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Axiomatizing Quantum MV-Algebras

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Abstract

We introduce the notion of *p-ideal* of a QMV-algebra and we prove that the class of all p-ideals of a QMV-algebra $\mathcal{M}$ is in one-to-one correspondence with the class of all congruence relations of $\mathcal{M}$.
Keywords: QMV-algebra, MV-algebra, quantum logic.

1 Introduction

Quantum MV-algebras (QMV-algebras) were introduced in [5] as a non lattice-theoretic generalization of MV-algebras ([2],[3]). The prototypical model of MV-algebras is based on the real interval $[0, 1]$. The introduction of QMV-algebras was mainly motivated by the search for an adequate algebraic structure for the "quantum counterpart" of the real interval $[0, 1]$, i.e., the class $E(\mathcal{H})$ of all *effects* of a Hilbert space $\mathcal{H}$. An effect of $\mathcal{H}$ is a bounded and positive linear operator of $\mathcal{H}$ whose spectrum is contained in $[0, 1]$. In physical terms, effects are the mathematical representatives of "unsharp properties" of a quantum physical system in that their possible values are contained in $[0, 1]$.

From an algebraic point of view, MV and QMV-algebras share a "core" set of axioms, which S. Gudder [6] has called *supplement algebra* (S-algebra). What makes an S-algebra an *MV-algebra* is the addition of the *Lukasiewicz axiom* ($(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$). This axiom is precisely what makes lattice-theoretic an MV-algebra. Effects of a Hilbert space *do not* determine a lattice. So, for an algebraic structure to be a faithful abstraction of unsharp properties, the Łukasiewicz axiom has to be weakened. Accordingly, *QMV-algebras* are obtained by adding to S-algebras some consequences of the Łukasiewicz axiom. How much this axiom has to be weakened is still an open problem. Solving this question, amounts to axiomatizing the logic of unsharp properties of a quantum physical system. The QMV-algebras determined by effects of a Hilbert space belong to the class of *quasi-linear* QMV-algebras, which is a particular (proper) subclass of QMV-algebras. An MV-algebra is quasi-linear iff it is linear.

The first step in order to axiomatize the QMV-algebra of all effects is to show that the variety generated by the class of all quasi-linear QMV-algebras is finitely based. However, as we will show in Section 4, the usual correspondence ideals/congruences (used by Chang to prove that every MV-algebra is the subdirect product of linear MV algberas) fails in QMV-algebras. In this paper we will present a new definition of ideal that allows us to recover this correspondence.

2 Quantum weakenings of MV-algebras: definitions and examples

Definition 2.1. A *supplement algebra* (*S-algebra*) is a structure $\mathcal{M} = (M, \oplus, {}^*, \mathbf{1}, \mathbf{0})$, where M is a non-empty set, $\mathbf{0}$ and $\mathbf{1}$ are constant elements of M, $\oplus$ is a binary operation and * is a unary operation, satisfying the following axioms $\forall a, b, c \in M$:

(S1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,

(S2) $a \oplus b = b \oplus a$,

(S3) $a \oplus a^* = \mathbf{1}$,

(S4) $a \oplus \mathbf{0} = a$,

(S5) $a \oplus \mathbf{1} = \mathbf{1}$,

(S6) $a^{**} = a$.

The class of all S-algebras will be denoted by $\mathbb{S}$.

By (S4) and (S3), we get $\mathbf{0}^* = \mathbf{0} \oplus \mathbf{0}^* = \mathbf{1}$; hence, by (S6), $\mathbf{1}^* = \mathbf{0}^{**} = \mathbf{0}$. Given any S-algebra $\mathcal{M} = (M, \oplus, {}^*, \mathbf{1}, \mathbf{0})$, we can define the following binary operations and relation:

$$a \odot b = (a^* \oplus b^*)^*, \tag{2.1}$$

$$a \Cap b = (a \oplus b^*) \odot b, \tag{2.2}$$

$$a \Cup b = (a \odot b^*) \oplus b, \tag{2.3}$$

$$a \preceq b \iff a = a \Cap b. \tag{2.4}$$

We assume $\odot$ to be more binding than $\oplus$.

Definition 2.2. An *MV-algebra* is an S-algebra $\mathcal{M} = (M, \oplus, {}^*, \mathbf{1}, \mathbf{0})$ that satisfies the following condition $\forall a, b \in M$:

(LA) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

The class of all MV-algebras will be denoted by $\mathbb{MV}$.

Theorem 2.1. *Let* $\mathcal{M} = (M, \oplus, {}^*, \mathbf{1}, \mathbf{0})$ *be an MV-algebra. The structure* $(M, \Cap, \Cup, \mathbf{1}, \mathbf{0})$ *is a bounded distributive lattice.*

Definition 2.3. A *quantum MV-algebra* (*QMV-algebra*) is an S-algebra $\mathcal{M} = \langle M, \oplus, {}^*, \mathbf{1}, \mathbf{0}\rangle$ that satisfies the following conditions $\forall a, b \in M$:

(QMV1) $a \Cup (b \Cap a) = a$,

(QMV2) $(a \Cap b) \Cap c = (a \Cap b) \Cap (b \Cap c)$,

(QMV3) $a \oplus (b \Cap (a \oplus c)^*) = (a \oplus b) \Cap (a \oplus (a \oplus c)^*)$,

(QMV4) $a \oplus (a^* \Cap b) = a \oplus b$,

(QMV5) $(a^* \oplus b) \Cup (b^* \oplus a) = \mathbf{1}$.

The class of all QMV-algebras will be denoted by $\mathbb{QMV}$. (QMV1) and (QMV2) represent a weak formulation of the absorption and of the associativity laws, respectively. Generally, $\Cap$ and $\Cup$ are not lattice-theoretic operations. (QMV3) and (QMV4) represent a kind of conditional distributivity law of $\oplus$ over $\Cap$.

As proved in [5], any MV-algebra is a QMV-algebra, but not the other way around (cf. Counterexample 2.3).

Definition 2.4. A *quantum involution algebra* (*QI-algebra*) is an S-algebra $\mathcal{M} = \langle M, \oplus, ^*, \mathbf{1}, \mathbf{0}\rangle$ that satisfies the following condition $\forall a, b, c \in M$:

(QI) $a \oplus b \neq \mathbf{1} \implies a \prec b^*$

The class of all QI-algebras will be denoted by $\mathbb{QI}$.

Definition 2.5. A QMV-algebra is said to be *quasi-linear* iff it satisfies (QI).

The class of all quasi-linear QMV-algebras will be denoted by $\mathbb{QLQMV}$.

Theorem 2.2. *Let $\mathcal{M}$ be a QMV-algebra. The following conditions are equivalent:*

(i) $\mathcal{M}$ *is quasi-linear.*

(ii) $\forall a, b \in M$: $a \not\preceq b \implies a \Cap b = b$.

(iii) $\forall a, b, c \in M$: *if* $a \oplus c = b \oplus c \neq \mathbf{1}$, *then* $a = b$.

Theorem 2.3. [6] *An S-algebra $\mathcal{M}$ is a QI-algebra iff $\mathcal{M}$ is a quasi-linear QMV-algebra.*

According to Theorem 2.3 , QI-algebras and quasi linear QMV-algebras are equivalent structures. Clearly, an MV-algebra is quasi-linear iff it is totally ordered. The class of all totally ordered MV-algebras will be denoted by $\mathbb{TMV}$. The notion of quasi-linearity can be further weakened as follows:

Definition 2.6. A *weakly-linear QMV-algebra* is a QMV-algebra $\mathcal{M} = (M, \oplus, ^*, \mathbf{1}, \mathbf{0})$ that satisfies the following condition $\forall a, b \in M$:

(WL) $a \oplus b^* = \mathbf{1}$ or $b \oplus a^* = \mathbf{1}$.

The class of weakly-linear QMV-algebras will be denoted by $\mathbb{WLQMV}$. Every quasi-linear QMV-algebras is a weakly-linear QMV-algebra, but not the other way around (cf. Counterexample 2.4)

Theorem 2.4. *Let $\mathcal{M}$ be a QMV-algebra. The following conditions are equivalent:*

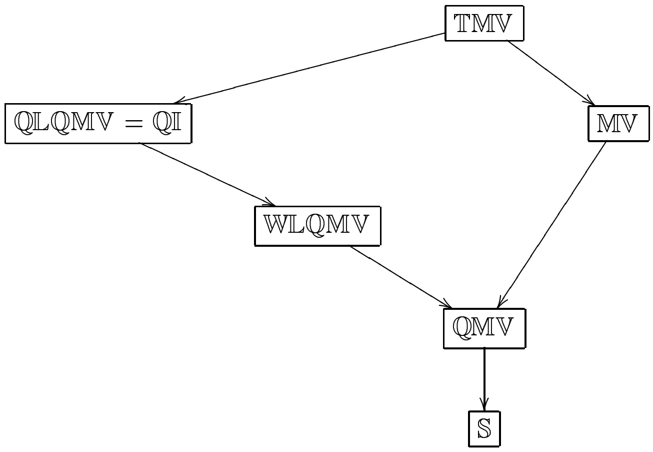

Figure 1:

(i) $\mathcal{M}$ *is weakly-linear.*

(ii) $\forall a, b \in M$: $a \Cap b = b$ *or* $b \Cap a = b$.

Example 2.1. (*S-algebra*)
Let $\mathcal{L}$ be any ortholattice. An ortholattice is a lattice $(L, \sqcap, \sqcup)$ with maximum ($\mathbf{1}$) and minimum ($\mathbf{0}$), equipped with a unary operation $'$ s.t. $\forall a, b \in L$: (i) $a'' = a$; (ii) $(a \sqcup b)' = a' \sqcap b'$; (iii) $a \sqcap a' = \mathbf{0}$.
The structure $(L, \sqcup, ', \mathbf{1}, \mathbf{0})$ is an S-algebra.

Example 2.2. (*Standard MV-algebra*)
Let $[0,1]$ be the unit real interval. For all $a, b \in [0,1]$, let

$$a \oplus b := \text{Min}\,\{a + b\,, 1\} \qquad (\textit{truncated sum})$$

and

$$a^* := 1 - a. \tag{2.5}$$

The structure $\mathcal{M}_{[0,1]} = ([0,1], \oplus, {}^*, 1, 0)$ is an MV-algebra, called *standard MV-algebra*. It turns out that the relation $\preceq$ (cf. 2.4) coincides with the restriction to $[0,1]$ of the usual order of $\mathbb{R}$. Consequently, $\mathcal{M}_{[0,1]}$ is *linear* (*totally ordered*), i.e. $\forall a, b \in [0,1]$: $a \preceq b$ or $b \preceq a$.
It turns out that $a \odot b = \text{Max}\,\{a + b - 1\,, 0\}$, $a \Cap b = \text{Min}\,\{a, b\}$ and $a \Cup b = \text{Max}\,\{a, b\}$.

Example 2.3. (*MV-algebra of fuzzy sets*)
Let X be a non-empty set and let $[0,1]^X$ be the set all $[0,1]$–valued functions on X (*fuzzy sets*) . Let us define the following operations on $[0,1]^X$, $\forall f, g \in [0,1]^X$

and $\forall x \in X$:

$$(f \oplus g)(x). = \mathrm{Min}\{1, f(x) + g(x)\} \tag{2.6}$$

and

$$(f^*)(x) = 1 - f(x). \tag{2.7}$$

Let **1** and **0** be the fuzzy sets s.t. $\forall x \in X$: $\mathbf{1}(x) = 1$ and $\mathbf{0}(x) = 0$.
The structure $[0,1]^{\mathcal{X}} := \left([0,1]^X, \oplus, ^*, \mathbf{1}, \mathbf{0}\right)$ is an MV-algebra.
It turns out that $\forall f, g \in [0,1]^X$ and $\forall x \in X$:

$$(f \Cap g)(x) = \mathrm{Min}\{f(x), g(x)\} \tag{2.8}$$

and

$$(f \Cup g)(x) = \mathrm{Max}\{f(x), g(x)\}. \tag{2.9}$$

According to an important theorem proved by Belluce [1], an MV algebra $\mathcal{M}$ is embeddable into an MV algebra of fuzzy sets iff $\mathcal{M}$ is *semisimple* (where an MV algebra is said to be semisimple iff the intersection of all its maximal ideals is $\{\mathbf{0}\}$).

Example 2.4. (*Standard QMV-algebra*)
Let $E(\mathcal{H})$ be the class of all effects of a Hilbert space $\mathcal{H}$. $E(\mathcal{H})$ coincides with the class of all bounded linear operators between $\mathbb{O}$ and $\mathbb{I}$, where $\mathbb{O}$ and $\mathbb{I}$ are the null and the identity operators, respectively. The operations $\oplus$ and * are defined as follows, for any $E, F \in E(\mathcal{H})$:

$$E \oplus F := \begin{cases} E + F & \text{if } E + F \in E(\mathcal{H}); \\ \mathbb{I} & \text{otherwise,} \end{cases} \tag{2.10}$$

where $+$ is the usual operator-sum.

$$E^* := \mathbb{I} - E. \tag{2.11}$$

The structure $\mathcal{E}(\mathcal{H}) := (E(\mathcal{H}), \oplus, ^*, \mathbb{I}, \mathbb{O})$ is a QMV-algebra, called *standard QMV-algebra*[4]. It turns out that the relation $\preceq$ (cf. 2.4) coincides with the usual partial order of $E(\mathcal{H})$, induced by the class of all density operators of $\mathcal{H}$. In other words, $\forall E, F \in E(\mathcal{H})$: $E \preceq F$ iff for any density operator D of $\mathcal{H}$: $\mathrm{Tr}(DE) \leq \mathrm{Tr}(DF)$, where "Tr" is the trace functional. Moreover:

$$E \Cap F = \begin{cases} E & \text{if } E \preceq F; \\ F & \text{otherwise,} \end{cases} \tag{2.12}$$

and

$$E \Cup F = \begin{cases} E & \text{if } F \preceq E; \\ F & \text{otherwise.} \end{cases} \tag{2.13}$$

As a consequence, $\mathcal{E}(\mathcal{H})$ is a quasi-linear QMV-algebra and therefore a QI-algebra by Theorem 2.3.

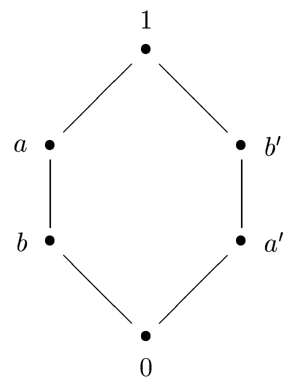

Figure 2: $\mathcal{O}_6$

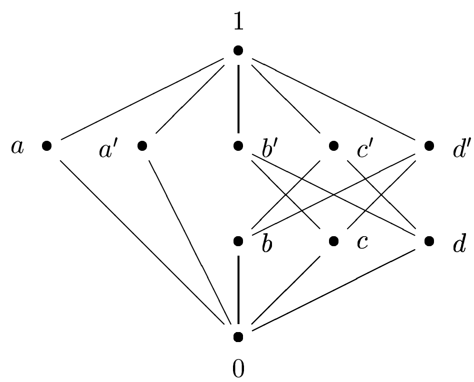

Figure 3: $\mathcal{G}_{10}$

None of the arrows of Figure 1 can be reversed as the following counterexamples show.

Counterexample 2.1. ($\mathbb{S} \subset \mathbb{QMV}$)
Let $\mathcal{O}_6$ be the ortholattice of six elements (see Figure 2). Let us consider the elements a, b. We have: $a \oplus (a^* \Cap b) = a \neq b = a \oplus b$. Thus, (QMV4) fails in $\mathcal{O}_6$.

Counterexample 2.2. ($\mathbb{WL}Q\mathbb{MV} \subset \mathbb{QMV}$)
Let us consider the *orthomodular lattice* $\mathcal{G}_{10}$ (see Figure 3). An orthomodular lattice is an ortholattice $\mathcal{L} = (L\, , \sqcap\, , \sqcup\, ,'\, , \mathbf{1}\, , \mathbf{0})$ that satisfies the following condition $\forall a, b \in L$:

$$(a \sqcap (a' \sqcup (a \sqcap b))) \sqcup b = b \qquad (\textit{orthomodularity})$$

As proved in [5], every orthomodular lattice is a QMV-algebra, taking $\oplus$ as $\sqcup$ and * as $'$. In $\mathcal{G}_{10}$ we have $a \Cap a^* = a^* \Cap a = \mathbf{0}$. Thus, $\mathcal{G}_{10}$ is not weakly-linear.

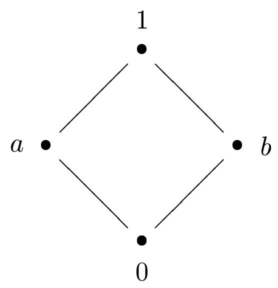

Figure 4: $\mathcal{M}_4$

Counterexample 2.3. ($\mathbb{MV} \subset \mathbb{QMV}$)
Take the QMV-algebra $\mathcal{E}(\mathcal{H})$ of the Example 2.4. Let us consider two non trivial effects $E, F \in E(\mathcal{H})$ s.t. $E + F^* \notin E(\mathcal{H})$ and $F \oplus E^* \notin E(\mathcal{H})$. By definition of $\oplus$: $E \oplus F^* = \mathbb{I}$ and $F \oplus E^* = \mathbb{I}$. Hence: $(E^* \oplus F)^* \oplus F = \mathbb{O} \oplus F \neq E = \mathbb{O} \oplus E = (E \oplus F^*)^* \oplus F$. Thus, the axiom (LA) (cf. Definition 2.2) fails in $\mathcal{E}(\mathcal{H})$.

The smallest QMV-algebra that is not an MV-algebra is determined by the set $M_4 = \{\mathbf{0}, \mathbf{1}, a, b\}$ with the operations $\oplus$ and * defined in the following way:

		$\oplus$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{0}$	a	a
$\mathbf{0}$	b	b
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$
a	$\mathbf{0}$	a
a	a	$\mathbf{1}$
a	b	$\mathbf{1}$
a	$\mathbf{1}$	$\mathbf{1}$
b	$\mathbf{0}$	b
b	b	$\mathbf{1}$
b	a	$\mathbf{1}$
b	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1}$	a	$\mathbf{1}$
$\mathbf{1}$	b	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

	*
$\mathbf{0}$	$\mathbf{1}$
a	a
b	b
$\mathbf{1}$	$\mathbf{0}$

It is easy the check that $a \Cup b = b \neq a = b \Cup a$. Thus the axiom (LA) (cf. Definition 2.2) fails in $\mathcal{M}_4$. This example shows that QMV-algebras, differently from MV-algebras, can contain more than one fixed point of the operation *.

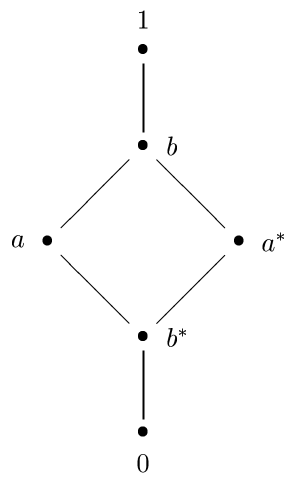

Figure 5: $\mathcal{M}_{wl}$

Counterexample 2.4. ($\mathbb{QLQMV} \subset \mathbb{WLQMV}$)
Let us consider the QMV-algebra $\mathcal{M}_{wl}$ (Figure 5) where the operation $\oplus$, apart the obvious conditions, is defined as follows:

		$\oplus$
a	a	$\mathbf{1}$
a	b	$\mathbf{1}$
a	b^*	$\mathbf{1}$
a^*	a^*	b
a^*	b	$\mathbf{1}$
a^*	b^*	b
b	b	$\mathbf{1}$
b	a	$\mathbf{1}$
b	a^*	$\mathbf{1}$
b^*	b^*	a
b^*	a	$\mathbf{1}$
b^*	a^*	b

One can check that $\mathcal{M}_{wl}$ is a weakly-linear QMV-algebra. However, $a^* \oplus a^* = b \neq \mathbf{1}$ and $a^* \not\leq a$. Thus, $\mathcal{M}_{wl}$ is not quasi-linear.

Counterexample 2.5. ($\mathbb{TMV} \subset \mathbb{MV}$)
Take any Boolean algebra containing more than two elements (where $\oplus$ is the *sup* and * is the complement).

Since the QMV-algebra of all effects is quasi-linear, it follows that $\mathbb{MV}$ and $\mathbb{QLMV}$ are unrelated.

3 Basic properties of QMV-algebras

In this Section, we will present some basic properties of the structures introduced in Section 2, starting from the properties that are already derivable in $\mathbb{S}$. The proof of the Theorems of this Section can be found in [5].

Theorem 3.1. *Let* $\mathcal{S} = (S\,, \oplus\,,^*\,, \mathbf{1}\,, \mathbf{0})$ *be an* S*-algebra. The following properties hold:*

(i) $a \odot b = b \odot a$.

(ii) $a \odot (b \odot c) = (a \odot b) \odot c$.

(iii) $a \odot a^* = \mathbf{0}$.

(iv) $a \odot \mathbf{0} = \mathbf{0}$.

(v) $a \odot \mathbf{1} = a$.

(vi) $a \Cap \mathbf{1} = a = \mathbf{1} \Cap a$.

(vii) $a \Cap \mathbf{0} = \mathbf{0} = \mathbf{0} \Cap a$.

(viii) $a = a \Cap a$.

(ix) $(a \Cup b)^* = a^* \Cap b^*$.

(x) $(a \Cap b)^* = a^* \Cup b^*$.

(xi) *If* $a \preceq b$*, then* $a = b \Cap a$.

In general, $a = b \Cap a$ *does not* imply $a = a \Cap b$.

Theorem 3.2. *Let* $\mathcal{S} = (S\,, \oplus\,,^*\,, \mathbf{1}\,, \mathbf{0})$ *be an* S*-algebra. The following properties hold:*

(i) *If* $a \oplus b = \mathbf{0}$*, then* $a = b = \mathbf{0}$.

(ii) *If* $a \odot b = \mathbf{1}$*, then* $a = b = \mathbf{1}$.

(iii) *If* $a \Cup b = \mathbf{0}$*, then* $a = b = \mathbf{0}$.

(iv) *If* $a \Cap b = \mathbf{1}$*, then* $a = b = \mathbf{1}$.

Theorem 3.3. (*Cancellation law*)
Let $\mathcal{S} = (S\,, \oplus\,,^*\,, \mathbf{1}\,, \mathbf{0})$ *be an* S*-algebra. For any* $a, b, c \in M$*: if* $a \oplus c = b \oplus c$*,* $a \preceq c^*$ *and* $b \preceq c^*$*, then* $a = b$.

Theorem 3.4. *Let* $\mathcal{S} = (S\,, \oplus\,,^*\,, \mathbf{1}\,, \mathbf{0})$ *be an* S*-algebra. If* $a \preceq b$*, then* $a^* \oplus b = \mathbf{1}$.

It should be noticed that, in general, $a^* \oplus b = \mathbf{1}$ *does not* imply $a \preceq b$. The property "$\forall a, b \in M: \ a^* \oplus b = \mathbf{1}$ implies $a \preceq b$ " characterizes MV-algebras w.r.t. QMV-algebras as the following Theorem asserts

Theorem 3.5. *Let* $\mathcal{M} = (M, \oplus, ^*, \mathbf{1}, \mathbf{0})$ *be a QMV-algebra. The following conditions are equivalent:*

(i) $\mathcal{M}$ *is an MV-algebra.*

(ii) $\forall a, b \in M$: *If* $a^* \oplus b = \mathbf{1}$, *then* $a \preceq b$.

As a corollary of Theorem 3.5, we obtain that no "genuine" QMV-algebra admits of an *implication* $\rightarrow$ being the right adjoint to the operation $\odot$, i.e.

$$a \odot b \preceq c \iff b \preceq a \rightarrow c. \tag{3.1}$$

More precisely, there exists no QMV-algebra $\mathcal{M} = \langle M, \oplus, ^*, \mathbf{1}, \mathbf{0}\rangle$ which satisfies the following conditions:

(i) $\mathcal{M}$ is *not* an MV-algebra;

(ii) $\langle M, \preceq, \odot\rangle$ is a (commutative) autonomous poset (cf. [7], p. 26);

(iii) $a^* = a \rightarrow \mathbf{0}$.

Therefore, MV-algebras are precisely quantum QMV-algebras with implication in the sense of (3.1). Similarly, Boolean algebras can be characterized as those *orthomodular lattices* admitting an implication in the sense of (3.1), whenever $\odot$ and * are replaced by the lattice-theoretic operations of *infimum* and of *orthocomplementation*, respectively. Hence the relationship between QMV-algebras and MV-algebras is the same as between orthomodular lattices and Boolean algebras.

Theorem 3.6. *Let* $\mathcal{M} = (M, \oplus, ^*, \mathbf{1}, \mathbf{0})$ *be a QMV-algebra. The following properties hold:*

(i) *If* $a \preceq b$, *then* $b^* \preceq a^*$.

(ii) $a \preceq b$ *iff* $b = b \Cup a = a \Cup b$.

(iii) $a \Cap (b \Cup a) = a$.

Theorem 3.7. $\langle M, \preceq, ^*, \mathbf{1}, \mathbf{0}\rangle$ *is an involutive bounded poset.*

Theorem 3.8. *Let* $\mathcal{M} = (M, \oplus, ^*, \mathbf{1}, \mathbf{0})$ *be a QMV-algebra. The following properties hold:*

(i) *If* $a \preceq b$, *then* $\forall c \in M$: $a \Cap c \preceq b \Cap c$. (weak monotony of $\Cap$)

(ii) *If* $a \preceq b$, *then* $\forall c \in M$: $a \Cup c \preceq b \Cup c$. (weak monotony of $\Cup$)

It should be noticed that, in general, $a \Cap b \not\preceq a$, $a \not\preceq a \Cup b$ and $a \Cap b \not\preceq b \Cup a$.

Theorem 3.9. (*Monotony of* $\oplus$ *and* $\odot$)
Let $\mathcal{M} = (M, \oplus, ^*, \mathbf{1}, \mathbf{0})$ *be a QMV-algebra. The following properties hold:*

(i) *If* $a \preceq b$, *then* $\forall c \in M$: $a \oplus c \preceq b \oplus c$.

(ii) *If* $a \preceq b$, *then* $\forall c \in M$: $a \odot c \preceq b \odot c$.

(iii) *If* $a \preceq b$ *and* $c \preceq d$ *then* $a \oplus c \preceq b \oplus d$.

(iv) $a \preceq b$ *and* $c \preceq d$ *then* $a \odot c \preceq b \odot d$.

Theorem 3.10. *Let* $\mathcal{M} = (M, \oplus, ^*, \mathbf{1}, \mathbf{0})$ *be a QMV-algebra. The following properties hold:*

(i) $a \odot b \preceq a$.

(ii) $a \preceq a \oplus b$.

(iii) $a \odot b \preceq a \Cap b$, $a \odot b \preceq b \Cap a$.

(iv) $a \Cup b \preceq a \oplus b$, $b \Cup a \preceq a \oplus b$.

Corollary 3.1. *If* $\mathcal{M}$ *is a linear (or totally ordered) QMV-algebra, then* $\mathcal{M}$ *is an MV-algebra.*

Theorem 3.11. *Let* $\mathcal{M}$ *be a QMV-algebra. The following conditions are equivalent:*

(i) $b = b \oplus a^*$.

(ii) $a \Cup b = \mathbf{1}$.

(iii) $a = a \oplus b^*$.

(iv) $b \Cup a = \mathbf{1}$.

Definition 3.1. Let $\mathcal{M}$ be a QMV-algebra. Let us define the following binary operations, for all $a \in M$ and for all $n \in \mathbb{N}$:

(i) $0 \cdot a = \mathbf{0}$, $\quad (n+1) \cdot a = n \cdot a \oplus a$

(ii) $a^0 = \mathbf{1}$, $\quad a^{n+1} = (a^n) \odot a$.

Clearly: $(n \cdot a)^* = (a^*)^n$; $(a^n)^* = (n \cdot a^*)$; $m \cdot (n \cdot a) = (m \cdot n) \odot a$; $a^{m+n} = (a)^m \odot (a)^n$; $a^{mn} = (a^m)^n$.

Theorem 3.12. *Let* $\mathcal{M}$ *be a QMV-algebra. If* $a \Cup b = \mathbf{1}$, *then* $\forall n \in \mathbb{N}$: $a^n \Cup b^n = \mathbf{1}$.

Lemma 3.1. *Let* $\mathcal{M}$ *be a weakly-linear QMV-algebra. The following property holds* $\forall a, b, c \in M$:

(i) $a \Cup (b \Cup c) = \mathbf{1} \implies a \Cup (c \Cup b) = \mathbf{1}$.

Proof. Suppose $a \Cup (b \Cup c) = \mathbf{1}$. By Theorem 3.11, $(b \Cup c) \Cup a = \mathbf{1}$. Since $\mathcal{M}$ is weakly-linear, we have either $a = \mathbf{1}$ or $b \Cup c = \mathbf{1}$. If $a = \mathbf{1}$, then we are done. If $b \Cup c = \mathbf{1}$, then again by Theorem 3.11, $c \Cup b = \mathbf{1}$; hence, $a \Cup (c \Cup b) = \mathbf{1}$. □

4 QMV-algebras, p-ideals and congruences

Let $\mathbb{K}$ be a class of algebraic structures of the same type. The *variety* generated by $\mathbb{K}$ will be denoted by $HSP(\mathbb{K})$. Chang has proved [3] that every MV-algebras can be represented as a subdirect product of linear MV-algebras. Consequently, $HSP(\mathbb{TMV}) = HSP(\mathbb{MV})$. Thus, the logic based on $HSP(\mathbb{TMV})$ is the same logic as the logic based on $HSP(\mathbb{MV})$. Chang has proved also [3] that an equation holds in $\mathbb{TMV}$ iff it holds in the standard MV-algebra $\mathcal{M}_{[0,1]}$ (cf. Example 2.3). Consequently, the $\aleph_0$-valued Łukasiewicz logic ($\aleph_0$-Ł) can be equivalently characterized by $\mathbb{MV}$, $\mathbb{TMV}$ or $\mathcal{M}_{[0,1]}$. For both MV and QMV-algebras, the partial ordering relation $\sqsubseteq$ can be seen to express a notion of *logical entailment*. Furthemore, inn $\aleph_0$-Ł, the notion of entailment is reducible to that of *logical truth*, for any MV-algebra $\mathcal{M} = (M\,, \oplus\,, {}^*\,, \mathbf{1}\,, \mathbf{0})$ admits a "good implication", i.e., a polynomial binary operation $\to$ s.t. $\forall a, b \in M$:

$$a \to b = \mathbf{1} \iff a \preceq b. \tag{4.1}$$

As Theorem 3.5 shows, $a \to b$ is just $a^* \oplus b$.

What about the axiomatizability of the logics based on $\mathbb{QMV}$, $\mathbb{WLQMV}$, and $\mathbb{QLQMV}$? The logic based on $\mathbb{QMV}$ is clearly axiomatizable since $HSP(\mathbb{QMV}) = \mathbb{QMV}$. In order to axiomatize the logics based on $\mathbb{WLQMV}$ and $\mathbb{QLQMV}$ one could try and generalize Chang's subdirect-product representation theorem. However, this is not possible, for one can prove [4] that:

- there exists an equation $\alpha = 1$ that holds in $\mathbb{WLQMV}$ but fails in $\mathbb{QMV}$;
- there exists an equation that holds in $\mathbb{QLQMV}$ but fails in $\mathbb{WLQMV}$.

Furthermore, one can prove that not every QMV-algebra admits of a "good" implication. [1]. For instance, the smallest genuine QMV-algebra $\mathcal{M}_4$ (Figure 4) does not admit of any good implication. Thus, from a logical perspective, entailment cannot be reduced to logical t ruth.

Accordingly, we are faced with the following problems:

1) is the notion of entailment of the logic based on $\mathbb{WLQMV}$ (finitely) axiomatizable?

2) If not, is the class of all logical truths of the logic based on $\mathbb{WLQMV}$ algebras (finitely) axiomatizable?

3) Is the notion of entailment of the logic based on $\mathbb{QLQMV}$ (finitely) axiomatizable?

4) If not, is the class of all logical truths of the logic based on $\mathbb{QLQMV}$ (finitely) axiomatizable?

[1]An elegant proof of this result has been given da P. Minari.

The first step to tackle these problems is to try and generalize some results concerning the usual correspondence between ideals and congruence relations. In order to present these results, it will be expedient to summarize the the main steps of Chang's subdirect-product representation theorem. These steps can be sketched as follows:

(i) there is a bijection between *ideals* and *congruence relations* of any MV-algebra $\mathcal{M}$;

(ii) the quotient algebra $\mathcal{M}_{\equiv_I}$ (where $\equiv_I$ is the congruence relation determined by a prime ideal I) is a linear MV-algebra;

(iii) for every non-zero element a of an MV-algebra, there exists at least one prime ideal I s.t. $a \notin I$.

The conclusion of the theorem then follows from a well known result of universal algebra. As proved in [5], the usual 1:1 correspondence between ideals and congruence relations breaks down in QMV-algebras. In this Section, we introduce a stronger notion of ideal (*p-ideal*), which allows us to generalize (i) and (ii) above. Resul t (iii) will be proved only for a particular quasi-variety of QMV-algebra and under the hypothesis that every ideal i s a p-ideal.

Definition 4.1. Let $\mathcal{M} = \langle M, \oplus, ^*, \mathbf{1}, \mathbf{0}\rangle$ be a QMV-algebra. An *ideal* of $\mathcal{M}$ is any non-empty subset $I \subseteq M$ s.t.

(i) $\forall a, b \in M : a, b \in I \Longrightarrow a \oplus b \in I$;

(ii) $a \in I \Longrightarrow \forall b \in M : \ a \odot b \in I$.

Definition 4.2. An ideal I is *prime* iff $\forall a, b \in M$: $a \odot b^* \in I$ or $a^* \odot b \in I$.

One can easily show that an ideal I is prime iff $\forall a, b \in M$:

$$a \Cap b \in I \Longrightarrow a \in I \text{ or } b \in I. \tag{4.2}$$

Definition 4.3. Two elements $a, b \in M$ are *perspective* ($P(a,b)$) iff they have a common complement, i.e., $\exists c \in M$ s.t. $a \odot c = b \odot c = \mathbf{0}$ and $a \oplus c = b \oplus c = \mathbf{1}$.

It should be noticed that the relation P in general is not transitive.

Definition 4.4. A *p-ideal* is an ideal I, which is closed under perspectivity: $\forall a, b \in I$: $b \in I$ and $P(a,b) \Longrightarrow a \in I$.

Differently from MV-algebras, in QMV-algebras not every ideal is a p-ideal.

Facts 4.1.

(i) Let I be an ideal. I is a p-ideal iff $a \in I \implies \forall b \in M$*:* $a \Cap b \in I$.

(ii) Let I be a p-ideal. $\forall a, b \in M$*:* $a \Cap b \in I \iff b \Cap a \in I$.

(iii) Let I be a p-ideal. $\forall a, b \in M$*:* $a \Cup b \in I \implies a \in I$.

(iv) Let I be a p-ideal. $a^* \oplus b = \mathbf{1}$ and $b \in I \implies a \in I$.

By Fact 4.1(i), it follows that in every MV-algebra, every ideal is a p-ideal.

It is easy to see that if $\equiv$ is a congruence relation on a QMV-algebra $\mathcal{M}$, then the set $I := \{a \mid a \equiv \mathbf{0}\}$ is a p-ideal.

Let $\mathcal{M}$ be a QMV-algebra. Let us define the *distance function* $d\colon M \times M \to M$ in the usual way:

$$d(a,b) := (a \odot b^*) \oplus (b \odot a^*)\,. \tag{4.3}$$

If I is an ideal, let us define the binary relation $\equiv_I$ on M in the following way:

$$a \equiv_I b \iff d(a,b) \in I. \tag{4.4}$$

If $\mathcal{M}$ is a MV-algebra, then $\equiv_I$ is a congruence relation. Furthermore,

$$I = \{a \in M \mid a \equiv_I \mathbf{0}\} \tag{4.5}$$

and $a \equiv b$ iff $d(a,b) = \mathbf{0}$. Thus, in the case of MV-algebras, the correspondence $I \mapsto \equiv_I$ is a bijection from the set of ideals of $\mathcal{M}$ onto the set of congruence relations on $\mathcal{M}$.

As we have seen, in the case of QMV-algebras, one can still prove that every congruence relation gives rise to a p-ideal; however, not every p-ideal determines a congruence relation according to (4.4) (cf. [5]). In order to recover this correspondence, we will define a new relation, which turns out to be stronger (in $\mathbb{QMV}$) than the relation based on the distance function.

Definition 4.5. Let $\mathcal{M}$ be a QMV-algebra and let I be an ideal on $\mathcal{M}$. $\forall a, b \in M$: $a \sim_I b$ iff the following conditions are satisfied:

(i) $\exists x \in M$ s.t. $x^* \preceq a, b$ and $x \odot a,\ x \odot b \in I$

(ii) $\exists y \in M$ s.t. $y^* \preceq a^*, b^*$ and $y \odot a^*,\ y \odot b^* \in I$.

Lemma 4.1. *Let $\mathcal{M}$ be a QMV-algebra. The following properties hold:*

(i) Let I be an ideal. $\forall a, b \in M$: $a \sim_I b \implies d(a,b) \in I$.

(ii) Let I be a p-ideal: $a \in I$ and $a \sim_I b \implies b \in I$.

(iii) If $\mathcal{M}$ is an MV-algebra, then $\forall a, b \in M$: $a \sim_I b \iff d(a,b) \in I$.

Proof. (i) Suppose $a \sim_I b$. Then, $\exists x^* \preceq a, b$ s.t. $a \odot x \in I$ and $b \odot x \in I$. Thus, $b \odot a^* \preceq b \odot x$ and $a \odot b^* \preceq a \odot x$. Since I is an ideal, we have $a \odot b^* \in I$ and $a^* \odot b \in I$. Hence, $d(a,b) \in I$.
(ii) Suppose $a \in I$ and $a \sim_I b$. By (i), $b \odot a^* \in I$. Thus, $b \Cup a = b \odot a^* \oplus a \in I$. By Fact 4.1(iii), it follows $b \in I$.
(iii) By (i), it suffices to show that $d(a,b) \in I \implies a \sim_I b$. Suppose $a \odot b^* \oplus a^* \odot b \in I$. Let $x := a^* \Cup b^*$. Clearly, $x^* \preceq a, b$. Moreover, $a \odot x = a \odot (a^* \Cup b^*) = a \odot b^* \in I$. Similarly, one can prove that $a^* \odot b \in I$. □

By Lemma 4.1(iii), the relation $\sim_I$ coincides (in $\mathbb{MV}$) with the relation $\equiv_I$ defined according to (4.4).

Theorem 4.1. *Let $\mathcal{M}$ be a QMV-algebra and let I be a p-ideal. The following properties hold $\forall a, b, c, d \in M$:*

(i) $a \sim_I a$,

(ii) $a \sim_I b \implies b \sim_I a$,

(iii) $a \sim_I b \implies a^* \sim_I b^*$,

(iv) $a \sim_I b$ *and* $c \sim_I d \implies a \oplus c \sim_I b \oplus d$.

Proof. (i)-(iii) The proof is straightforward.
(iv) Suppose $a \sim_I b$ and $c \sim_I d$. We want to prove that $\exists z \in M$ s.t. $z^* \preceq a \oplus c, b \oplus d$, and $z \odot (a \oplus c), z \odot (b \oplus d) \in I$. The proof of condition (ii) of Definition 4.5 is similar. By hypothesis, $\exists x, y \in M$ s.t. $x^* \preceq a, b, y^* \preceq c, d, x \odot a, x \odot b \in I$ and $y \odot c, y \odot d \in I$. Let $z := x \odot y$. By monotony of $\oplus$, we get $z^* = x^* \oplus y^* \preceq a \oplus c$. Similarly, $z^* \preceq b \oplus d$. Thus, it remains to prove that $z \odot (a \oplus c), z \odot (b \oplus d) \in I$. By hypothesis, $a \odot x, c \odot y \in I$ so that $a \odot x \oplus c \odot y \in I$. In order to prove that $z \odot (a \oplus c) \in I$, it suffices to show, by Fact 4.1(iv), that $(x \odot y \odot (a \oplus c))^* \oplus a \odot x \oplus c \odot y = \mathbf{1}$.

$$\begin{aligned} x^* \oplus y^* \oplus a^* \odot c^* \oplus a \odot x \oplus c \odot y &= a \Cup x^* \oplus c \Cup y^* \oplus a^* \odot c^* \\ &= a \oplus c \oplus a^* \odot c^* \qquad (x^* \preceq a, y^* \preceq c) \\ &= \mathbf{1}. \end{aligned}$$

The proof of $z \odot (b \oplus d) \in I$ is similar. □

Let I be an ideal on a QMV-algebra $\mathcal{M}$. Let $\equiv_I$ be the *transitive closure* of the relation $\sim_I$. Thus, $\forall a, b \in M$:

$$a \equiv_I b \text{ iff } \exists x_1, \ldots x_n \in M \text{ s.t. } x_1 = a,\ x_i \sim_I x_{i+1} \text{ (with } 1 \leq i \leq n-1) \text{ and } x_n = b \quad (4.6)$$

By Theorem 4.1, $\equiv_I$ is a congruence relation on M. Conversely, every congruence relation determines a p-ideal in the usual manner (cf. 4.5).

Lemma 4.2. *Let $\mathcal{M}$ be a QMV-algebra and let I be a p-ideal on $\mathcal{M}$. The following conditions are equivalent $\forall a \in M$:*

(i) $a \equiv_I \mathbf{0}$,

(ii) $a \in I$,

(iii) $a \sim_I \mathbf{0}$.

Proof. (i) $\Rightarrow$ (ii) Suppose $a \equiv_I \mathbf{0}$. Then $\exists x_1, \ldots, x_n \in M$. s.t. $x_1 = a, \ldots, x_i \sim_I x_{i+1}, \ldots, x_n = \mathbf{0}$. Now, $\mathbf{0} \in I$ and therefore, by Lemma 4.1(ii), $x_{n-1} \in I$. By iteration, we get $a \in I$.
The proof of (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) is straightforward. □

According to Lemma 4.2 and Theorem 4.1, if I is a p-ideal, then $\equiv_I$ is a congruence relation and the set $\{a \mid a \equiv_I \mathbf{0}\}$ is easily seen to be equal to I. Conversely, given a congruence relation $\equiv$, the set $I := \{a \mid a \equiv \mathbf{0}\}$ is still a p-ideal and $\equiv_I \leq \equiv$; however, in general, $\equiv \not\leq \equiv_I$ as the following counterexample shows.

Let us consider the QMV-algebra $\mathcal{M}_4$ (Figure 4). The set $I = \{\mathbf{0}\}$ is clearly a p-ideal. The congruence relation $\equiv_I$ determined by $\sim_I$ is the identity relation. I is the only proper p-ideal on $\mathcal{M}_4$; thus, the identity relation is the only congruence relation determined by a p-ideal. We want to show that there exists a congruence relation $\equiv$ on $\mathcal{M}_4$ and two elements $x, y \in M_4$ s.t. $x \not\equiv_I y$ (i.e., $x \neq y$) and $x \equiv y$. Let us define $x \equiv y$ iff $d(x,y) = \mathbf{0}$ (cf. (4.3)). It is easy to check that $\equiv$ is a congruence relation on $\mathcal{M}_4$. Take $x = a$ and $y = b$. Then, $d(a,b) = a^* \odot b \oplus a \odot b^* = \mathbf{0}$. Thus, $a \equiv b$ and $a \neq b$.

Let $\mathcal{M}$ be a QMV-algebra and let I be a p-ideal on $\mathcal{M}$. Let us consider the quotient algebra $\mathcal{M}_{/\equiv_I}$. Clearly, $\mathcal{M}_{/\equiv_I}$ is a QMV-algebra. The equivalence class determined by any element a of M will be denoted by $[a]$.

Theorem 4.2. *$\mathcal{M}_{/\equiv_I}$ is weakly-linear iff I is prime.*

Proof. Suppose $\mathcal{M}_{/\equiv_I}$ is weakly linear. Thus, $\forall [a],[b] \in M_{/\equiv_I}$: $a \Cap b \equiv_I b$ or $b \Cap a \equiv_I a$. Then, $b^* \oplus (a \Cap b) \equiv_I \mathbf{1}$ or $a^* \oplus (b \Cap a) \equiv_I \mathbf{1}$. By (QMV4), $b^* \oplus a \equiv_I \mathbf{1}$ or $a^* \oplus b \equiv_I \mathbf{1}$. Thus, by Lemma 4.2, $a^* \odot b \in I$ or $b^* \odot a \in I$.
Suppose that I is a prime p-ideal. We have to show that $a \Cap b \equiv_I b$ or $b \Cap a \equiv_I a$. Suppose $a \Cap b \not\equiv_I b$. Then, $a \oplus b^* \not\equiv_I \mathbf{1}$. By Lemma 4.2, $a^* \odot b \notin I$. Since I is prime, we have that $b^* \odot a \in I$. By Lemma 4.2, $b^* \odot a \equiv_I \mathbf{0}$. Hence, $b \Cap a \equiv_I a$. □

Theorem 4.3. *Let $\mathcal{M}$ be a QMV-algebra, which satisfies condition (i) of Lemma 3.1. $\forall a \in M$ s.t. $a \neq \mathbf{0}$, there exists an ideal I on $\mathcal{M}$ s.t. $a \notin I$ and s.t. if I is a p-ideal, then I is prime.*

Proof. Let I be the ideal, which is maximal w.r.t. the property "$a \notin I$". Suppose, by contradiction, that I is not prime. Thus, $\exists x, y \in M$ s.t. $x \odot y^* \notin I$ and $y \odot x^* \notin I$. Since I is maximal, we obtain that $\exists c, d \in I$ and $\exists m, n \in \mathbb{N}$ s.t. $a \preceq c \oplus m \cdot (x \odot y^*)$ and $a \preceq d \oplus n \cdot (y \odot x^*)$. Let $u := c \oplus d$ and $p := \mathrm{Max}(m,n)$. By monotonicity of $\Cap$ (Theorem 3.8(i)), we obtain

$$a \preceq (u \oplus p \cdot (x \odot y^*)) \Cap (u \oplus p \cdot (y \odot x^*)) \,. \tag{4.7}$$

We want to prove that

$$[(u \oplus p \cdot (x \odot y^*)) \Cap (u \oplus p \cdot (y \odot x^*))]^* \oplus u = \mathbf{1}. \tag{4.8}$$

$$\begin{aligned}
&(u \oplus p \cdot (x \odot y^*))^* \Cup (u \oplus p \cdot (y \odot x^*))^* \oplus u = \\
&\quad (u \oplus p \cdot (x \odot y^*))^* \odot (u \oplus p \cdot (y \odot x^*)) \oplus u^* \odot (p \cdot (y \odot x^*))^* \oplus u = \\
&\quad u^* \odot \left((x \odot y^*)^*\right)^p \odot (u \oplus p \cdot (y \odot x^*)) \oplus u^* \odot \left((y \odot x^*)^*\right)^p \oplus u = \\
&\quad \left((x \odot y^*)^*\right)^p \Cup \left(u^* \odot \left((y \odot x^*)^*\right)^p \oplus u\right) = \\
&\quad \left((x \odot y^*)^*\right)^p \Cup \left(\left((y \odot x^*)^*\right)^p \Cup u\right) .
\end{aligned} \tag{4.9}$$

By (QMV5) and Theorem 3.12, $\big((x \odot y^*)^*\big)^p \Cup \big((y \odot x^*)^*\big)^p = \mathbf{1}$. Now, $\big((y \odot x^*)^*\big)^p \preceq u \Cup \big((y \odot x^*)^*\big)^p$; hence, by weak monotonicity of $\Cup$ (Theorem 3.8(ii)), we obtain

$$\big((x \odot y^*)^*\big)^p \Cup \big(u \Cup \big((y \odot x^*)^*\big)^p\big) = \mathbf{1} \tag{4.10}$$

By hypothesis, $\mathcal{M}$ satisfies condition (i) of Lemma 3.1 so that

$$\big((x \odot y^*)^*\big)^p \Cup \big(\big((y \odot x^*)^*\big)^p \Cup u\big) = \mathbf{1}. \tag{4.11}$$

By (4.9), $[(u \oplus p \cdot (x \odot y^*)) \Cap (u \oplus p \cdot (y \odot x^*))]^* \oplus u = \mathbf{1}$.

By hypothesis, I is a p-ideal; since $u \in I$ we obtain, by Fact 4.1(iv),

$$(u \oplus p \cdot (x \odot y^*)) \Cap (u \oplus p \cdot (y \odot x^*)) \in I. \tag{4.12}$$

By (4.7), we obtain $a \in I$, contradiction. □

As a corollary, we obtain that if $\mathcal{M}$ is a QMV-algebra (satisfying condition (i) of Lemma 3.1) s.t. every ideal is a p-ideal, then $\forall a \in M$ ($a \neq zero$), there exists a prime p-ideal I s.t. $a \notin I$.

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