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Convex Isomorphisms of Archimedean Lattice Ordered Groups

Ján Jakubík
Matematický ústav SAV, Grešákova 6,
040 01 Košice, Slovakia

Abstract

This paper contains a result of Cantor-Bernstein type concerning archimedean lattice ordered groups.

Sikorski [9] and Tarski [11] (cf. also Sikorski [10]) proved a theorem of Cantor-Bernstein type for σ-complete Boolean algebras.

Further, theorems of such type were proved by the author for some classes of complete lattice ordered groups (cf. [5, 6]) and for a class of complete MV-algebras (cf. [7]).

Let G be a lattice ordered group. The underlying lattice of G will be denoted by $\ell(G)$. Next, let G^D and G^L be the Dedekind completion or the lateral completion of G, respectively.

An isomorphism ψ of a lattice L_1 into a lattice L_2 is said to be convex if $\psi(L_1)$ is a convex sublattice of L_2.

In the present paper the following result is proved:

(A_1) Let G_1 and G_2 be archimedean lattice ordered groups. Suppose that

(i) there exists a convex isomorphism of the lattice $\ell(G_1)$ into $\ell(G_2)$;

(ii) there exists a convex isomorphisms of the lattice $\ell(G_2)$ into $\ell(G_1)$.

Then the lattice ordered groups G_1^{DL} and G_2^{DL} are isomorphic.

This generalizes Theorem (A) of [5].

1 Preliminaries

For lattice ordered groups we apply the standard definitions and notations (cf., e.g., Darnel [3], Kopytov and Medvedev [8]). The group operation in a lattice ordered group will be written additively.

In this section we recall some relevant notions and results.

Let G be a lattice ordered group. G is called complete (σ-complete) if each nonempty bounded subset of G (or each nonempty denombrable bounded subset of G) has the supremum and the infimum in G.

A nonempty subset $\{x_i\}_{i\in I}$ of G is said to be disjoint (or orthogonal) if $x_i \geqq 0$ for each $i \in I$, and $x_{i(1)} \wedge x_{i(2)} = 0$ whenever $i(1), i(2) \in I$ and $x_{i(1)} \neq x_{i(2)}$.

G is called orthogonally complete if each disjoint subset of G has the supremum in G.

1.1. Theorem (Cf. [5], Theorem (A).) *Let G_1 and G_2 be lattice ordered groups which are complete and orthogonally complete. Suppose that*

(i) there exists a convex isomorphism of $\ell(G_1)$ into $\ell(G_2)$;

(ii) there exists a convex isomorphism of $\ell(G_2)$ into $\ell(G_1)$.

Then G_1 and G_2 are isomorphic.

If G is an subgroup of a lattice ordered group H such that for each $h \in H$ with $0 < h$ there exists $g \in G$ with $0 < g \leqq h$, then G is called a dense ℓ-subgroup of H.

We recall (cf. Conrad [2]) that a lattice ordered group H is said to be a lateral completion of G if the following conditions are satisfied:

(i) H is orthogonally complete;

(ii) G is a dense ℓ-subgroup of H;

(iii) if H_1 is an ℓ-subgroup of H such that $G \subseteq H_1$ and H_1 is orthogonally complete, then $H_1 = H$.

If H is a lateral completion of G then we express this fact by writing $H = G^L$. In view of Bernau [1], each lattice ordered group possesses a lateral completion.

For the notion of the Dedekind completion G^D of a lattice ordered group G cf., e.g., Darnel [3].

2 Auxiliary results

Let L be a lattice. For $X \subseteq L$ we denote by X^u and X^ℓ the set of all upper bounds or the set of all lower bounds of X in L, respectively.

Next let $\mathcal{L}$ be the system of all subsets X of L such that $X \neq \emptyset$ and $X^u \neq \emptyset$. Put

$$D(L) = \{X^{u\ell} : X \in \mathcal{L}\}.$$

The system $D(L)$ is partially ordered by inclusion.

It is easy to verify that $D(L)$ is a conditionally complete lattice. If $\{X_i\}_{i\in I}$ is a nonempty subsystem of $D(L)$ and if it is bounded in $D(L)$, then

$$\wedge_{i\in I} X_i = \cap_{i\in I} X_i, \quad \vee_{i\in I} X_i = (\cup_{i\in I} X_i)^{u\ell}.$$

Suppose that M is a convex sublattice of the lattice L. For $X \subseteq M$ let $X^{u(M)} = X^u \cap M$, $X^{\ell(M)} = X^\ell \cap M$. Further let $\mathcal{L}_M$ be defined analogously as $\mathcal{L}$ (with L replaced by M), and

$$D(M) = \{X^{u(M)\ell(M)} : X \in \mathcal{L}_M\}.$$

Similarly as in the case of $D(L)$ we consider $D(M)$ as to be partially ordered by inclusion.

If $X \in \mathcal{L}_M$, then there exists $m \in M$ such that $x \leqq m$ for each $x \in X$. Thus for each $t \in X^u$ we have

$$m \wedge t \in X^{u(M)},$$

and this yields that

(i) X^u is a filter of L which is generated by the set $X^{u(m)}$,

(ii) $X^{u(M)}$ is an ideal of X^u.

From (i) and (ii) we obtain the relation

$$X^{u\ell} = X^{u(M)\ell}, \tag{1}$$

whence

$$X^{u(M)\ell(M)} = X^{u\ell} \cap M. \tag{2}$$

Clearly $D(L) \subseteq \mathcal{L}$ and $D(M) \subseteq \mathcal{L}_M$. Further, if $X \in D(M)$, then

$$X^{u(M)\ell(M)} = X,$$

hence according to (2) we have

$$X = X^{u\ell} \cap M. \tag{3}$$

We define a mapping $\psi : D(M) \to D(L)$ such that

$$\psi(X) = X^{u\ell}$$

for each $X \in D(M)$.

Let $X, Y \in D(M)$. If $X \leqq Y$, then celarly $\psi(X) \leqq \psi(Y)$, hence the mapping ψ is isotone. If $\psi(X) \leqq \psi(Y)$, then in view of (3) we get $X \leqq Y$, thus ψ is an isomorphism of $D(M)$ onto the partially ordered subset $\psi(D(M))$ of $D(L)$.

Let $X, Y \in D(M)$, $Z \in D(L)$ and suppose that

$$X^{u\ell} \leqq Z \leqq Y^{u\ell}.$$

Put $Z_1 = Z \cap M$. We have $X \subseteq Z \cap M$, whence $Z_1 \neq \emptyset$. There exists $y_0 \in M$ such that $y \leqq y_0$ for each $y \in Y$. Further, in view of (3) we have

$$Y = Y^{u\ell} \cap M \supseteq Z \cap M,$$

whence $z_1 \leqq y_0$ for each $z_1 \in Z_1$. Therefore $Z_1 \in \mathcal{L}_M$ and thus $Z_1^{u(M)\ell(M)} \in D(M)$.

By using (1) we get $Y^{u\ell} = Y^{u(M)\ell}$. Thus if $t \in Y^{u\ell}$, then $t \leqq s$ for each $s \in Y^{u(M)}$, hence, in particular, $t \leqq y_0$. Thus $z \leqq y_0$ for each $z \in Z$.

Let $z_1 \in Z_1$, $v \in Z_1^u$ and $z \in Z$. Then

$$z_1 \leqq z_1 \vee z \leqq y_0,$$

whence $z_1 \vee z \in M$. From the relation $Z \in D(L)$ we infer that Z is a sublattice of L, thus $z_1 \vee z \in Z$. We obtain $z_1 \vee z \in Z_1$, which yields that $z \leqq z_1 \vee z \leqq v$. Then

$$Z_1^u \subseteqq Z^u.$$

Since $Z_1 \subseteq Z$, the relation $Z_1^u \supseteqq Z^u$ holds. Therefore $Z_1^u = Z^u$ and so

$$Z_1^{u\ell} = Z^{u\ell} = Z.$$

In view of (2) we have

$$Z_1^{u(M)\ell(M)} = Z_1^{u\ell} \cap M,$$

hence

$$Z_1^{U(M)\ell(M)} = Z \cap M = Z_1.$$

Therefore $Z_1 \in D(M)$. At the same time, $\psi(Z_1) = Z$. We conclude that $\psi(D(M))$ is a convex subset of the lattice L.

Let $\psi(P) = P_1$, $\psi(Q) = Q_1$. Since $D(M)$ is a lattice, there are $U, V \in D(M)$ such that both P and Q belong to the interval $[U, V]$ of $D(M)$. Then $\psi(U) \leqq \psi(V)$ for $T \in \{P_1, Q_1\}$. In view of the convexity of $\psi(D(M))$ in $D(L)$ we infer that $\psi(D(M))$ is a sublattice of $D(L)$.

By summarizing, we obtain

2.1. Lemma *Let M be a convex sublattice of a lattice L. Then there exists a convex isomorphism of the lattice $D(M)$ into the lattice $D(L)$.*

2.2. Lemma *Let L_1 and L_2 be lattices such that there exists a convex isomorphism of L_1 into L_2. Then there exists a convex isomorphism of $D(L_1)$ into $D(L_2)$.*

Proof This is an immediate consequence of 2.1. □

2.3. Lemma *Let G be an archimedean lattice ordered group. Then the underlying lattice of its Dedekind completion G^D is equal to $D(\ell(G))$.*

This is well-known; cf., e.g., Darnel [3].

For a lattice ordered group G we denote by $\ell(G^+)$ the underlying lattice of the lattice ordered semigroup G^+.

2.4. Lemma *Let G_1 and G_2 be lattice ordered groups. Then the following conditions are equivalent:*

(i) There exists a convex isomorphisms of the lattice $\ell(G_1)$ into $\ell(G_2)$.

(ii) There exists a convex isomorphism of the lattice $\ell(G_1^+)$ into $\ell(G_2^+)$.

Proof This is a consequence of Lemma 1.4 in [6]. □

3 Archimedean lattice ordered groups

3.1. Lemma *Let G_1 and G_2 be archimedean lattice ordered groups. Assume that there exists a convex isomorphism of the lattice $\ell(G_1)$ into $\ell(G_2)$. Then there exists a convex isomorphism of the lattice $\ell(G_1^D)$ into the lattice $\ell(G_2^D)$.*

Proof The assertion follows from 2.2 and 2.3. □

3.2. Lemma *Let G be a complete lattice ordered group and $H = G^L$, $0 \leqq h \in H$. Then there exists a disjoint subset $\{x_i\}_{i\in I}$ of G such that the relation $h = \vee_{i\in I} h_i$ is valid in H.*

Proof Cf. [4], Section 2. □

The following lemma is easy to verify, the proof will be omitted.

3.3. Lemma *Let G be a lattice ordered group. Suppose that $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ are disjoint subsets of G and that $x = \vee_{i\in I} x_i$, $y = \vee_{j\in J} y_j$. Then the following conditions are equivalent:*

(i) $x \leqq y$;

(ii) *for each $i \in I$ the relation*

$$x_i = \vee_{j\in J}(x_i \wedge y_j)$$

is valid.

Now assume that G_1 and G_2 are complete lattice ordered groups and that φ is a convex isomorphism of the lattice $\ell(G_1^+)$ into $\ell(G_2^+)$. Put

$$\varphi_0(t) = \varphi(t) - \varphi(0)$$

for each $t \in G_1$. Then φ_0 is a convex isomorphism of $\ell(G_1^+)$ into $\ell(G_2^+)$ such that $\varphi_0(0) = 0$.

Denote $H_i = G_i^L$ $(i = 1, 2)$. Let $x \in G_1^L$, $x \geqq 0$. In view of 3.2 there exists a disjoint subset $\{x_i\}_{i\in I}$ of G_1 such that

$$x = \vee_{i\in I} x_i$$

is valid in H_1. If, at the same time, there is another disjoint subset $\{x'_j\}_{j\in J}$ in G_1 such that

$$x = \vee_{j\in J} x'_j$$

holds in H_1, then according to 3.3 we have in G_1 the relations

$$x_i = \vee_{j\in J}(x_i \wedge x'_j) \quad \text{for each } i \in I,$$

$$x'_j = \vee_{i\in I}(x'_j \wedge x_i) \quad \text{for each } j \in J.$$

We obtain that $\{\varphi_0(x_i)\}_{i\in I}$ and $\{\varphi_0(x'_j)\}_{j\in J}$ are disjoint subsets of G_2 such that the relations

$$\varphi(x_i) = \vee_{j\in J}(\varphi_0(x_i) \wedge \varphi_0(x'_j)) \quad \text{for each } i \in I,$$

$$\varphi_0(x'_j) = \vee_{i\in I}(\varphi_0(x'_j) \wedge \varphi_0(x_i)) \quad \text{for each } j \in J$$

are valid in G_2. Then in view of 3.3,

$$\vee_{i\in I}\varphi_0(x_i) = \vee_{j\in J}\varphi_0(x'_j)$$

holds in H_2.

We put $\psi(x) = \vee_{i\in I}\varphi_0(x_i)$. Then according to the above mentioned relations, ψ is a correctly defined mapping of the set H_1^+ into H_2^+ such that $\psi(t) = \varphi_0(t)$ for each $t \in G_1$.

Further, from 3.3 we infer that if $x, y \in H_1$ and $x \leqq y$, then $\psi(x) \leqq \psi(y)$. Moreover, from the fact that φ_0 is a convex isomorphism and by using 3.3 again we obtain that

$$\psi(x) \leqq \psi(y) \Rightarrow x \leqq y.$$

Thus, in particular, ψ is a monomorphism.

Let $\{x_i\}_{i\in I}$ and let x be as above. Suppose that $z \in H_2$, $0 \leqq z \leqq \psi(x)$. Hence

$$z = z \wedge \psi(x) = \vee_{i\in I}(z \wedge \varphi_0(x_i)).$$

The element z can be expressed in the form

$$z = \vee_{k\in K} z_k,$$

where $\{z_k\}_{k\in K}$ is a disjoint subset of G_2. Hence in H_2 we have

$$z = \vee_{i\in I} \vee_{k\in K} (z_k \wedge \varphi_0(x_i)).$$

Denote $t_{ki} = z_k \wedge \varphi_0(x_i)$ for each $k \in K$ and each $i \in I$. We have $t_{ki} \in [0, \varphi_0(x_i)]$ (where the interval is taken with respect to G_2), whence there is $x_{ki} \in G_1$ such that $\varphi_0(x_{ki}) = t_{ki}$. Moreover, $\{t_{ki}\}_{k\in K, i\in I}$ is a disjoint system in G_2 and thus $\{x_{ki}\}_{k\in K, i\in I}$ is a disjoint system in G_1. Hence there exists $t \in H_1$ such that

$$t = \vee_{k\in K, i\in I} x_{ki}$$

is valid in H_1, and then we have

$$\psi(t) = \vee_{k\in K, i\in I}\varphi_0(x_{ki}) = z.$$

In this way we verified that $\psi(H_1)$ is a convex subset of H_2. It is easy to show that for $z_1, z_2 \in \psi(H_1)$ there exists $z_3 \in \psi(H_1)$ with $z_3 \geqq z_1$, $z_3 \geqq z_2$. By using this and the convexity of $\psi(H_1)$ we obtain that ψ is a convex isomorphism of $\ell(H_1^+)$ into $\ell(H_2^+)$.

Hence we have

3.4. Lemma *Let G_1 and G_2 are complete lattice ordered groups. Suppose that there exists a convex isomorphism of $\ell(G_1^+)$ into $\ell(G_2^+)$. Then there exists a convex isomorphism of $\ell(H_1^+)$ into $\ell(H_2^+)$, where $H_i = G_i^L$ $(i = 1, 2,)$.*

3.5. Lemma *Let G_1 and G_2 be archimedean lattice ordered groups. Suppose that there exists a convex isomorphism of the lattice $\ell(G_1)$ into $\ell(G_2)$. Then there exists a convex isomorphism of $\ell(H_1)$ into $\ell(H_2)$, where $H_i = G_i^{DL}$ $(i = 1, 2)$.*

Proof This is a consequence of 2.4, 3.1 and 3.4. □

Proof of (A_1)

Let G_1 and G_2 be as in (A_1). From 3.5 we obtain that

(i) there exists a convex isomorphism of $\ell(G_1^{DL})$ into $\ell(G_2^{DL})$;

(ii) there exists a convex isomorphism of $\ell(G_2^{DL})$ into $\ell(G_1^{DL})$.

Both G_1^{DL} and G_2^{DL} are laterally complete lattice ordered groups. Moreover, since G_1^D and G_2^D are complete lattice ordered groups, according to [4], Section 2, both G_1^{DL} and G_2^{DL} are complete lattice ordered groups as well. Now it suffices to apply Theorem 1.1. □

If G_1 and G_2 are complete and orthogonally complete, then $G_1^D = G = G_1^L$, hence $G_1^{DL} = G_1$, and similarly for G_2. Thus (A_1) is a generalization of (A).

We remark that each σ-complete lattice ordered group is archimedean; therefore in (A_1) the archimedean property can be replaced by σ-completeness.

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