

Representation of a Boolean Algebra by its Triangular Norms

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Abstract

Given a complete and atomic Boolean algebra B , there exists a family τ_γ of triangular norms on B such that, under the partial ordering of triangular norms, τ_γ is a Boolean algebra isomorphic to B , where γ is the set of all atoms in B . In other words, as we have shown in this note, every complete and atomic Boolean algebra can be represented by its own triangular norms. What we have not shown in this paper is our belief that τ_γ is not unique for B and that, for such a representation, B needs neither to be complete, nor to be atomic.

Keywords: Triangular norm; complete and atomic Boolean algebra; fuzzy group.

1 Introduction

In order to generalize the triangle inequality in a metric space to the more general probabilistic metric spaces, a class of associative binary compositions on the closed unit interval $[0, 1]$ of the real line, called *triangular norms* or briefly *t-norms*, were first introduced by Menger [3]. Subsequently, triangular norms were extensively studied by Schweizer and Sklar ([5], [6], [7]) with the same purpose.

In fact, triangular norms provide an immediate generalization of the minimum function on $[0, 1]$, wherever the minimum resides in any axiomatic system. For example, Anthony and Sherwood [1] have already tried to generalize the *fuzzy group* of Rosenfeld [4] by replacing the minimum function there with a general triangular norm.

We hasten to recommend that before replacing the minimum with a *t*-norm one must keep in mind the specific need of the system to be upgraded. Simply cutting an axiom that involves the minimum function and pasting the corresponding axiom that involves a *t*-norm may obstruct the flow of the system and destroy its elegance.

One may argue that a general triangular norm is useless so far as its acceptability by the computer is concerned. We agree that if $a \leq b$ in $[0, 1]$, then $\min\{a, b\} = a$, which is a computer-friendly result that cannot be accessed by a *t*-norm. The situation is, however, not so gloomy. We know that $\min\{c, c\} = c$

for any real c . In other words, every real number is idempotent under the minimum function. For a triangular norm there is no dearth of such idempotent numbers in $[0, 1]$, and on the set of these idempotent numbers the t -norm simulates the behaviour of the minimum. As a consequence, with a simple *if-then* statement, any general triangular norm can be introduced into the environment of a computer programme.

Although there are innumerable triangular norms on $[0, 1]$, and we already know how to generate infinitely many of them, the interval $[0, 1]$ appears to be a very narrow space on which such generalized compositions should be studied. In fact, $[0, 1]$ is merely a complete chain. One can profitably study triangular norms when they are defined on a lattice in general and a complete Boolean algebra in particular.

Endowed with the partial ordering on the class τ of all triangular norms on a lattice L , as suggested by Schweizer and Sklar in [5], the poset τ has emerged as a very rich structure.

In this note we have shown that a particular subset of τ forms a Boolean algebra which is isomorphic to L , where L is a complete and atomic Boolean algebra. We strongly believe that such a representation is traceable in τ even if L is any general Boolean algebra.

In what follows, L denotes a lattice containing 0 and 1.

2 Triangular norms on a lattice

Definition 1. Let (L, \leq, \wedge, \vee) be any lattice with 0 and 1. A binary composition T on L is called a *triangular norm* or a *t -norm* on L if, for all $x, y, z \in L$, T satisfies the following four axioms:

$$\begin{array}{lll}
 \text{(T1)} & (xTy)Tz = xT(yTz) & \text{(associativity)} \\
 \text{(T2)} & xTy = yTx & \text{(commutativity)} \\
 \text{(T3)} & y \leq z \Rightarrow xTy \leq xTz & \text{(monotonicity)} \\
 \text{(T4)} & xT1 = x & \text{(boundary condition).}
 \end{array}$$

Example 2. *The meet \wedge is a t -norm on L . The composition N on L defined by*

$$xNy = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

is a t -norm on L .

Definition 3. [5]. Suppose that T and S are two triangular norms on L . If $xTy \leq xSy$ for all $x, y \in L$, then one writes $T \leq S$. If $xTy = xSy$ for all $x, y \in L$, then one writes $T = S$. Under the ordering \leq , the set τ of all triangular norms on L forms a partially ordered set.

Theorem 4. [5]. *For any triangular norm T on L , $N \leq T \leq \wedge$.*

Thus, the meet \wedge on L is the greatest t -norm on L . We have $xT0 \leq x \wedge 0 = 0$, and so $xT0 = 0$ for any t -norm T and any $x \in L$. With the help of the following theorem, which can easily be proved, one can identify the meet \wedge on L completely.

Theorem 5. *Let T be a triangular norm on L . Then $T = \wedge$ if and only if $xTx = x$ for all $x \in L$.*

Thanks to this theorem, in order to construct a t -norm on L which does not equal \wedge , we have only to pick up some c in L with $0 < c < 1$, and prescribe some $cTc < c$. Let us take an example.

Example 6. *On the pentagonal lattice the composition T given by Table 1 is a t -norm which is strictly less than the meet, because $aTa = 0 < a = a \wedge a$.*

| | | | | | |
|-----|---|-----|-----|-----|-----|
| T | 0 | a | b | c | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | 0 | a |
| b | 0 | 0 | b | 0 | b |
| c | 0 | 0 | 0 | c | c |
| 1 | 0 | a | b | c | 1 |

Table 1

Definition 7. A subset α of L , which does not contain 0, is a J -set in L , if α satisfies the following condition:

$$0 < y \leq x \in \alpha \Rightarrow y \in \alpha; \quad x, y \in L. \quad (J)$$

We note that the empty set is a J -set in L . The importance of a J -set in L can be gauged from the following hot results, which indicate how to characterize a general t -norm on L .

Theorem 8. *Let T be a triangular norm on L and let*

$$\omega = \{x \in L : x \neq 0, xTx = 0\}.$$

Then

- (1) ω is a J -set in L .
- (2) If $a \in \omega$ and $b \leq a$, then $aTb = 0$.
- (3) If $a \in \omega$, $b \leq a$, $c \leq a$, then $bTc = 0$.

Let us pick up the thread of our main objective with the family T_α of t -norms on L presented in the following theorem.

Theorem 9. *Let α be a J -set in L . Then the binary composition T_α on L defined by*

$$xT_\alpha y = \begin{cases} 0 & \text{if } x \neq 1 \neq y \quad \text{and } x \wedge y \in \alpha, \\ x \wedge y & \text{otherwise,} \end{cases}$$

is a triangular norm on L .

Proof. In the very beginning we note that $xT_\alpha y \in \alpha$ implies that either $x = 1$ or $y = 1$. For the sake of convenience, we write T for T_α . Clearly, T satisfies (T2) and (T4).

(T3): Let $x, y, z \in L$ and $y \leq z$. We show that $xTy \leq xTz$. If $x = 1$ or $y = 1$, the result is obvious. So, let $x \neq 1 \neq y$. If $xTy = 0$, then there remains nothing to prove. So, let $xTy \neq 0$. Then $xTy = x \wedge y \neq 0$. If $x \wedge z \in \alpha$, then $x \wedge y \leq x \wedge z \in \alpha$, which implies that $x \wedge y \in \alpha$. Hence, $xTy = 0$, which is a contradiction. We must therefore have $x \wedge z \notin \alpha$. Then we get $xTz = x \wedge z \geq x \wedge y = xTy$.

(T1): Let $x, y, z \in L$. We show that $(xTy)Tz = xT(yTz)$. If $1 \in \{x, y, z\}$, then the result is obvious. So, let $1 \notin \{x, y, z\}$. We now break the problem into four cases.

Case (1): Let $xTy = 0 = yTz$. Then $(xTy)Tz = 0 = xT(yTz)$.

Case (2): Let $xTy = 0 < yTz$. Then we get

$$xT(yTz) = xT(y \wedge z) \leq xTy = 0 = (xTy)Tz.$$

Case (3): Let $yTz = 0 < xTy$. This is same as Case (2).

Case (4): Let $xTy > 0 < yTz$. Then $(xTy)Tz = (x \wedge y)Tz$ and $xT(yTz) = xT(y \wedge z)$. If $x \wedge y \wedge z \in \alpha$, then $(x \wedge y)Tz = 0 = xT(y \wedge z)$, and we are done.

So, let $x \wedge y \wedge z \notin \alpha$. Then we get $(x \wedge y)Tz = (x \wedge y) \wedge z$ and $xT(y \wedge z) = x \wedge (y \wedge z)$. This completes the proof of (T1) as well as the theorem. \square

3 Representation of a Boolean algebra

Let B be an atomic Boolean algebra which is complete. If γ is the set of all atoms of B , then B is isomorphic to the Boolean algebra $P(\gamma)$ of all subsets of γ under the set inclusion. Further, γ is a J -set in B such that if $\alpha \subseteq \gamma$, then both α and $\gamma - \alpha$ are J -sets in B .

Lemma 10. *Let B be an atomic Boolean algebra in which the set γ of all its atoms has cardinality ≥ 2 . Let $\alpha \subseteq \gamma$ and $\beta \subseteq \gamma$. Then*

$$\alpha \subseteq \beta \Leftrightarrow T_\beta \leq T_\alpha.$$

Proof. Firstly, let $\alpha \subseteq \beta$. Let $x, y \in B$. If $x = 1$, then $xT_\beta y = y = xT_\alpha y$. If $y = 1$, then $xT_\beta y = x = xT_\alpha y$. If $x \neq 1 \neq y$ and $x \wedge y \in \beta$, then $xT_\beta y = 0 \leq xT_\alpha y$. If $x \neq 1 \neq y$ and $x \wedge y \notin \beta$, then $xT_\beta y = x \wedge y = xT_\alpha y$. Conversely, let $T_\beta \leq T_\alpha$. If possible, let there be $c \in \alpha$ such that $c \notin \beta$. Then $cT_\beta c = c \wedge c = c > 0 = cT_\alpha c$, which is a contradiction. \square

We are now ready to state our main theorem.

Theorem 11. *Let B be a complete and atomic Boolean algebra in which the set γ of all its atoms has cardinality ≥ 2 . Then, under the partial ordering \leq for triangular norms, the collection*

$$\tau_\gamma = \{\tau_\alpha : \alpha \subseteq \gamma\}$$

of triangular norms T_α on B , as defined in Theorem 9, forms a Boolean algebra with meet \wedge , join \vee , and complement T'_α given by

$$\begin{aligned} T_\alpha \wedge T_\beta &= T_{\alpha \cup \beta'} \\ T_\alpha \vee T_\beta &= T_{\alpha \cap \beta'} \quad \text{and} \\ T'_\alpha &= T_{\alpha'}, \end{aligned}$$

where $\alpha' = \gamma - \alpha$. Furthermore, B is isomorphic to τ_γ under the identification map

$$\alpha \longrightarrow T_{\alpha'}, \quad \alpha \subseteq \gamma.$$

Proof. It is clear that both \wedge and \vee are commutative and that each operation is distributive over the other. By Lemma 10, we see that T_\emptyset is the greatest member of τ_γ which is the identity element for \wedge , and that T_γ is the least member of τ_γ , which is the identity element for \vee . Lastly, we have

$$\begin{aligned} T_\alpha \vee T_{\alpha'} &= T_{\alpha \cap \alpha'} = T_\emptyset, \quad \text{and} \\ T_\alpha \wedge T_{\alpha'} &= T_{\alpha \cup \alpha'} = T_\gamma. \end{aligned}$$

According to Huntington [2], τ_γ is a Boolean algebra. Under the map

$$\alpha \longrightarrow T_{\alpha'}, \quad \alpha \subseteq \gamma,$$

$P(\gamma)$ is isomorphic to τ_γ . Consequently, B is isomorphic to τ_γ . \square

Example 12. *Tables 2-5 give the four triangular norms, $T_{\{a,b\}} = N, T_{\{a\}}, T_{\{b\}}$ and $T_\emptyset = \wedge$ on B_2 , the Boolean algebra with 2 atoms.*

| N | 0 | a | b | 1 |
|-----|---|-----|-----|-----|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | a |
| b | 0 | 0 | 0 | b |
| 1 | 0 | a | b | 1 |

Table 2

| \wedge | 0 | a | b | 1 |
|----------|---|-----|-----|-----|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a |
| b | 0 | 0 | b | b |
| 1 | 0 | a | b | 1 |

Table 3

| $T_{\{a\}}$ | 0 | a | b | 1 |
|-------------|---|-----|-----|-----|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | 0 | a |
| b | 0 | 0 | b | b |
| 1 | 0 | a | b | 1 |

Table 4

| $T_{\{b\}}$ | 0 | a | b | 1 |
|-------------|---|-----|-----|-----|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a |
| b | 0 | 0 | 0 | b |
| 1 | 0 | a | b | 1 |

Table 5

It is easy to see that, besides these four, there exist no other triangular norms on B_2 . However, for B_3 , the Boolean algebra with 3 atoms, besides the eight triangular norms contained in τ_γ , $\gamma = \{a, b, c\}$, there exist other t -norms on B_3 . For example, Table 6 gives a t -norm on B_3 , which does not lie in τ_γ .

| T | 0 | a | b | c | d | e | f | 1 |
|-----|---|-----|-----|-----|-----|-----|-----|-----|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | 0 | a | a | 0 | a |
| b | 0 | 0 | 0 | 0 | b | 0 | 0 | b |
| c | 0 | 0 | 0 | 0 | 0 | c | 0 | c |
| d | 0 | a | b | 0 | d | a | b | d |
| e | 0 | a | 0 | c | a | e | c | e |
| f | 0 | 0 | 0 | 0 | b | c | 0 | f |
| 1 | 0 | a | b | c | d | e | f | 1 |

Table 6

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