*Mathware & Soft Computing 5 (1998) 5-12*

# Some Properties Concerning the Quasi-inverse of a $t$-norm

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**Abstract**

Some properties of the quasi-inverse operators are presented. They are basic tools in order to reduce complex expressions involving several of such operators. An effective calculation for the quasi-inverse of a continuous t-norm is also provided.

## 1 Introduction.

The aim of this paper is to provide the reader with a set of elementary properties, which are useful tools in order to reduce complex expressions where a t-norm and its associated quasi-inverse appear several times.

Some of the results presented here are not new and they can be found disseminated in the literature, mainly under two different forms: as specific properties concerning a restrictive class of t-norms, (even a particular t-norm like $T$ =Min, $T$ = Ł...) or into the setting of more general logic and algebraic structures (mainly GL-Monoids and MV-Algebras [3]).

The properties are arranged into three different classes depending on the continuity of the chosen t-norm: the general case –arbitrarily t-norms–, left continuous t-norms and continuous t-norms.

Let us recall some elementary concepts.

**Definition 1.1.** *A t-norm is an operation $T : [0,1]\times[0,1] \to [0,1]$ which is associative, commutative, non-decreasing in both variables and that satisfies $T(1,x) = x$, $T(0,x) = 0$ for any $x \in [0,1]$.*

**Definition 1.2.** *Given a t-norm $T$, its quasi-inverse $\hat{T}$ is defined by*

$$\hat{T}(x|y) = \sup\{\alpha \in [0,1] \ / \ T(\alpha,x) \leq y\}, \quad \textit{for any } x,y \in [0,1].$$

**Definition 1.3.** *Given a t-norm $T$, its symmetrized quasi-inverse $E_T$ is defined by* $E_T(x,y) = Min\{\hat{T}(x|y), \hat{T}(y|x)\} = \hat{T}(Max\{x,y\} \mid Min\{x,y\})$ *for any* $x, y \in [0,1]$.

In the same way that, in the setting of fuzzy logic, $T$ can be interpreted as an extension of the classical ($\{0,1\}$) conjunction $\wedge$ to the whole unit interval, $\hat{T}$ can be viewed as the residuated implication associated to $T$, and it is very common to note $\hat{T}(x|y)$ by $x \underset{T}{\longrightarrow} y$ and $E_T(x,y)$ by $x \underset{T}{\longleftrightarrow} y$ (the natural equivalence). However, care is needed when dealing with arbitrarily chosen t-norms, because, in this case, $\hat{T}(x|y)$ could not define neither an implication function [5] nor a T-preorder [6], which are the most common ways to generalize the classical implication to the fuzzy framework. In the same way, $E_T(x,y)$ could not define a fuzzy equivalence relation (T-indistinguishability, similarity,...). As we will see later, the left-continuity of the t-norm $T$ is needed in order to ensure that $\hat{T}$ acomplishes with these basic structures.

Examples of t-norms and its associated quasi-inverses are:

(1) $T(x,y) = \text{Min}\{x,y\}$, and $\hat{T}(x|y) = \begin{cases} \text{Min}\{x,y\}, & \text{if } x \geq y, \\ 1, & \text{in other case.} \end{cases}$

(2) $T(x,y) = \text{Ł}(x,y) = \text{Max}\{x+y-1, 0\}$, (The Luckasiewicz t-norm), and $\hat{T}(x|y) = \begin{cases} 1-x+y & \text{if } x \geq y, \\ 1 & \text{in other case.} \end{cases}$

(3) $T(x,y) = x \cdot y$ and $\hat{T}(x|y) = \begin{cases} y/x & \text{if } x \geq y \\ 1 & \text{in other case.} \end{cases}$

(4) $T(x,y) = Z(x,y) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{in other case} \end{cases}$ and $\hat{T}(x,y) = \begin{cases} 1 & \text{if } x < 1 \\ y & \text{if } x = 1. \end{cases}$

**Definition 1.4.** *A continuous t-norm $T$ is archimedean if $T(x,x) < x$ for any $x \in (0,1)$.*

**Definition 1.5.** *An archimedean t-norm is strict if $T^n(x) > 0$ for any $x \in (0,1]$ and for any $n \in \mathbb{N}$.*

**Note.** $T^n(x)$ is defined in a recurrent way by $T^1(x) = x$,

$$T^n(x) = T(x, T^{n-1}(x)), \quad n \geq 1.$$

Next, representation theorem characterizes the archimedian t-norms.

**Theorem 1.6.** (Ling). *$T$ is an archimedean t-norm if, and only if, there exists a continuous decreasing function $f : [0,1] \to [0,+\infty]$, such that $f(1) = 0$ and $T(x,y) = f^{[-1]}(f(x) + f(y))$.*

*Usually $f$ is termed the additive generator of $T$, and $f^{[-1]}$ denotes the pseudo-inverse of $f$, defined by*

$$f^{[-1]}(x) = \begin{cases} 1, & \text{if } x < 0 \\ f^{-1}(x), & \text{if } x \in [0, f(0)] \\ 0, & \text{in other case} \end{cases}$$

*and $T$ is strict if, and only if, $f(0) = +\infty$.*

In a more general way, we have:

**Therorem 1.7.** *$T$ is a continuous t-norm if, and only if, there exists a family $\{(a_i, b_i)\}_{i\in I}$ of disjoint intervals of [0,1], and $f_i : [a_i, b_i] \to [0, +\infty]$, such that $f_i(b_i) = 0$, and*

$$T(x,y) = \begin{cases} f_i^{[-1]}(f_i(x) + f_i(y)) & \text{if } (x,y) \in (a_i, b_i) \times (a_i, b_i) \\ \min\{x, y\} & \text{in other case.} \end{cases}$$

*Here, $f^{[-1]}$ is defined by*

$$f_i^{[-1]}(x) = \begin{cases} b_i & \text{if } x \leq 0 \\ f^{-1}(x) & \text{if } x \in [0, f(a_i)] \\ a_i & \text{in other case.} \end{cases}$$

Note that $T$ =Min is obtained when $I = \emptyset$, and archimedean t-norms when $I = \{i_0\}$ and $(a_{i_0}, b_{i_0}) = (0,1)$. In any other case, we say that $T$ is an ordinal sum.

A proof of theorem 1.7 as well as further reading on these topics can be found in [4].

## 2 Basic properties

Let us start with the most general case, in which no hypothesis about the continuity of the t-norm $T$ is assumed.

**Proposition 2.1.** *Let $T$ be a t-norm. For any $x, y, z \in [0,1]$, we have:*

a) If $x \leq y$ then $\hat{T}(x|y) = 1$.

b) If $T(x,z) \leq y$ then $\hat{T}(x|y) \geq z$.

c) If $T \leq T'$ then $\hat{T} \geq \hat{T}'$

d) $\hat{T}(x|T(x,y)) \geq y$

e) If $T(\hat{T}(x|y), x) \geq y$ then $x \geq y$

f) $T(x,y) \geq \text{Inf}\{\alpha \ / \ \hat{T}(x|\alpha) \geq y\}$

*Proof.* Evident. ■

**Proposition 2.2.** *Let $T$ be a t-norm. For any $x, y, z \in [0, 1]$, we have:*

$$\hat{T}(T(x,z)|T(y,z)) \geq \hat{T}(x|y).$$

*Proof.* It is sufficient to show that $A_1 \subseteq A_2$ being $A_1 = \{\alpha \in [0,1] \ / \ T(\alpha, x) \geq y\}$ and $A_2 = \{\alpha \in [0,1] \ / \ T(\alpha, T(x,z)) \leq T(y,z)\}$. ■

The concept of right (or left) continuity is applied only to functions depending on a single variable, and it does not make sense in the case of several variables. However, we will say that a function $F(x, y)$ is right (or left) continuous with respect to the variable $x$ (resp. $y$) if $F(x, y_0)$ is right (or left) continuous for any fixed $y_0 \in [0, 1]$ (resp. $F(x_0, y)$ for any fixed $x_0 \in [0, 1]$).

Obviously, since a t-norm is a commutative operation, $T(x, y)$ is right (or left) continuous with respect to the variable $x$ if, and only if, it is right (or left) continuous with respect to the variable $y$. We will refer to these t-norms as right (or left) continuous (without any reference to the variables).

**Proposition 2.3.** *For any t-norm $T$, its quasi-inverse $\hat{T}(x|y)$ is a non-decreasing and right continuous function with respect to the variable $y$.*

*Proof.* If $\{y_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ is decreasing and such that $\lim_{n\to\infty} y_n = y$, we can consider for any $x \in [0,1]$, $A_n = \{\alpha \in [0,1] \ / \ T(a, x) \leq y_n\}$, and $A = \{\alpha \in [0,1] \ / \ T(\alpha, x) \leq y\}$. It is evident that $A = \bigcap_{n\in\mathbb{N}} A_n$, and the $\lim_{n\to\infty} \hat{T}(x|y_n) = \hat{T}(x|y)$. ■

# 3 Quasi-inverses of left continuous t-norms

Left continuity plays a crucial role in order to relate the quasi-inverse with logical and algebraic structures. In this case, $([0,1], \leq, T)$ is a GL-monoid where its residuated structure is given by $\hat{T}$ [3].

In this section we prove some relevant properties such as proposition 3.4.a and Theorem 3.2.c –they ensure that $\hat{T}(x|y)$ defines an implication function and that it acomplishes with multivalued Modus Ponens [2], [5]–, and theorem 3.2.b (T-transitivity) which relates $\hat{T}$ and $E_T$ with T-preorders and T-indistinguishabilitites [6].

**Proposition 3.1.** *If $T$ is a left continuous t-norm, then $\hat{T}(x|y)$ is a non increasing and left continuous function with respect to $x$.*

*Proof.* Analogous to Proposition 2.3. ■

**Theorem 3.2.** *For any t-norm $T$, these are equivalent statements:*

*a)* $T$ *is left continuous.*

*b)* $T(\hat{T}(x|y), \hat{T}(y|z)) \leq \hat{T}(x|z)$ *(T-transitivity)*

*c)* $T(x, \hat{T}(x|y)) \leq y$ *(Modus Ponens)*

*d)* $T(x, y) \leq z$ *if, and only if,* $x \leq \hat{T}(y|z)$

*e)* $Inf\{\alpha \in [0,1] \ / \ \hat{T}(x|\alpha) \geq y\} = T(x, y)$.

*Proof.* It is straightforward showing that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$, $(d) \Rightarrow (e)$ and $(e) \Rightarrow (c)$. ■

Theorem 3.2.e. has an interesting meaning from a structural point of view. It stablishes that the map that sends each t-norm $T$ to its quasi-inverse is an injective one when only left continuous t-norms are considered [2]. This is not true for arbitrarily t-norms, as it is shown in next example.

**Example 3.3.** Let us consider $T_1$ and $T_2$ t-norms defined by:

$$T_1(x,y) = \begin{cases} 0, & \text{if } (x,y) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \\ \text{Min}\{x,y\}, & \text{otherwise.} \end{cases}$$

$$T_2(x,y) = \begin{cases} T_1(x,y), & \text{if } (x,y) \neq \left(\frac{1}{2}, \frac{1}{2}\right) \\ \frac{1}{2}, & \text{if } (x,y) = \left(\frac{1}{2}, \frac{1}{2}\right). \end{cases}$$

Obviously, $\hat{T}_1 = \hat{T}_2$ and $T_1 \neq T_2$. ■

**Proposition 3.4.** *Let* $T$ *be a left-continuous t-norm. For any* $x, y, z \in [0,1]$ *we have:*

*a)* $\hat{T}\left(x \mid \hat{T}(y|z)\right) = \hat{T}\left(y \mid \hat{T}(x|z)\right) = \hat{T}\left(T(x,y)|z\right)$

*b)* $T\left(x, \hat{T}(y|z)\right) \leq \hat{T}(y \mid T(x,z))$

*c)* $\hat{T}(\hat{T}(z|x) \mid \hat{T}(z|y)) \geq \hat{T}(x|y)$

*d)* $\hat{T}(\hat{T}(y|z) \mid \hat{T}(x|z)) \geq \hat{T}(x|y)$

*e)* $\hat{T}(\hat{T}(x|y)(z) \geq T(x, \hat{T}(y|z))$.

*Proof.*

(a) From Theorem 3.2 it is easy to show that

$$\begin{aligned} \{\alpha \in [0,1] \ / \ T(\alpha, x) \leq \hat{T}(y|z)\} &= \{\alpha \in [0,1] \ / \ T(\alpha, y) \leq \hat{T}(x|z)\} = \\ &= \{\alpha \in [0,1] \ / \ T(\alpha, T(x,y)) \leq z\}, \end{aligned}$$

and (a) is obtained by taking the suprema of this set.

(b) Let us consider $A = \{\alpha \in [0,1] \ / \ T(\alpha, y) \le z\}$

$$\begin{aligned} T\left(T(x, \hat{T}(y|z)), y\right) &= T(T(x, \sup A), y) = \\ &= \sup_{\alpha \in A} T(x, T(\alpha, y)) \le T(x, z), \end{aligned}$$

so

$$T(x, \hat{T}(y, z)) \le \hat{T}(y|T(x, z)).$$

(c) and (d) are elementary consequences of Theorem 3.2.b.

(e) From theorem 3.2.c it follows
$T\left(T(x, \hat{T}(y|z)), \hat{T}(x|y)\right) = T\left(x, T(\hat{T}(x|y), \hat{T}(y|z))\right) \le T(x, \hat{T}(x|z)) \le z.$ ■

Any property in proposition 3.4 gives sufficient condition in order to ensure the left continuity of $T$: the t-norm $T = Z$, that clearly is not left continuous satisfies all them.

**Proposition 3.5.** $\hat{T}(x|y) = \sup\{\alpha \in [0,1] \ / \ \hat{T}(\alpha|y) \ge x\}$ *for any left continuous t-norm* $T$.

*Proof.* It is evident since $\{\alpha \in [0,1] \ / \ T(\alpha, x) \le y\} = \{\alpha \in [0,1] \ / \ \hat{T}(\alpha|y) \ge x\}$ (Theorem 3.2.d). ■

Proposition 3.5 does not characterize left continuous t-norms, as it is shown in next example.

**Example 3.6.** Let us consider $T_1$ and $T_2$ the t-norms defined in Example 3.3.

$T_1$ is a left continuous t-norm, and $T_2$ is not. Clearly $\hat{T}_1 = \hat{T}_2$, since both $\hat{T}_1$ and $\hat{T}_2$ satisfy $\hat{T}(x|y) = \sup\{\alpha \in [0,1] \ / \ \hat{T}(\alpha|y) \ge x\}$.

Next corollary can be obtained by applying Proposition 3.4.a,c and d recurrently.

**Corollary 3.5.** *Given a left continuous t-norm. For any* $x, y, z, x_1, ..., x_n, z_1, ..., z_n \in [0,1]$ *we have:*

(a) $\hat{T}\left(x_1|\hat{T}(x_2|...|\hat{T}(x_{n-1}|x_n))...\right) = \hat{T}\left(T(...T(x_1, x_2), ..., x_{n-1})|x_n\right)$, $n \ge 4$

(b) $\hat{T}\left(\hat{T}(...\hat{T}(x|z_1)|...|z_n)|\hat{T}(\hat{T}(...\hat{T}(y|z_1)|...|z_n)\right) \ge \begin{cases} \hat{T}(x|y) & \text{if } n = 2m \\ \hat{T}(y|x) & \text{if } n = 2m+1 \end{cases}$

(c) $\hat{T}\left(\hat{T}(...\hat{T}(x_1|x_2)|...)|x_n\right) \ge$

$$\ge \begin{cases} T\left(...T(\hat{T}(x_1|x_2), \hat{T}(x_3|x_4), ...), \hat{T}(x_{n-1}|x_n)\right) & \text{if } n = 2m \\ T\left(...T(x_1, \hat{T}(x_2|x_3))...), \hat{T}(x_{n-1}|x_n)\right) & \text{if } n = 2m+1, (m \ge 2). \end{cases}$$

# 4 Quasi-inverses of continuous t-norms

Let us recall that any continuous t-norm $T$ is either archimedean, $T$ =Min or an ordinal sum (Theorem 1.7). From an algebraic point of view, it is worth noting that, if $T$ is a non strict archimedean t-norm, then $([0,1], \leq, T)$ is a MV-algebra.

**Proposition 4.1.** *If $T$ is an archimedean t-norm with additive generator $f$, then $\hat{T}(x|y) = f^{[-1]}(f(y) - f(x))$, for any $x, y \in [0,1]$.*

*Proof.* [6]. ■

Next theorem provides us with an effective way to calculate the quasi-inverse of any continuous t-norm.

**Theorem 4.2.** *A function $F : [0,1] \times [0,1] \to [0,1]$ is the quasi-inverse of a continuous t-norm $T$ (i.e. $F = \hat{T}$) if, and only if, there exists a family $\{(a_i, b_i)\}_{i \in I}$ of disjoint intervals of [0,1], and a family of continuous and decreasing functions $f_i : [a_i, b_i] \to [0, f_i(a_i)]$ such that $f(b_i) = 0$ satisfying*

$$F(x|y) = \begin{cases} 1 & \text{if } x \leq y \\ f^{[-1]}(f_i(y) - f_i(x)), & \text{if } (x,y) \in [a_i, b_i) \times [a_i, b_i) \\ y & \text{in any other case.} \end{cases}$$

*Proof.* It is a consequence of Theorem 1.7. ■

Note that, in this case, $T$ is the ordinal sum associated to $\{(a_i, b_i)\}_{i \in I}$ and to $\{f_i\}_{i \in I}$.

Under the hypothesis of continuity for the t-norm $T$, there are some inequalities in Section 3 that become equalities.

**Corollary 4.3.** *Given a continuous t-norm $T$, for any $x, y, z \in [0,1]$ such that $z \leq y \leq x$, we have:*

*(a)* $T\left(\hat{T}(x|y), \hat{T}(y|z)\right) = \hat{T}(x|z)$

*(b)* $\hat{T}\left(\hat{T}(x|y) \mid \hat{T}(x|z)\right) = \hat{T}(y|z)$.

*Proof.* It can be easily obtained from Theorem 4.2 by considering four different cases, namely: $x, y, z \in (a_i, b_i)$ for some $i \in I$; $y, z \in (a_i, b_i)$ but $x \notin (a_i, b_i)$; $x, y \in (a_i, b_i)$ but $z \notin (a_i, b_i)$; any other case.

It is worth noting that $\hat{T}\left(\hat{T}(y|z) \mid \hat{T}(x|z)\right) \geq \hat{T}(y|z)$, but the equality does not hold ($T$ =Min is an easy counterexample).

By applying Corollary 4.3 recurrently, we obtain:

**Corollary 4.4.** *Given a continuous t-norm $T$, for any $x, z, y_1, ..., y_n$ such that $z \leq y_1 \leq ... \leq y_n \leq x$ have:*

$$T\left(\hat{T}(x|y_n), \hat{T}(y_n|y_{n-1}), ..., \hat{T}(y_1|z)\right) = \hat{T}(x|z).$$

**Corollary 4.5.** *Given a continuous t-norm $T$, for any $x_1, ..., x_n, y, z \in [0,1]$ such that $z \leq y \leq x_1 \leq ... \leq x_n$, we have:*

$$\hat{T}\left(\hat{T}(...|x_1), ..., x_n)|y) \mid \hat{T}(...|x_1)...x_n)|z)\right) = \hat{T}(y|z).$$

**Proposition 4.4.** *(Modus Ponens) If $T$ is a continuous t-norm and $x, y \in [0,1]$, then $T\left(x, \hat{T}(x|y)\right) = y$ if, and only if, $x \geq y$.*

*Proof.* [6]. ■

**Proposition 4.5.** *Let $T$ be a continuous t-norm, and $A \subseteq [0,1] \times [0,1]$ the set containing all points $(x, y)$ where $\hat{T}(x|y)$ is a continuous function. We have:*

*(a) If $A = [0,1] \times [0,1]$, then $T$ is a non strict archimedean t-norm.*

*(b) If $A = [0,1] \times (0,1]$, then $T$ is a strict archimedean t-norm.*

*(c) In any other case, $T$ is an ordinal sum or $T =$Min.*

*Proof.* It is a consequence of Theorem 4.2. ■

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