (top right) is not monotone and, therefore, not a t-norm. The t-reverse (bottom right) of the ordinal sum $\{\langle 0.4, 1, T\rangle\}$ (bottom left), however, is a t-norm, namely, the ordinal sum $\{\langle 0, 0.6, T_{\mathbf{L}}\rangle\}$.

Theorem 4.4. *Let T be the ordinal sum $\{\langle \alpha_k, \beta_k, T_k\rangle\}_{k\in K}$ such that T is t-reversible and $T_{k_0} < T_{\mathbf{L}}$ for some $k_0 \in K$. Then we have $\beta_{k_0} = 1$ (as a consequence, there is at most one summand T_k with $T_k < T_{\mathbf{L}}$).*

Proof. Let $(x,y) \in]0,1[^2$ be a point such that $T_{k_0}(x,y) < T_{\mathbf{L}}(x,y)$, i.e.,

$$x + y - 1 - T_{k_0}(x,y) > 0. \tag{5}$$

Assume that $\beta_{k_0} < 1$ is true. Then, on the one hand, we have

$$T^*(1-\beta_{k_0}, 1-\beta_{k_0}) = 1-\beta_{k_0}. \tag{6}$$

On the other hand, observe that

$$1-\alpha_{k_0} + (\alpha_{k_0}-\beta_{k_0})\cdot x > 1-\beta_{k_0}, \tag{7}$$
$$1-\alpha_{k_0} + (\alpha_{k_0}-\beta_{k_0})\cdot y > 1-\beta_{k_0}, \tag{8}$$

implying that

$$\begin{aligned}
&T^*(1-\alpha_{k_0} + (\alpha_{k_0}-\beta_{k_0})\cdot x, 1-\alpha_{k_0} + (\alpha_{k_0}-\beta_{k_0})\cdot y) \\
&\quad= \max(0, 1-\alpha_{k_0} + (\alpha_{k_0}-\beta_{k_0})\cdot x + 1-\alpha_{k_0} + (\alpha_{k_0}-\beta_{k_0})\cdot y - 1 \\
&\qquad\qquad +\alpha_{k_0}(\alpha_{k_0}-\beta_{k_0})\cdot T_{k_0}(x,y)) \\
&\quad= \max(0, 1-\alpha_{k_0} + (\beta_{k_0}-\alpha_{k_0})\cdot (T_{k_0}(x,y) - x - y)) \\
&\quad= \max(0, 1-\beta_{k_0} - (\beta_{k_0}-\alpha_{k_0})\cdot (x+y-1-T_{k_0}(x,y))) \\
&\quad< 1-\beta_{k_0},
\end{aligned}$$

where the inequality follows from (5). This, together with (6), (7) and (8), violates the monotonicity of the t-norm T^*, and therefore $\beta_{k_0} < 1$ cannot be true. ∎

Conversely, it is not difficult to see that the each ordinal sum of some special form is t-reversible allowing us to formulate the following result:

Corollary 4.5. *Let the t-norm T be the ordinal sum $\{\langle\alpha_k,\beta_k,T_k\rangle\}_{k\in K}$ of Frank t-norms up to possibly one summand, say T_{k_0}, with $T_{k_0} < T_{\mathbf{L}}$ and $\beta_{k_0} = 1$. Then T is t-reversible and its t-reverse T^* equals the t-reverse of $\tilde{T}$, where $\tilde{T}$ is the ordinal sum $\{\langle\alpha_k,\beta_k,\tilde{T}_k\rangle\}_{k\in K}$ with $\tilde{T}_k = T_k$ for all $k \neq k_0$ and $\tilde{T}_{k_0} = T_{\mathbf{L}}$.*

5 Self-reverse t-norms

We are now interested in studying t-norms which are self-reverse, i.e., satisfy the equality $T^* = T$. From Remark 4.3(iii) we know that all Frank t-norms $T_s^{\mathbf{F}}$, $s \in [0,+\infty]$ have this property. We are now able to characterize all continuous self-reverse t-norms.

Theorem 5.1. *Let T be a continuous t-norm. Then $T^* = T$ if and only if T is an ordinal sum $\{\langle\alpha_k,\beta_k,T_{s_k}^{\mathbf{F}}\rangle\}_{k\in K}$ of Frank t-norms such that for each $k \in K$ with $T_{s_k}^{\mathbf{F}} \neq T_{\mathbf{M}}$ there is a $j \in K$ with $s_j = s_k$, $\alpha_j = 1-\beta_k$ and $\beta_j = 1-\alpha_k$.*

Proof. Assuming $T^* = T$ then we have $T^{**} = T$ and, by Theorem 3.2, $T \geq T_{\mathbf{L}}$. Then for the dual t-conorm S_T of T we obtain

$$S_T(x,y) \leq S_{\mathbf{L}}(x,y) \leq x+y,$$

implying

$$x + y - S_T(x, y) \geq 0$$

and, taking into account $T^* = T$,

$$T(x, y) = x + y - S_T(x, y).$$

Because of [2], this means that T must be an ordinal sum $\{\langle \alpha_k, \beta_k, T^{\mathbf{F}}_{s_k} \rangle\}_{k \in K}$ of Frank t-norms. From Remark 4.3(ii) we know that T has to be symmetric in the sense that for each $k \in K$ with $T^{\mathbf{F}}_{s_k} \neq T_{\mathbf{M}}$ ($T_{\mathbf{M}}$ acts like a neutral element when constructing ordinal sums and does not influence this symmetry) there exists a $j \in K$ such that $s_j = s_k$, $\alpha_j = 1 - \beta_k$ and $\beta_j = 1 - \alpha_k$. ■

Recall that in the trivial case $K = \{1\}$, $\alpha_1 = 0$ and $\beta_1 = 1$, i.e., if T itself is a Frank t-norm, the symmetry condition is always satisfied. In the light of this theorem we can give the following variation of the results of [2]:

Corollary 5.2 . *For a continuous t-norm T the function $G : [0,1]^2 \to [0,1]$ given by $G(x, y) = x + y - T(x, y)$ is a t-conorm if and only if T is an ordinal sum $\{\langle \alpha_k, \beta_k, T^{\mathbf{F}}_{s_k} \rangle\}_{k \in K}$ of Frank t-norms, in which case the t-conorm G is dual to the t-reverse T^*, i.e.,*

$$G(x, y) = 1 - T^*(1 - x, 1 - y).$$

6 Concluding remarks

Some questions concerning t-reverses of t-norms remain still open. The most important open problem is the complete characterization of all t-reversible t-norms. Other related questions can be formulated as follows:

Question 1. *Is a continuous t-norm T t-reversible if and only if T is an ordinal sum whose summands are Frank t-norms up to possibly one summand in the upper right corner of the unit square which is weaker than $T_{\mathbf{L}}$?*

Question 2. *If T is a t-reversible t-norm, is T^* necessarily t-reversible?*

Question 3*. *If T is a t-reversible t-norm, is T^* necessarily continous?*

Question 4. *If T is a t-reversible t-norm, is T^* necessarily an ordinal sum of Frank t-norms?*

We conjecture that there is an affirmative answer to each of these questions. However, we have not proven this claim so far (nor do we have counterexamples). Obviously, if there is a positive answer to Question 4, this would imply positive answers to both Questions 2 and 3.

* **Note added in proof:** An affirmative answer to Question 3 was given in M. Šabo, On the continuity of t-reverse of t-norms, *Tatra Mountains Math. Publ.* 6 (1995), 173-178.

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