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Maximal MV-algebras

A. Filipoiu[1], G. Georgescu[2] & A. Lettieri[3]
[1] Dept. of Mathematics 1, Univ. Politechnica,
Str. Spl. Independentei 313, Bucharest, Romania.
[2] Inst. Mathematics, Str. Academiei 14, Bucharest, Romania.
e-mail: george@main.atm.ro
[3] Ist. di Matematica, Fac. di Architettura, Univ. di Napoli,
Via Monteoliveto 3, 80134 Napoli, Italia.
e-mail: lettieri@ds.unina.it

Abstract

In this paper we define maximal MV-algebras, a concept similar to the maximal rings and maximal distributive lattices. We prove that any maximal MV-algebra is semilocal, then we characterize a maximal MV-algebras as finite direct product of local maximal MV-algebras.

We recall that an MV-algebra $A = (A, +, \cdot, -, 0, 1)$ is a system such that $(A, +, 0)$ is a commutative monoid with identity, $x + 1 = 1$, $\bar{\bar{x}} = x$, $\bar{0} = 1$, $x \cdot y = (\overline{\bar{x} + \bar{y}})$ and $x \cdot \bar{y} + y = y \cdot \bar{x} + x$ for all x, $y \in A$; by setting $x \vee y = x + \bar{x} \cdot y$, $x \wedge y = (x + \bar{y}) \cdot y$ and $x \leq y$ iff $x \wedge y = x$ for all x, $y \in A$, we induce on A the structure $(A, \vee, \wedge, 0, 1)$ which is a bounded distributive lattice. In the sequel, for brevity, we write xy instead of $x \cdot y$.

We refer to [1], [7] and [8] for all the unexplained notions on MV-algebras. The following definitions and results are well known and drawn from [8].

A nonempty subset $I \subseteq A$ is an *ideal* of A if I is closed under "+" and $x \in I$, $y \in A$, $y \leq x$ imply $y \in I$. I is *proper* iff $1 \neq I$. A proper ideal P of A is prime iff for x, $y \in A$, $x \wedge y \in P$ implies either $x \in P$ or $y \in P$. A *maximal* ideal of A is a proper ideal of A which is maximal with respect to the inclusion. We denote by $\operatorname{Spec} A$ the set of all prime ideals of A and by $\operatorname{Max} A$ the set of maximal ideals of A.

The ideals of an MV-algebra A are in bijective correspondence with the congruence relations on A.

The relation "$x \equiv y \pmod{I}$ iff $d(x, y) = x\bar{y} + \bar{x}y \in I$" is a congruence relation on A for every ideal I, and each congruence relation on A is of this form; we denote by A/I the quotient algebra obtained in this way. An MV-algebra A is *linearly ordered* if the underlying lattice is linearly ordered and A is locally finite iff for each $x \in A$, $1 = nx = x + \ldots + x$ (n-times) for some positive integer n. An ideal P

of A is prime if and only if A/P is linearly ordered and an ideal M is maximal if and only if A/M is locally-finite. For brevity, we put $a_1 + a_2 + \ldots + a_n = \sum_{i=1}^n a_i$.

We denote by $\operatorname{Rad} A$ the intersection of all maximal ideals of the MV-algebra A; A is *semisimple* iff $\operatorname{Rad} A = 0$, i.e. A is a subdirect product of locally-finite MV-algebras [1].

Furthermore, A is local if $\operatorname{Max} A$ is singleton and in this case $\operatorname{Max} A = \operatorname{Rad} A$.

If A is an MV-algebra, we denote by $B(A) = \{x \in A/x+x = x\}$; $B(A)$ is called *the center* of A and has a natural structure of Boolean algebra with respect to the operations induced by those of A. Moreover, it is the largest Boolean subalgebra of A.

If $e \in B(A)$, then the ideal generated by $\{e\}$ is equal to $(e] = \{x \in A/x \leq e\}$ and in this case the system $((e], +, \sim, 0, e)$ is an MV-algebra, where $\tilde{x} = \bar{x} \cdot e$ (see [2]). Further, the map defined by $h_e(x) = xe$ for any $x \in A$ is an MV-homomorphism of A onto $(e]$.

Lemma 1. *[2]. Let e, $f \in B(A)$. Then*

i) *for all x, $y \in A$, $e(x + y) = ex + ey$;*

ii) *for all $x \in A$, $ex = e \wedge x$;*

iii) *if $e \wedge f = 0$, then $x(e + f) = xe + xf$.*

The following is the MV-version of an elementary result in ring theory.

Lemma 2. *For an MV-algebra A the following are equivalent:*

(1) *A is isomorphic to a finite product of MV-algebras $\Pi_{i=1}^n A_i$;*

(2) *there exist $e_1, \ldots, e_n \in B(A)$ such that $e_i e_j = 0$ for $i \neq j$ and $e_1 + \ldots + e_n = 1$.*

Proof. Let $\varphi : A \to \Pi_{i=1}^n A_i$ be an isomorphism, then $\varphi(B(A)) = \Pi_{i=1}^n B(A_i)$. For $i = 1, \ldots, n$, we remark that the elements $e_i = \varphi^{-1}(0, \ldots, 1, \ldots, 0)$, with 1 at the i-th place, satisfy (2). Thus (1) implies (2).

Conversely, let $\varphi : A \to \Pi_{i=1}^n (e_i]$ such that $\varphi(x) = (xe_1, \ldots, xe_n)$ for every $x \in A$. It is plain to prove that φ is an MV-homomorphism; φ is surjective since if $(x_1, \ldots, x_n) \in \Pi_{i=1}^n (e_i]$, using (i) of Lemma 1 and by setting $x = x_1 + \ldots + x_n$, we get $\varphi(x) = (x_1, \ldots, x_n)$. Using (iii) of Lemma 1, it is easily seen that φ is one-one. □

Remark 1. Under the hypothesis (1) of Lemma 2, A_i is MV-isomorphic to $(e_i]$ for every $i = 1, \ldots, n$. Indeed, let $\varphi : A \to \Pi_{i=1}^n A_i$ be the isomorphism, used in the previous proof and $h_i : A_i \to \Pi_{i=1}^n A_i$ be defined by $h_i(x) = (0, \ldots, x, \ldots, 0)$, with $x \in A_i$ at the i-th place. Then $\varphi^{-1} \circ h_i : A_i \to A$ is an MV-isomorphism which applies A_i onto $(e_i]$.

A well known result in ring theory is the Chinese remainder theorem.

The following MV-version of such theorem was proved in [12]. We also remark that it can be deduced from a general result [11]. If I, J are two ideals of the MV-algebra A, the ideal generated by $I \cup J$ is denoted by $I + J$ and $I + J = \{x \in A/x \leq y + z \quad \text{for some} \quad y \in I, z \in J\}$.

Proposition 1. *Let $I_1, \dots, I_n$ be ideals of the MV-algebra A such that $I_i + I_j = A$ for $i \neq j$, $i, j \in \{1, \dots, n\}$. Then, for every $x_1, \dots, x_n \in A$, there is $x \in A$ such that $x \equiv x_i$ (mod I_i) for $i = 1, \dots, n$.*

Proof. For $n = 2$, if $I_1 + I_2 = A$, then there exist $a_{12} \in I_1$ and $a_{21} \in I_2$ such that $a_{12} + a_{21} = 1$. From that, $\overline{a_{21}} \leq a_{12}$, that implies $a_{21} \equiv 1$ (mod I_1). If we consider $x = x_1 a_{21} + x_2 a_{12}$, we get

$$\begin{aligned} x/I_1 &= x_1 I_1 \cdot a_{21}/I_1 + x_2/I_1 \cdot a_{12}/I_1 = \\ &= x_1/I_1 \cdot 1/I_1 + x_2/I_1 \cdot 0/I_1 = x_1/I_1 \end{aligned}$$

i.e. $x \equiv x_1$ (mod I_1) and similarly $x \equiv x_2$ (mod I_2).

For an arbitrary n, for $i \neq j$, $i, j \in \{1, \dots, n\}$ there exist $a_{ij} \in I_i$, $a_{ji} \in I_j$ such that $a_{ij} + a_{ji} = 1$. Then the thesis follows by considering $x = \sum_{i=1}^{n} x_i a_{1i} \dots a_{i-1,i}$ $a_{i+1,i} \dots a_{ni}$ and reasoning as above, it is seen that $x \equiv x_i$ (mod I_i) for $i = 1, \dots, n$. □

As in maximal rings [6] and maximal distributive lattices [10], we introduce maximal MV-algebras connecting this notion with the one of semilocal MV-algebra.

Definition 1. A is called semilocal if $\operatorname{Max} A$ is a finite set.

Example 1.

1) A local MV-algebra [5] is semilocal because $\operatorname{Max} A$ is singleton.
2) A perfect MV-algebra A [5] is semilocal, A being generated by its radical, i.e., $A = \operatorname{Rad} A \cup \overline{\operatorname{Rad} A}$, where $\overline{\operatorname{Rad} A} = \{x \in A / \bar{x} \in \operatorname{Rad} A\}$.

Proposition 2. *For an MV-algebra A, the following are equivalent:*

(1) *A is semilocal;*

(2) *$A/\operatorname{Rad} A$ is isomorphic to a finite direct product of locally-finite MV-algebras.*

Proof.

(1) ⇒ (2).
If $\operatorname{Max} A = \{M_1, \dots, M_n\}$, then $\operatorname{Rad} A = \cap_{i=1}^{n} M_i$ and the map $\varphi : A/\operatorname{Rad} A \to \Pi_{i=1}^{n} A/M_i$, given by $\varphi(x/\operatorname{Rad} A) = (x/M_1, \dots, x/M_n)$ is an MV-isomorphism. Indeed, it is clearly an MV-homomorphism. Further, φ is surjective by Proposition 1, because $M_i + M_j = A$ for $i \neq j$, and φ is injective because $\varphi(x/\operatorname{Rad} A) = \varphi(y/\operatorname{Rad} A)$ means $x/M_i = y/M_i$, i.e. $d(x, y) \in M_i$ for any $i = 1, \dots, n$, hence $x/\operatorname{Rad} A = y/\operatorname{Rad} A$.

(2) ⇒ (1).
Let A be isomorphic to $\Pi_{i=1}^{n} A_i$, where each A_i is a locally-finite MV-algebra.
Denoting by 0_i the zero element of A_i, by [4, Prop. 2.2], $\operatorname{Spec}(A/\operatorname{Rad} A) = \operatorname{Max}(A/\operatorname{Rad} A) = \{\{0_1\} \times A_2 \times \dots \times A_n, \dots, A_1 \times \dots \times \{0_i\} \times \dots \times A_n, \dots, A_1 \times \dots \times A_{n-1} \times \{0_n\}\}$. Thus $\operatorname{Max} A$ is finite, i.e. A is semilocal.

□

Corollary 1. *For an MV-algebra A, the following are equivalent:*

(1) *A is semisimple and semilocal;*

(2) *A is isomorphic to a finite direct product of locally-finite MV-algebras.*

Definition 2. Let $\{a_i\}_{i\in I}$ be a family of elements of an MV-algebra A and $\{Pi\}_{i\in I}$ be a family of ideals in A. We shall say that the family $\{(a_i, P_i)/i \in I\}$ has the property ($\bullet$) if for any finite subset $\triangle$ of I, there exists $x_\triangle \in A$ such that $x_\triangle \equiv a_i$ (mod P_i) for any $i \in \triangle$.

Definition 3. The MV-algebra A will be called maximal if for any family $\{(a_i, P_i)/i \in I\}$ with the property ($\bullet$), there exists $x \in A$ such that $x \equiv a_i$ (mod P_i) for any $i \in I$.

Remark 2. Let $A = \{0, c, 2c, \dots, 1-2c, 1-c, 1\}$ be the MV-algebra defined in ([7], pag. 474). A is a perfect MV-algebra and it has only three ideals: 0, $\mathrm{Rad}\, A = \{0, c, 2c, \dots\}$ and A. Clearly A is maximal because has a finite number of ideals.

Lemma 3. *Any finite direct product of maximal MV-algebras is a maximal MV-algebra.*

Proof. It suffices to prove that $A = A_1 \times A_2$ is maximal if A_1 and A_2 are maximal. By Lemma 2 and Remark 1, $A_1 \tilde{\to} (e]$, $A_2 \tilde{\to} (\bar{e}]$ with $e \in B(A)$. If P is an ideal of A and $x \equiv a$ (mod P), it follows $xe \equiv ae$ (mod $P \cap (e]$) since, using Lemma 1 and elementary facts from [7], $xe\overline{ae} + \overline{xe}ae = xe(\bar{a} \vee \bar{e}) + (\bar{x} \vee \bar{e})ae = (xe\bar{a} \vee xe\bar{e}) + (\bar{x}ae \vee \bar{e}ae) = (x\bar{a} + \bar{x}a)e$. Similarly $x\bar{e} \equiv a\bar{e}$ (mod $P \cap (\bar{e}]$). Now let $\{(a_i, P_i)/i \in I\}$ be a family in A with property ($\bullet$). Then the families $\{(a_ie, P_i \cap (e])/i \in I\}$ and $\{(a_i\bar{e}, P_i \cap (\bar{e}])/i \in I\}$ verify the property ($\bullet$) in the maximal MV-algebras $(e]$ and $(\bar{e}]$, respectively. Let $y \in (e]$ and $z \in (\bar{e}]$ such that $y \equiv a_ie$ (mod $P_i \cap (e]$) and $z \equiv a_i\bar{e}$ (mod $P_i \cap (\bar{e}]$) for any $i \in I$. Then $y + z \equiv a_ie + a_i\bar{e}$ (mod P_i), i.e. $y + z \equiv a_i$ (mod P_i) for any $i \in I$, since $a_ie + a_i\bar{e} = a_i(e + \bar{e}) = a_i$ by (iii) of Lemma 1. □

Proposition 3. *If A is a maximal MV-algebra, then A is a semilocal MV-algebra.*

Proof. Any family $\{(x_M, M)/M \in \mathrm{Max}\, A\}$ has the property ($\bullet$) because for any finite family $\{M_1, \dots, M_n\} \subseteq \mathrm{Max}\, A$, if $i \neq j$, we have $M_i + M_j = A$, and by Prop. 1 there is $x* \in A$ such that $x* \equiv x_{M_i}$ (mod M_i) for $i = 1, \dots, n$. Since A is maximal, there exists $x \in A$ such that $x \equiv x_M$ (mod M) for any $M \in \mathrm{Max}\, A$.

Let $I = \{a \in A/\{M \in \mathrm{Max}\, A/a \notin M\}$ is finite $\}$. Then I is an ideal of A. We shall prove that the family

$$\{(0, I)\} \cup \{(1, M)/M \in \mathrm{Max}\, A\} \tag{α}$$

has the property ($\bullet$). If we take a finite subfamily $\{(0, I), (1, M_1), \dots, (1, M_n)\}$, using the definition of I, we obtain $\cap\{M/M \in \mathrm{Max}\, A \backslash \{M_1, \dots, M_n\}\} \subseteq I$ and considering the family

$$\{(1, M_1), \dots, (1, M_n)\} \cup \{(0, M)/M \in \mathrm{Max}\, A \backslash \{M_1, \dots, M_n\}\},$$

as at the beginning of the proof, there exists $x \in A$ such that $x \equiv 1 \pmod{M_i}$, $i = 1, \dots, n$ and $x \equiv 0 \pmod{M}$ for $M \in \operatorname{Max} A/\{M_1, \dots, M_n\}$.

It follows that $x \in I$ and thus the family (α) has the property $(\bullet)$. By hypothesis, there is $y \in A$ such that $y \equiv 0 \pmod{I}$ and $y \equiv 1 \pmod{M}$ for any $M \in \operatorname{Max} A$; it follows that $y \in I$ and for any $M \in \operatorname{Max} A$, $y \notin M$. Then we deduce $\operatorname{Max} A = \{M \in \operatorname{Max} A / y \in M\}$. Since $y \in I$, $\operatorname{Max} A$ is finite, i.e. A is semilocal. □

Proposition 4. *For an MV-algebra A, the following are equivalent:*

(1) *A is semisimple and maximal;*

(2) *A is isomorphic to finite direct product of locally-finite MV-algebras.*

Proof.

(1) $\Rightarrow$ (2).

By Prop. 3 and Corollary 1.

(2) $\Rightarrow$ (1).

It suffices to observe that any locally-finite MV-algebra is maximal, because it has only two ideals, and then to use Lemma 3.

□

Corollary 2. *Let A be a semisimple MV-algebra. Then A is maximal iff A is semilocal.*

Remark 3.

a) There exist semisimple MV-algebras which are not maximal. By example, if $A = \Pi_{i \in I} A_i$ where I is infinite and every A_i is locally-finite, we observe that A is semisimple but not semilocal ($\operatorname{Max} A$ is infinite), hence, by Prop. 3, A is not maximal.

b) There exist maximal MV-algebras which are not semisimple: by example, the perfect MV-algebra A of Remark 2.

Lemma 4. *Let A be an MV-algebra and $x \in A$. If $x \wedge \bar{x} \in \operatorname{Rad} A$, then $(2x)^2 = 2x^2 \in B(A)$ and $\overline{2x^2} = 2\bar{x}^2$.*

Proof. By [2, Thm. 1] and from the hypothesis,

(i) $$(2x)^2 \wedge (2\bar{x})^2 = (2x \wedge 2\bar{x})^2 = [2(x \wedge \bar{x})]^2 = 0$$

and

$$2x^2 \wedge 2\bar{x}^2 = 2(x^2 \wedge \bar{x}^2) = 2(x \wedge \bar{x})^2 = 0.$$

This last equality implies that

(ii) $$(2x)^2 \vee (2\bar{x})^2 = \overline{2\bar{x}^2} \vee \overline{2x^2} = 1.$$

Thus, from (i) and (ii), it follows that $(2x)^2 \in B(A)$. Further, $(2x)^2 = \overline{(2\bar{x})^2} = 2x^2$. Then $\overline{2x^2} = \overline{(2x)^2} = 2\bar{x}^2$.

□

Proposition 5. *Let A be an MV-algebra. For any $f \in B(A\backslash \operatorname{Rad} A)$, there is $e \in B(A)$ such that $f = e\backslash \operatorname{Rad} A$.*

Proof. If $f = g\backslash \operatorname{Rad} A \in B(A\backslash \operatorname{Rad} A)$, then $g^2 \equiv g \pmod{\operatorname{Rad} A}$; hence $g \cdot \overline{g^2} = g(\bar{g} + \bar{g}) = g \wedge \bar{g} \in \operatorname{Rad} A$. By Lemma 4, $e = 2g^2 \in B(A)$ and $g\backslash \operatorname{Rad} A = 2g^2\backslash \operatorname{Rad} A = e\backslash \operatorname{Rad} A$. □

Lemma 5. *If an MV-algebra A is such that $A = \Pi_{i=1}^n A_i$ and $\operatorname{Max} A$ has n elements, then $A_1, \ldots, A_n$ are local MV-algebras.*

Proof. Let $\operatorname{Max} A = \{M_1, M_2 \ldots M_n\}$. By [4], for any $i = 1, 2, \ldots, n$, $M_i = A_1 \ldots \times A_{i-1} \times N_i \times A_{i+1} \ldots A_n$, for some $N_i \in \operatorname{Max} A_i$.

If one of the MV-algebras $A_1, \ldots, A_n$ is not local, then $\operatorname{Max} A$ has more than n elements. □

Theorem 1. *For an MV-algebra, the following are equivalent:*

(1) *A is isomorphic to a a finite direct product of local maximal MV-algebras;*

(2) *A is maximal.*

Proof.

(1) ⇒ (2).

By Lemma 3.

(2) ⇒ (1).

Let $\operatorname{Max} A = \{M_1, \ldots, M_n\}$. By Prop. 3 and Prop. 2, $A/\operatorname{Rad} A$ is isomorphic to $\Pi_{i=1}^n A/M_i$. By Lemma 2, there exist $f_1, \ldots, f_n \in B(A/\operatorname{Rad} A)$ such that $f_1 + \ldots + f_n = 1/\operatorname{Rad} A$ and $f_i f_j = 0/\operatorname{Rad} A$ for $i \neq j$. By Prop. 5, there exist $e_1, \ldots, e_n \in B(A)$ such that $f_i = e_i/\operatorname{Rad} A$, $i = 1, \ldots, n$. Since $e_i e_j/\operatorname{Rad} A = 0/\operatorname{Rad} A$, it follows that $e_i e_j \in B(A) \cap \operatorname{Rad} A = \{0\}$ for $i \neq j$. Similarly $\sum_{i=1}^n e_i/\operatorname{Rad} A = 1/\operatorname{Rad} A$ implies $e_1 + \ldots + e_n = 1$. By Lemma 2 and Remark 1 is isomorphic to $\Pi_{i=1}^n (e_i]$. Because $\operatorname{Max} A$ has n elements, it follows via Lemma 5 that $(e_i]$, $i = 1, \ldots, n$ are local MV-algebras. Now we shall prove that any MV-algebra of the form $(e]$, $e \in B(A)$, is maximal. Assume that the family $\{(x_i, P_i)/i \in I\}$ has property (•) in $(e]$. Since P_i are ideals also in A and, since A is maximal, there is $x \in A$ such that $x \equiv x_i \pmod{P_i}$ for any $i \in I$. Thus $d(x, x_i) = x\bar{x}_i + \bar{x}x_i \in P_i$ for any $i \in I$, hence $x\bar{x}_i + \bar{x}x_i \leq e$ for any $i \in I$. But $x_i \leq e$ and by [8, Th. 3.1] $x = x1 = x(x_i + \bar{x}_i) \leq x_i + x\bar{x}_i \leq e + e = e$, hence $x \leq e$. Thus $x \in (e]$ and $x \equiv x_i \pmod{P_i}$ in $(e]$ because, $d_{(e]}(x, x_i) = x\tilde{x}_i + \tilde{x}x_i = xe\bar{x}_i + \bar{x}ex_i = e(x\bar{x}_i + \bar{x}x_i) \leq x\bar{x}_i + \bar{x}x_i \in P_i$ by i) of Lemma 1.

□

Definition 4. An MV-algebra A has the property **P** if for every M, $N \in \operatorname{Max} A$ such that $M \cap B(A) = N \cap B(A)$, it is $M = N$.

Proposition 6. *If A is a maximal MV-algebra, then A has the property **P**.*

Proof. By Theorem 1, A is isomorphic to $\Pi_{i=1}^n A_i$, where every A_i is a local MV-algebra; so $\operatorname{Max} A_i = \{\operatorname{Rad} A_i\}$ for $i = 1, 2, \dots, n$. Let us denote by φ an isomorphism from A onto $\Pi_{i=1}^n A_i$. If M, $N \in \operatorname{Max} A$, then $\varphi(M) = A_1 \times \dots \times \operatorname{Rad} A_h \times \dots A_n$ for some $h \in \{1, \dots, n\}$ and $\varphi(N) = A_1 \times \dots \times \operatorname{Rad} A_k \times \dots A_n$ for some $k \in \{1, \dots, n\}$.

If we suppose $M \cap B(A) = N \cap B(A)$, then $\varphi(M) \cap \Pi_{i=1}^n B(A_i) = \varphi(N) \cap \Pi_{i=1}^n B(A_i)$. Let $(a_1, a_2, \dots, a_n) \in \varphi(M) \cap \Pi_{i=1}^n B(A_i)$ such that $a_h = 0$ e $a_j = 1$ for $j \neq h$. Such element must belong also to $\varphi(N) \cap \Pi_{i=1}^n B(A_i)$. Hence $a_k = 0$ and $h = k$, i.e. $\varphi(M) = \varphi(N)$, which implies $M = N$. □

Remark 4. The property **P** is not sufficient for an MV-algebra to be maximal. Indeed, the Boolean algebra $\{0, 1\}^N$ obviously has property **P**, but it is not maximal by Corollary 1.

We shall end the paper with two examples. The first one is an example of a semilocal MV-algebra which is not maximal, while the second one is an example of a maximal MV-algebra with infinite many ideals.

Let ${}^*\mathbb{R}$ be a non-standard model of real numbers with natural order and ϵ be a positive infinitesimal element of ${}^*\mathbb{R}$. Let $\epsilon^2 = \epsilon \cdot \epsilon, \dots, \epsilon^n = \epsilon \cdot \dots \cdot \epsilon$ (n-times) where $\cdot$ is the usual product in the field ${}^*\mathbb{R}$; then $\epsilon^i > 0$ for any $i \in \mathbb{N}$ and $\epsilon^i \ll \epsilon^j$ for $i > j$.

The unit interval ${}^*[0, 1] \subseteq^* \mathbb{R}$ is an MV-algebra under the operations: $x + y = \min\{1, x + y\}$, $\bar{x} = 1 - x$ and $xy = \max\{0, x + y - 1\}$. Let $\mathbb{N}$ be the ordered set of positive natural numbers. For every $n \in \mathbb{N}$, let E_n be the subalgebra of $*[0, 1]$ generated by $\{\epsilon, \epsilon^2, \dots, \epsilon^n\}$ and E be the subalgebra $\cup_{n \in \mathbb{N}} E_n$. E is a perfect MV-algebra. We recall from [3] the following results:

(a) $E =< (\epsilon^i)/i \in \mathbb{N} >$;

(b) Every $x \in \operatorname{Rad} E \backslash \{0\}$ is a finite linear combination of ϵ^i, $i \in \mathbb{N}$ by integer coefficients n_i such that if $x = \sum_{i=1}^n n_i \epsilon^i$ and if $i_0 = \min\{i/n_i \neq 0\}$, then $n_{i_o} > 0$. Thus $x = n_r \epsilon^r + n_{r+1} \epsilon^r + \dots + n_t \epsilon^t$ with $r \geq 1$, $r \geq t$ and $n_r > 0$.

(c) Every $x \in \overline{\operatorname{Rad} E}$ is of the form $x = 1 - y$ with $y \in \operatorname{Rad} E$.

(d) The set of all ideals of E is $\{0\}$, $< \epsilon >, \dots, < \epsilon^i >, \dots$, where $i \in \mathbb{N}$.

Set $Pi =< \epsilon^i >$ for any $i \in \mathbb{N}$; thus $P_i \subset P_j$ for any $i > j$ and $\varepsilon^i \in P_i \backslash P_{i+1}$. Denote $a_1 = 0$, $a_2 = \epsilon$, $a_3 = \epsilon + \epsilon^2$, $\dots$, $a_i = \epsilon + \epsilon^2 + \dots + \epsilon^{i-1}$ for any $i \in \mathbb{N}$, thus $(a_i)_{i \in \mathbb{N}}$ is an increasing sequence of elements in E.

Proposition 7. $\{(a_i, P_i)/i \in \mathbb{N}\}$ *has the property* $(\bullet)$.

Proof. Let $\triangle = \{i_1, \dots, i_s\}$ be a finite subset of $\mathbb{N}$ where $i_1 < i_2 < \dots < i_s$. Set $x = a_{i_s}$, then $x \equiv a_{i_k} \pmod{P_{i_k}}$ for every $k = 1, \dots, s$. Indeed, we have $d(x, a_{i_k}) = a_{i_s} \bar{a}_{i_k} = (\epsilon + \dots + \epsilon^{i-1} + \epsilon^{i_k} + \dots + \epsilon^{i_s - 1})(\epsilon + \dots + \epsilon^{i_k - 1})^- = (\epsilon + \dots + \epsilon^{i_k - 1})^- \wedge (\epsilon^{i_k} + \dots + \epsilon^{i_s - 1}) = \epsilon^{i_k} + \dots + \epsilon^{i_s - 1} \in P_{i_k}$. □

Proposition 8. *E is not a maximal MV-algebra.*

Proof. Suppose E is maximal. Then if we consider the family $\{(a_i, P_i)/i \in \mathbb{N}\}$ of Prop. 7 there is an $x \in E$ such that $d(x, a_i) \in P_i$ for every $i \in \mathbb{N}$.

Claim 1. $a_i < x$ for every $i \in \mathbb{N}$.

If $x \le a_k$ for some $k \in \mathbb{N}$, then $d(x, a_{k+1}) = a_{k+1}\bar{x} \ge a_{k+1}\bar{a}_k = \epsilon + \epsilon^2 + \ldots + \epsilon^k)(\epsilon+\epsilon^2+\ldots+\epsilon^{k-1})^- = (\epsilon+\epsilon^2+\ldots+\epsilon^{k-1})^- \wedge \epsilon^k = \epsilon^k \notin P_{k+1}$, therefore $x \not\equiv a_{k+1}$ (mod P_{k+1}). We have a contradiction , so Claim 1 holds.

Claim 2. For every $i \in \mathbb{N}$ there is $k_i \in \mathbb{N}$ such that $x \le a_i + k_i\epsilon^i$.

By hypothesis and by Claim 1, $d(x, a_i) = x\bar{a}_i \in P_i =< \epsilon^i >$. Thus $x\bar{a}_i \le k_i\epsilon^i$ for some $k_i \in \mathbb{N}$, hence $x\bar{\epsilon}\overline{\epsilon^2}\ldots\overline{\epsilon^{i-1}} \le k_i\epsilon^i$. By adding ϵ to both sides of this inequality we have:

$$\epsilon \vee x\overline{\epsilon^2}\ldots\overline{\epsilon^{i-1}} \le k_i\epsilon^i + \epsilon. \qquad (\beta)$$

By Claim 1, $\epsilon + \epsilon^2 + \ldots + \epsilon^{i-1} < x$, then

$$(\epsilon^2 + \ldots + \epsilon^{i-1})^-(\epsilon + \epsilon^2 + \ldots + \epsilon^{i-1}) \le x(\epsilon^2 + \ldots + \epsilon^{i-1})^-.$$

Thus

$$(\epsilon^2 + \ldots + \epsilon^{i-1})^- \wedge \epsilon \le x(\epsilon^2 + \ldots + \epsilon^{i-1})^-,$$

hence

$$\epsilon \le x(\epsilon^2 + \ldots + \epsilon^{i-1})^- = x\overline{\epsilon^2}\cdot\overline{\epsilon^3}\ldots\overline{\epsilon^{i-1}}.$$

It follows, by (β), that $x\overline{\epsilon^2}\ldots\overline{\epsilon^{i-1}} \le k_i\epsilon^i + \epsilon$.

Again, by adding ϵ^2 to both sides of the above inequality, we have $\epsilon^2 \vee x\overline{\epsilon^3}\ldots\epsilon^{i-1} \le k_i\epsilon^i + \epsilon + \epsilon^2$. By finite iterations of this procedure, one obtains $x < k_i\epsilon^i + \epsilon + \ldots + \epsilon^{i-1} = k_i\epsilon^i + a_i$, and Claim 2 holds.

By Claims 1 and 2, $x \in \operatorname{Rad} E\backslash\{0\}$, thus from (b), $x = n_r\epsilon^r + n_{r+1}\epsilon^{r+1} + \ldots + n_t\epsilon^t$, with $r \ge t$ and $n_r > 0$. By Claim 1, $x > a_2 = \epsilon$, thus $r = 1$ and by Claim 2, there exists k_2 such that $x \le a_2 + k_2\epsilon^2 = \epsilon + k_2\epsilon^2$. Thus $n_r = 1$ and so $x = \epsilon + n_2\epsilon^2 + \ldots + n_t\epsilon^t$. By Claim 1, $x > a_3 = \epsilon + \epsilon^2$, thus $n_2 > 0$, so by Claim 2 there exists k_3 such that $x < a_3 + k_3\epsilon^3 = \epsilon + \epsilon^2 + k_3\epsilon^3$.

Thus $n_2 = 1$ and after finite iterations, we have $x = \epsilon + \epsilon^2 + \ldots + \epsilon^t = a^{t+1}$ which contradicts Claim 1. □

We proved that E is a semilocal MV-algebra which is not maximal.

Now we shall give an example of a maximal MV-algebra with infinite many ideals.

Let $G = \oplus\{Z_i/i \in \mathbb{N}\}$ be the lexicographic product of denumerable infinite copies of the abelian ℓ-group $\mathbb{Z}$ of the relative integers and $e^i \in G$ such that $e^i_k = 0$ if $k \ne i$ and $e^i_k = 1$ if $k = i$.

Consider the perfect MV-algebra $A = \mathcal{G}(G)$, where $\mathcal{G}$ is a functor from the category of Abelian ℓ-groups to the category of perfect MV-algebras [9].

If we set $P_i =< (0, e^i) >$, then $P_i \subset P_j$ for $i > j$, hence the prime spectrum of A is $\operatorname{Spec} A = \{P_i/i \in \mathbb{N}\}$. Every infinite subfamily of $\{P_i/i \in \mathbb{N}\}$ will be denoted $\{P_{i_k}/k \in \mathbb{N}\}$, where $\{i_k/k \in \mathbb{N}\}$ is a stricty increasing sequence of elements in $\mathbb{N}$.

Lemma 6. *With the above notations, let $\{P_{i_k}/k \in \mathbb{N}\}$ be an infinite subfamily of $\{P_i/i \in \mathbb{N}\}$ and $\{b^{i_k}/k \in \mathbb{N}\} \subseteq A$ such that $\{(b^{i_k}, P_{i_k})/k \in N\}$ has the property $(\bullet)$. Then the following properties hold:*

(i) $\{b^{i_k}/k \in \mathbb{N}\} \subseteq \mathrm{Rad}\,A$ *or* $\{b^{i_k}/k \in \mathbb{N}\} \subseteq \overline{\mathrm{Rad}\,A}$.

(ii) *If $\{i_1, \ldots, i_r\}$ is a finite subset of $\{i_k/k \in \mathbb{N}\}$ and $i_o = \max\{i_1, \ldots, i_r\}$ then $b^{i_o} \equiv b^{i_k}$ (mod P_{i_k}) for any $k = 1, \ldots, r$.*

(iii) *If $b^{i_h} = (x, a_i^{i_h})$ and $b^{i_k} = (y, a_i^{i_k})$, then $x = y$ and $a_i^{i_h} = a_i^{i_k}$ for every $i < \min\{i_h, i_k\}$.*

Proof.

(i) Suppose there is $k \in \mathbb{N}$ such that $b^{i_k} \in \overline{\mathrm{Rad}\,A}$. If $b^{i_h} > b^{i_k}$ then $b^{i_h} \in \overline{\mathrm{Rad}\,A}$. Assume $b^{i_h} < b^{i_k}$. By hypothesis, there is $x \in A$ such that $x \equiv b^{i_h}$ (mod P_{i_h}) and $x \equiv b^{i_k}$ (mod P_{i_k}), so $d(b^{i_h}, b^{i_k}) \leq d(x, b^{i_h}) + d(x, b^{i_k}) \in P_{i_h} + Pi_k \subseteq \mathrm{Rad}\,A$; from $b^{i_k} = b^{i_h} + d(b^{i_h}, b^{i_k})$, it follows $b^{i_k} \in \overline{\mathrm{Rad}\,A}$.

(ii) By hypothesis, there is $x \equiv b^{i_k}$ (mod Pi_k) for $k = 1, \ldots, r$. Thus $d(b^{i_o}, b^{i_k}) \leq d(b^{i_o}, x) + d(x, b^{i_k}) \in P_{i_o} + P_{i_k} = P_{i_k}$, because $i_o > i_k$ implies $P_{i_o} \subseteq P_{i_k}$.

(iii) Suppose now $h < k$. By (ii), $b^{i_h} \equiv b^{i_k}$ (mod P_{i_h}), so $(0, z) = d(b^{i_h}, b^{i_k}) \in P_{i_h}$, hence there is $r \in \mathbb{N}$ such that $z \leq re^{i_h}$ and $z_i = 0$ for $i < i_h$. It follows that $a_i^{i_h} = a_i^{i_k}$ for any $i < i_h$.

□

Proposition 9. *$\mathcal{G}(G)$ is a maximal MV-algebra.*

Proof. Suppose $\{(b^{i_k}, P_{i_k})/k \in \mathbb{N}\}$ has the property $(\bullet)$ and $b^{i_k} = (0, a^{i_k})$, where $a^{i_k} \in \oplus\{Z_i/i \in \mathbb{N}\}$. Define $y = (0, (x_i)_{i \in \mathbb{N}}) \in A$ by $x_i = a_i^{i_h}$ for $i < i_h$; by (iii) of the previous lemma, x is well defined. Now we shall prove that $y \equiv b^{i_k}$ (mod P_{i_k}) for any $k \in \mathbb{N}$. Let $(0, w) = d(y, b^{i_k})$. For every $i < i_k$, $x_i = a_i^{i_k}$, hence $w_i = 0$ for $i < i_k$ and there is $r \in \mathbb{N}$ such that $w_i \leq re^{i_k}$, therefore $(0, w) \in P_{i_k}$.

If $b^{i_k} = (1, a^{i_k})$, then we choose $y = (1, (x_i)_{i \in \mathbb{N}})$ with $x_i = a_i^{i_k}$ for $i < i_k$ and the proof is analogue. □

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