Mathware & Soft Computing 5 (1998) 39-47

On Some Inexact Relations in Probabilized Boolean Algebras

Susana Cubillo
Dpto. de Matemática Aplicada. Facultad Informática
Universidad Politécnica de Madrid.
Campus de Montegancedo. 28660 Boadilla del Monte. Madrid.
e-mail: scubillo@fi.upm.es

Abstract

This paper is devoted to characterize monotonicity, conditionality and transitivity of some rational relations defined in a probabilized Boolean Algebra

1 Introduction.

Given a probabilized Boolean Algebra $(E,+,\cdot,p)$ there are several well known inexact relations that are Conditionals. For example $I_p\,(b/a) = p\,(a \to b) = p\,(a' + b) = 1 - p\,(a) + p\,(a \cdot b)$ is basic in Nilsson's probabilistic logic ([3]); this relation is a W-preorder ($W(a,b) = Max\,(0, x+y-1)$ the Lukasiewicz t-norm), but not a Prod-preorder. In plausible reasoning, G. Pólya ([5]) used the conditional probability $p^*\,(b/a) = \frac{p(a\cdot b)}{p(a)}$ which is not a T-preorder for any continuous t-norm T; this relation was also used by J. Pearl ([4]). This paper, on some way inspired in [7], is devoted to characterize monotonicity, conditionality and transitivity of some rational relations defined by means of a given probability. The choosen relation should agree with the problem we are working in, and from which we will induce the desirable properties.

We will only consider the case in which $(E,+,\cdot,p)$ is a probabilized Boolean Algebra such that for any $\alpha \in [0,1]$ it exists some $a \in E$ with $p\,(a) = \alpha$, and the relation has the form

$$R\,(b/a) = \frac{a_0 + a_1 p(a) + a_2 p(b) + a_3 p(a\cdot b)}{b_0 + b_1 p(a) + b_2 p(b) + b_3 p(a\cdot b)} \qquad (*)$$

Some theorems are enunciated without proof, because of both its easiness and the limited lengh of this paper. Proofs can be found in [2].

Let us begin by remembering some definitions to be used in the following sections. From now on, T will be a continuous t-norm.

Definition. A fuzzy relation $R : E \times E \to [0,1]$ is reflexive if for all $a \in E$ it is $R(a/a) = 1$, and it is T-transitive if for all a, b, c in E the inequality $T(R(b/a), R(c/b)) \leq R(c/a)$ holds.

A T-preorder is a reflexive and T-transitive fuzzy relation.

For example, relation $I_p(b/a) = 1 - p(a) + p(a \cdot b)$ is a W-preorder. The reflexivity is trivial, and let's prove that $W(R(b/a), R(c/b)) =$
$Max(0, 1 - p(a) + p(a \cdot b) - p(b) + p(b \cdot c)) \leq R(c/a) = 1 - p(a) + p(a \cdot c)$.

As $0 \leq R(c/a)$ it is sufficient to see $1 - p(a) + p(a \cdot b) - p(b) + p(b \cdot c) \leq 1 - p(a) + p(a \cdot c)$, or $p(a \cdot b \cdot c) + p(a \cdot b \cdot c') + p(b \cdot c \cdot a) + p(b \cdot c \cdot a') \leq p(b \cdot a \cdot c) + p(b \cdot a \cdot c') + p(b \cdot a' \cdot c) + p(b \cdot a' \cdot c') + p(a \cdot c \cdot b) + p(a \cdot c \cdot b')$, equivalent to $0 \leq p(b \cdot a' \cdot c') + p(a \cdot c \cdot b')$, which is always true.

But I_p is not a Prod-preorder: Choosing a, b, c such that $p(a) = 1$, $0 < p(b) < 1$ and $p(c) = 0$, it is $I_p(b/a) \cdot I_p(c/b) > I_p(c/a)$. So, I_p is not a T-preorder for any $T \geq \Pr od$.

The relation $p^*(b/a) = \frac{p(a \cdot b)}{p(a)}$ is not a T-preorder for any T. Taking a, b, c with $a \leq b$, $p(a) > 0$, $p(a \cdot c) = 0$, $p(b \cdot c) > 0$, it is $T(p^*(b/a), p^*(c/b)) = T\left(\frac{p(b \cdot a)}{p(a)}, \frac{p(b \cdot c)}{p(b)}\right) = T\left(1, \frac{p(b \cdot c)}{p(b)}\right) = \frac{p(b \cdot c)}{p(b)} > p^*(c/a)$.

Definition. The relation R is monotonic if $R(b/a) \leq R(b/a \cdot c)$ for all a, b and c. R si T-restricted monotonic if $T(R(b/a), R(c/a)) \leq R(b/a \cdot c)$, for all a, b and c.

Definition. Given a fuzzy set $\mu : E \to [0,1]$, a fuzzy relation $R : E \times E \to [0,1]$ is a μ-T-conditional, if for any a, b in E, it is $T(\mu(a), R(b/a)) \leq \mu(b)$. It is also said that μ is a T-Logical State for R.

For instance, the Kleene-Dienes' Implication $R_\mu^{KD}(b/a) = Max(1 - \mu(a), \mu(b))$ is a μ-W-conditional: $W(\mu(a), R_\mu^{KD}(b/a)) = Max(0, \mu(a) + Max(1 - \mu(a), \mu(b)) - 1) \leq \mu(b)$. Nevertheless it is not μ-Prod-conditional: choosing a, b such that $0 < \mu(a) < 1$, $\mu(b) = 0$, it will be $\mu(a) \cdot Max(1 - \mu(a), \mu(b)) > \mu(b)$. So, it is not a μ-T-conditional for any $T \geq Prod$.

Furthermore, in a probabilized Boolean Algebra (E, p), I_p and p^* are a p-W-conditional and a p-$Prod$-conditional, respectively.

It is well known ([9]) that R is a μ-T-conditional if and only if $R \leq I_\mu^T$, with $I_\mu^T(b/a) = sup\{z \in [0,1]; T(z, \mu(a)) \leq \mu(b)\}$.

2 Some Nilsson's Pattern probabilistic logics.

If the relation should model the logical implication, it seems addecuate to impose reflexivity. So,

Theorem 2.1. Given the Algebra (E, p), the relation $(*)$ is reflexive if and only if $a_0 = b_0$ and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$.

Proof.- R is reflexive if $R(a/a) = 1$ for all a, that is, if for any a , $a_0 + (a_1 + a_2 + a_3)\, p(a) = b_0 + (b_1 + b_2 + b_3)\, p(a)$, which is true only if $a_0 = b_0$ and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$.

It should be pointed out that if $a_0 \neq 0$, it always can be taken $a_0 = 1$. In this section it will be studied the case $a_0 = b_0 = 1$, and $b_1 = b_2 = b_3 = 0$; that is,

$$R(b/a) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)$$

and in order to get reflexivity, $a_1 + a_2 + a_3 = 0$.

Theorem 2.2. $R(b/a) \leq 1$ if and only if $a_1, a_2 \leq 0$.

Proof.- If $R(b/a) \leq 1$, it is $1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b) \leq 1$ for all a, b and $a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b) \leq 0$.

Taking a such that $p(a) = 0$, then $a_2 p(b) \leq 0$ for all b, and $a_2 \leq 0$. And, similarly, with $p(b) = 0$, it is $a_1 \leq 0$.

Conversely, if a_1 , $a_2 \leq 0$, as $a_3 = -a_1 - a_2$, it is $a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b) = a_1 (p(a) - p(a \cdot b)) + a_2 (p(b) - p(a \cdot b)) \leq 0$, and $R(b/a) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b) \leq 1$.

Similarly, it follows ([2])

Theorem 2.3. $R(b/a) \geq 0$ if and only if a_1 , $a_2 \geq -1$.

Therefore, to work with well defined relations, it should be $-1 \leq a_1$, $a_2 \leq 0$.

Let's now find the relations which are a T- preorder.

Theorem 2.4. $R(b/a) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)$ is a W-preorder.

Proof.- It should be $W(R(b/a), R(c/b)) \leq R(c/a)$ for any a, b, c, or equivalently $Max\,(0, 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b) + 1 + a_1 p(b) + a_2 p(c) + a_3 p(b \cdot c) - 1)$ $\leq 1 + a_1 p(a) + a_2 p(c) + a_3 p(a \cdot c)$.

$R(c/a)$ is always greater or equal to zero, and then it is sufficient to see that $a_1p(a)+a_2p(b)+a_3p(a\cdot b)+1+ a_1p(b)+a_2p(c)+a_3p(b\cdot c) \leq 1+ a_1p(a)+a_2p(c)+ a_3p(a\cdot c)$ that is equivalent to $(a_1+a_2)(p(b)-p(a\cdot b)-p(b\cdot c)+p(a\cdot c)) \leq 0$, because of $a_3=-a_1-a_2$. But $a_1+a_2 \leq 0$ and $p(b)-p(a\cdot b)-p(b\cdot c)+p(a\cdot c) = p(b)-p(a\cdot b\cdot c)-p(a\cdot b\cdot c')-p(b\cdot c)+p(a\cdot c) \geq 0$ (as $p(a\cdot b\cdot c')+p(b\cdot c) \leq p(b)$ and $p(a\cdot b\cdot c) \leq p(a\cdot c)$); then $(a_1+a_2)(p(b)-p(a\cdot b)-p(b\cdot c)+p(a\cdot c)) \leq 0$. ■

Theorem 2.5. If $R(b/a) = 1+ a_1p(a)+a_2p(b)+a_3p(a\cdot b)$ is a $Prod$-preorder, then $R(b/a) \equiv 1$; that is, $a_1=a_2=a_3=0$.

Proof. If R is a $Prod$-preorder, for all a,b,c it is $Prod(R(b/a), R(c/b)) \leq R(c/a)$ and $(1+a_1p(a)+a_2p(b)+a_3p(a\cdot b))(1+a_1p(b)+a_2p(c)+a_3p(b\cdot c)) \leq 1+ a_1p(a)+a_2p(c)+a_3p(a\cdot c)$.

Taking a,b,c such that $p(a)=0$, $b\leq c$, $0<p(b)<p(c)$, because of $a_3 = -a_1-a_2$ and simplifying, it is obtained $a_2^2p(b)(p(c)-p(b)) \leq 0$. As $p(b)>0$, $p(c)-p(b)>0$, it is $a_2^2 \leq 0$ and $a_2=0$.

Now, choosing a,b and c with $p(c)=0$, $b<a$ and $0<p(b)<p(a)$, it results $a_1^2 \leq 0$; that is, $a_1=0$ and $a_3=0$. ■

Corollary 2.6. If $R(b/a) = 1+ a_1p(a)+a_2p(b)+a_3p(a\cdot b)$ is a Min-preorder, it is $R(b/a)=1$, for all a,b.

Corollary 2.7. $I_p(b/a) = 1-p(a)+p(a\cdot b)$ is a W-preorder, but it is neither $Prod$-preorder nor Min-preorder.

Theorem 2.8. $R(b/a) = 1+ a_1p(a)+a_2p(b)+a_3p(a\cdot b)$ is monotonic if and only if $a_2=0$.

Proof. If $R(b/a) \leq R(b/a\cdot c)$ for all a,b,c, it will be $1+ a_1p(a)+a_2p(b)+ a_3p(a\cdot b) \leq 1+ a_1p(a\cdot c)+a_2p(b)+a_3p(a\cdot b\cdot c)$. Taking $a=b$, $p(a)\neq 0$ and $p(c)=0$, it is $-a_2p(a) \leq 0$ and $a_2 \geq 0$, concluding $a_2=0$.

Reciprocally, if $a_2=0$, it should be proved that $1+ a_1p(a)+a_3p(a\cdot b) \leq 1+ a_1p(a\cdot c)+a_3p(a\cdot c\cdot b)$, with $a_3=-a_1$; that is, $a_1(p(a)-p(a\cdot b)-p(a\cdot c)+ p(a\cdot c\cdot b)) \leq 0$. But this is all right, because of $a_1 \leq 0$ and $p(a)-p(a\cdot b) - p(a\cdot c)+p(a\cdot c\cdot b) \geq 0$. ■

This result means that the monotonic reflexive relations $R(b/a) = 1+ a_1p(a)+ a_2p(b)+a_3p(a\cdot b)$ are just those belonging to the pattern $R(b/a) = 1- mp(a)+ mp(a\cdot b)$, with $0\leq m\leq 1$.

What about restricted monotonicity if R is non-monotonic?

Theorem 2.9. ([2]) $R(b/a) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)$, with $a_2 \neq 0$, is W-restricted monotonic if and only if $a_1 \leq a_2$, and it is not Min-restricted monotonic.

Theorem 2.10. $R(b/a) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)$ is a p-W-conditional if and only if $a_1 = -1$.

Corollary 2.11. The only relation $R(b/a) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)$ that is monotonic and p-W-conditional, is the $R(b/a) = I_p(b/a) = 1 - p(a) + p(a \cdot b)$ for any a, b.

Theorem 2.12. $R(b/a) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)$ is not a p-*Prod*-conditional.

Proof. Let's suppose $p(a) \cdot R(b/a) \leq p(b)$ for any a and b. Taking a, b such that $p(a) \neq 1, 0$, and $p(b) = 0$, it results $a_1 \leq \frac{-1}{p(a)} < -1$, in contradiction with $-1 \leq a_1 \leq 0$. ■

Corollary 2.13. $R(b/a) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)$ is not a p-Min-conditional.

3 Some Lukasiewicz's Pattern probabilistic logics.

Now let's consider the relations $R(b/a) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)$ with $a_1 + a_2 + a_3 = 0$ (so, R is reflexive), and $a_1, a_2 \geq -1$ ($R \geq 0$). Without having $a_1, a_2 \leq 0$ it is not $R \leq 1$. In these cases, it is convenient to use the new bounded relation

$$R(b/a) = Min(1, 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)).$$

It should be pointed out that if $a_1, a_2 \geq 0$, it is $1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b) = 1 + a_1 (p(a) - p(a \cdot b)) + a_2 (p(b) - p(a \cdot b)) \geq 1$ and $R(b/a) \equiv 1$.

On the other hand, if $a_1, a_2 \leq 0$, it is $R(b/a) = Min(1, 1 + a_1 (p(a) - p(a \cdot b)) + a_2 (p(b) - p(a \cdot b))) = 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)$, and these relations were studied in the former section.

Then, let's consider only the cases $a_1 > 0$, $a_2 < 0$ or $a_1 < 0$, $a_2 > 0$.

Theorem 3.1. If $R(b/a) = Min(1, 1 + a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b))$, then R is monotonic if and only if $a_1 < 0$ and $a_2 > 0$.

Proof. Let's suppose that R is monotonic and $a_1 > 0$ (so, $a_2 < 0$).

By the monotonicity, for any a, b and c, it will be $Min\,(1, 1 + a_1 p\,(a) + a_2 p\,(b) + a_3 p\,(a \cdot b)) \leq Min\,(1, 1 + a_1 p\,(a \cdot c) + a_2 p\,(b) + a_3 p\,(a \cdot b \cdot c))$.

Choosing a, b, c such that $p\,(a) \neq 0$, $p\,(b) \neq 0$, $p\,(a \cdot b) = 0$, $p\,(c) = 0$, it is obtained $a_1 p\,(a) \leq 0$, which is an absurd. So, it should be $a_2 > 0$ and $a_1 < 0$.

Now, let $a_2 > 0$, $a_1 < 0$ be. Let's see if it is $Min\,(1, 1 + a_1 p\,(a) + a_2 p\,(b) + a_3 p\,(a \cdot b)) \leq Min\,(1, 1 + a_1 p\,(a \cdot c) + a_2 p\,(b) + a_3 p\,(a \cdot b \cdot c))$, for all a, b, c.

If the second term is 1, it is true. If not, it is necessary $1 + a_1 p\,(a) + a_2 p\,(b) + a_3 p\,(a \cdot b) \leq 1 + a_1 p\,(a \cdot c) + a_2 p\,(b) + a_3 p\,(a \cdot b \cdot c)$, or equivalently, $a_1 (p\,(a \cdot c') - p\,(a \cdot b \cdot c')) - a_2 p\,(a \cdot b \cdot c') \leq 0$. But this is always true, because of $p\,(a \cdot c') - p\,(a \cdot b \cdot c') \geq 0$. ■

Then, for any relation $R\,(b/a) = Min\,(1, 1 + a_1 p\,(a) + a_2 p\,(b) + a_3 p\,(a \cdot b))$, with $a_1 > 0$ and $a_2 < 0$, monotonicity does not hold.

Theorem 3.2. If $R\,(b/a) = Min\,(1, 1 + a_1 p\,(a) + a_2 p\,(b) + a_3 p\,(a \cdot b))$ is non-monotonic, then it is not W-restricted monotonic.

Proof.- Let's suppose $W\,(R\,(b/a)\,, R\,(c/a)) \leq R\,(b/a \cdot c)$ for all a, b, c.

Choosing a, b, c such that $p\,(a) = p\,(b) = 1$, $p\,(c) = 0$, and because of $a_1 > 0$, $a_2 < 0$, it is $W\,(R\,(b/a)\,, R\,(c/a)) = Max\,(0, 1 + Min\,(1, 1 + a_1) - 1) = 1$ and $R\,(b/a \cdot c) = 1 + a_2 < 1$. Then $W\,(R\,(b/a)\,, R\,(c/a)) > R\,(b/a \cdot c)$, that breaks restricted monotonicity.

Corollary 3.3. Under the conditions of theorem 2.2., the relation R is neither $Prod$-restricted monotonic nor Min-restricted monotonic.

Theorem 3.4. $R\,(b/a) = Min\,(1, 1 + a_1 p\,(a) + a_2 p\,(b) + a_3 p\,(a \cdot b))$, with $-1 \leq a_1 < 0$ and $a_2 > 0$, is a p-W-conditional if and only if $a_1 = -1$ and $0 < a_2 \leq 1$.

Proof.- If R is a p-W-conditional, $W\,(p\,(a)\,, R\,(b/a)) \leq p\,(b)$, for each a, b, and $Max\,(0, p\,(a) + Min\,(1, 1 + a_1 p\,(a) + a_2 p\,(b) + a_3 p\,(a \cdot b)) - 1) \leq p\,(b)$.

If a, b are such that $p\,(a) = 1$, $p\,(b) = 0$, it is obtained that $a_1 = -1$.

If it is $a_2 > 1$, then $\frac{1}{2} < \frac{a_2}{1+a_2} < 1$, and we can take $\alpha \in \left(\frac{1}{2}, \frac{a_2}{1+a_2}\right)$. Choosing $a \in E$ such that $p\,(a) = \alpha$, it is $p\,(a') = 1 - \alpha$, and $1 - \alpha < \alpha$. In this case, $W\,(p\,(a)\,, Min\,(1, 1 + a_1 p\,(a) + a_2 p\,(a') + a_3 p\,(a \cdot a'))) =$
$W\,(\alpha, Min\,(1, 1 - \alpha + a_2\,(1 - \alpha))) = Max\,(0, Min\,(\alpha, a_2\,(1 - \alpha))) = \alpha > 1 - \alpha = p\,(a')$, that is absurd. Then it must be $a_2 \leq 1$.

On the other hand, if $a_1 = -1$ and $0 < a_2 \leq 1$, it is

$Max\,(0, p\,(a) + Min\,(1, 1 - p\,(a) + a_2 p\,(b) + p\,(a \cdot b) - a_2 p\,(a \cdot b)) - 1) \leq p\,(b)$.

Then $W\,(p\,(a)\,, R\,(b/a)) \leq p\,(b)$. ■

Theorem 3.5. Any relation $R\,(b/a) = Min\,(1, 1 - p\,(a) + mp\,(b) + (1-m)\,p\,(a \cdot b))$, $0 \leq m \leq 1$, is a W-preorder.

Corollary 3.6. The Lukasiewicz's implication $I_p^W\,(b/a) = Min\,(1, 1 - p\,(a) + p\,(b))$ is monotonic, is a W-preorder and is a p-W-conditional.

Proof.- In fact, $I_p^W\,(b/a)$ is equal to $Min\,(1, 1 + a_1 p\,(a) + a_2 p\,(b) + a_3 p\,(a \cdot b))$, with $a_1 = -1$, $a_2 = 1$ and $a_3 = 0$. ■

4 Some Polya's Pattern probabilistic logics.

In this section relations with $a_0 = b_0 = b_2 = b_3 = 0$, $b_1 \neq 0$ are studied; that is,

$$R\,(b/a) = \begin{cases} R\,(b/a) = \frac{a_1 p(a) + a_2 p(b) + a_3 p(a \cdot b)}{p(a)}, & \text{if } p\,(a) \neq 0 \\ 1, & \text{if } \ p\,(a) = 0 \end{cases}$$

as it is always possible to get $b_1 = 1$.

As $b_1 + b_2 + b_3 = 1$, by theorem 1.1., R is reflexive if and only if $a_1 + a_2 + a_3 = 1$ and

$$R\,(b/a) = \frac{a_1 (p(a) - p(a \cdot b)) + a_2 (p(b) - p(a \cdot b)) + p(a \cdot b)}{p(a)}.$$

From now on, we will suppose R reflexive.

Theorem 4.1. $R \geq 0$ if and only if $a_1, a_2 \geq 0$.

Proof.- If for all a, b it is $R\,(b/a) = \frac{a_1 (p(a) - p(a \cdot b)) + a_2 (p(b) - p(a \cdot b)) + p(a \cdot b)}{p(a)} \geq 0$, choosing a, b such that $p\,(b) = 0$, $p\,(a) \neq 0$, it results $a_1 \geq 0$. Taking a, b with $p\,(a) \neq 0$, $p\,(b) \neq 0$, $p\,(a \cdot b) = 0$, it is $a_2 \geq -\frac{a_1 p(a)}{p(b)}$, for any a, and so $a_2 \geq 0$.

On the other hand, if $a_1, a_2 \geq 0$, clearly $R\,(b/a) \geq 0$. ■

Theorem 4.2. If $R \geq 0$, it is: $R \leq 1$ if and only if $a_2 = 0$ and $a_1 \leq 1$.

Proof.- If for all a, b it is $R\,(b/a) = \frac{a_1 (p(a) - p(a \cdot b)) + a_2 (p(b) - p(a \cdot b)) + p(a \cdot b)}{p(a)} \leq 1$, taking b such that $p\,(b) = 0$, it is $a_1 \leq 1$. Furthermore, taking a, b, such that $p\,(a)\,, p\,(b) \neq 0, p\,(a \cdot b) = 0$, it is obtained $a_2 \leq \frac{(1 - a_1) p(a)}{p(b)}$ for all a. Choosing the values of $p\,(a)$ sufficiently small, it can be obtained $a_2 = 0$.

Reciprocally, if $a_2 = 0$ and $a_1 \leq 1$, $R\,(b/a) = \frac{a_1 (p(a) - p(a \cdot b)) + p(a \cdot b)}{p(a)} \leq 1$. ■

From now on, the relations will be well defined; that is $0 \leq a_1 \leq 1$, $a_2 = 0$, $a_3 = 1 - a_1$, and

$$R(b/a) = \frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)} = \frac{a_1 p\left(a \cdot b'\right)+p(a \cdot b)}{p(a)}$$

Theorem 4.3. The only relation $R(b/a) = \frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)}$ that is monotonic is the $R \equiv 1$.

Proof.- Obviously, $R(b/a) \equiv 1$ is monotonic.

If R is monotonic, it is $\frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)} \leq \frac{a_1(p(a \cdot c)-p(a \cdot b \cdot c))+p(a \cdot b \cdot c)}{p(a \cdot c)}$ for every a, b, c.

We can choose $a, b, c \in E$, such that $p(a \cdot b) \neq 0$, $p(a \cdot b') \neq 0$ and $c = b'$. Then $\frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)} \leq a_1$, $-a_1 p(a \cdot b) \leq -p(a \cdot b)$; it results $a_1 \geq 1$, and $a_1 = 1$. So, $R(b/a) = \frac{p(a)-p(a \cdot b)+p(a \cdot b)}{p(a)} = 1$. ■

So, in any relation $R(b/a) = \frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)}$, $0 \leq a_1 < 1$, monotonicity is broken.

Theorem 4.4. $R(b/a) = \frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)}$, $0 \leq a_1 \leq 1$, is W-restricted monotonic.

Theorem 4.5. In the conditions of last theorem, if $a_1 \neq 1$, R is not Min-restricted monotonic.

Proof.- If $a_1 < 1$, taking a, b, c such that $p(a \cdot b) \neq 0$, $p(a \cdot b') \neq 0$ and $c = b'$, it is $R(b/a \cdot c) < Min(R(b/a), R(c/a))$, and restricted monotonicity is broken. ■

Theorem 4.6. $R(b/a) = \frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)}$ is a W-preorder if and only if $R \equiv 1$. So, if $R(b/a) = \frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)}$ is a $Prod$-preorder, or a Min-preorder, then $R \equiv 1$.

Proof.- $R \equiv 1$ is clearly a W-preorder.

If R is a W-preorder, for any a, b, c it should be
$Max\left(0, \frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)} + \frac{a_1(p(b)-p(b \cdot c))+p(b \cdot c)}{p(b)} - 1\right) \leq \frac{a_1(p(a)-p(a \cdot c))+p(a \cdot c)}{p(a)}$.

If $p(a) = 0.5$, $c = a'$, $p(b) = 1$, it is got $1 \leq a_1$, and then $a_1 = 1$. ■

Theorem 4.7. $R(b/a) = \frac{a_1(p(a)-p(a \cdot b))+p(a \cdot b)}{p(a)}$ is a p-W-conditional if and only if $a_1 = 0$, that is, if $R(b/a) = \frac{p(a \cdot b)}{p(a)}$.

Note. The Conditional Probability $p^*(b/a) = \frac{p(a \cdot b)}{p(a)}$ $(a_1 = 0)$ is not p-Min-conditional in an enough rich Boolean Algebra: choosing a and b such that $b < a$ and $0 < p(b) < p(a) < 1$, it is $Min\left(p(a), \frac{p(a \cdot b)}{p(a)}\right) = Min\left(p(a), \frac{p(b)}{p(a)}\right) > p(b)$.

Furthermore, it is neither monotonic nor W-preorder (and so, it is neither $Prod$-preorder nor Min-preorder), and it is the only relation in this section being a p-W-conditional.

Acknowledgements. The author would like to thank Prof. Enric Trillas for his continuous support and valuable suggestions, not only in this paper, but in all my work during the last years; and to Prof. C. Alsina for his encouragement to continue in the study of rational relations defined by probability.

References

[1] Birkhoff, G. (1979), *Lattice Theory*, Amer. Math. Soc. Colloquium Pub. 25, Third Ed.

[2] Cubillo, S. (1993), *Contribución al estudio de la lógica y de los condicionales borrosos*, Ph. Thesis (in Spanish). Facultad de Informática, Universidad Politécnica de Madrid.

[3] Nilsson, N.J. (1986), *Probabilistic Logic*, Artificial Intelligence 28, 71-88.

[4] Pearl, J. (1988), *Probabilistic Reasoning in Intelligent Systems*, Morgan Kaufmann, California.

[5] Pólya, G. (1966), *Matemáticas y Razonamiento plausible*, Tecnos, Madrid.

[6] Schweizer, B., Sklar, A. (1983), *Probabilistic Metric Spaces*, Elsevier North-Holland, New York.

[7] Tomas, M.S. (1988), *Sobre t-norms racionals i negaciones*, Actes IV Congrés Català de Lògica (in Catalan), 119-121.

[8] Trillas, E. (1992), *On exact and inexact Conditionals*, Proceedings IPMU'92, Universidad de las Islas Baleares, 649-655.

[9] Trillas, E. (1993), *On Logic and Fuzzy Logic*, International Journal for Approximate Reasoning, Vol. 1, n.2, 107-137.

[10] Trillas, E., Alsina, C. (1993), *Logic: Going farther from Tarski?*, Fuzzy Sets and Systems 53, 1-13.

[11] Trillas,E., Cubillo, S. (1994), *On monotonic fuzzy conditionals*, Journal of Applied Non-Classical Logics, vol. 4, n.2, 201-214.