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Orthogonal Decompositions of MV-Spaces

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Abstract

A maximal disjoint subset S of an MV-algebra A is a basis iff $\{x \in A : x \leq a\}$ is a linearly ordered subset of A for all $a \in S$. Let $\operatorname{Spec} A$ be the set of the prime ideals of A with the usual spectral topology. A decomposition $\operatorname{Spec} A = \cup_{i \in I} T_i \cup X$ is said to be orthogonal iff each T_i is compact open and $S = \{a_i\}_{i \in I}$ is a maximal disjoint subset. We prove that this decomposition is unrefinable (i.e. no $T_i = \Theta \cap Y$ with Θ open, $\Theta \cap Y = \emptyset$, int $Y = \emptyset$) iff S is a basis. Many results are established for semisimple MV-algebras, which are the algebraic counterpart of Bold fuzzy set theory.

Keywords: MV-algebra, orthogonal decomposition, basis, closed ideal, semisimple MV-algebra, complete MV-algebra.

1 Introduction

Given a topological space T one may try to study T by examining the various quotient spaces that T admits. Equivalently one may look at the different types of decompositions T admits. One thus may obtain information about T by examining related structures.

We shall apply this method to the class of MV-spaces. These spaces are the prime ideal spaces of MV-algebras. To date there is no known purely topological description of these spaces and this work may be considered as an attempt to obtain more information towards that purpose.

This work may also be considered as a contribution to the duality between an MV-space and its associated algebra. In this regard we endeavour to obtain theorems of the form: an MV-space T admits a decomposition of a certain type iff the corresponding algebra A has certain properties. In order to obtain results with substance we shall in general restrict ourselves to a particular type of decomposition, namely what we shall call here "orthogonal" decompositions. These decompositions will naturally relate to "orthogonal" subsets of the corresponding MV-algebra and thereby set up an interchange between this type of decomposition and properties of the algebra.

We present some theorems already known in lattice-ordered group theory extendible to MV-algebras by using the Γ functor [17]; however, in order to get a self-contained paper, we exhibit the proofs using properly MV-machinery.

2 Definitions

An MV-algebra is a system $< A, +, \cdot, -, 0, 1 >$ where A is a non-empty set, "+", "·", are binary operations, $-a$ is an unary operation on A, $0, 1 \in A$, $0 \neq 1$. We denote this by its underlying set A. The following axioms are to be satisfied: $\langle A, +, 0\rangle$, $\langle A, \cdot, 1\rangle$ are commutative monoids with identity 0, 1, respectively; $\overline{(a+b)} = \bar{a} \cdot \bar{b}$, $\overline{a \cdot b} = \bar{a} + \bar{b}$, $\bar{\bar{a}} = a$, $\bar{0} = 1$; finally, $a + \bar{a}b = b + \bar{b}a$.

Introducing the operation "$\vee$" on A by defining $a \vee b = a + \bar{a} \cdot b$, the final axiom says $a \vee b = b \vee a$. We also introduce the operation "$\wedge$" by $a \wedge b = \overline{\bar{a} \vee \bar{b}} = a \cdot (\bar{a} + b)(= b \cdot (\bar{b} + a))$. The induced system $\langle A, \vee, \wedge, 0, 1\rangle$ becomes a distributive lattice with the least element 0, the greatest element 1 and the order defined by $a \leq b$ iff $a \wedge b = a$. An ideal I of the MV-algebra A is a subset with $0 \in I$, I is closed under $+$, and $a \in I$, $b \in A$ implies $a \cdot b \in I$. Equivalently $a \in I$, $b \leq a$ implies $b \in I$. An ideal I of A is always a lattice ideal in the induced distributive lattice $< A, \vee, \wedge, 0, 1 >$ but lattice ideals in general are not MV-ideals. An ideal $P \subseteq A$ is prime iff $a \wedge b \in P$ implies $a \in P$ or $b \in P$ or equivalently iff for each a, $b \in A$, $a\bar{b} \in P$ or $\bar{a}b \in P$. All maximal ideals are prime.

For definitions and concepts on MV-algebras used here, we refer the reader to [1] and [9]. Some of the next definitions can be found in detail in [7], however we recall those essential for our purposes.

We denote by $\operatorname{Spec} A$ the set of prime ideals of A with the usual spectral topology, i.e. $\Theta \subseteq \operatorname{Spec} A$ is open iff for some ideal (or subset) $I \subseteq A$, we have $\Theta = V(I) = \{P \in \operatorname{Spec} A : I \nsubseteq P\}$. It is known that this is a spectral space [1]. An MV-space Z is a spectral space such that $Z = \operatorname{Spec} A$ for some MV-algebra A. Not all spectral spaces are MV-spaces [4]. Now let $a \in A$, the open sets $V(a) = \{P \in \operatorname{Spec} A : a \notin P\}$ constitute a basis for the open sets of $\operatorname{Spec} A$. Each $V(a)$ is compact open, any compact open T is such that $T = V(a)$ for some $a \in A$. An orthogonal or disjoint subset $S \subseteq A$ is a nonempty set such that $0 \notin S$ and $a, b \in S$, $a \neq b$ implies $a \wedge b = 0$ ([8],7.3.1).

Given any subset $W \subseteq A$, $W \neq \emptyset$, we set $W^{\perp} = \{x \in A : x \wedge w = 0\ \forall w \in W\}$ and $W^{\perp}$ is always an ideal of A. For brevity, $(W^{\perp})^{\perp} = W^{\perp\perp}$ and $\{a\}^{\perp} = a^{\perp}$, $a \in A$. Let $B(A) = \{a \in A : a + a = a\}$ be the subalgebra of idempotents of A, then we note that $V(a)$ is clopen in $\operatorname{Spec} A$ iff $a \in B(A)$.

By decomposition of a topological space is meant an expression $T = \cup_{i \in I} T_i$, where each T_i is a nonempty subspace of T and $T_i \cap T_j = \emptyset$ whenever $i \neq j$.

In what follows the following theorem will be useful ("Riesz decomposition property"):

Theorem 1. *For x, y, $z \in A$, $(x + y) \wedge z \leq (x \wedge z) + (\bar{x} \wedge y \wedge z)$.*

Proof. Let $d = (x + y) \wedge z$. Then $d \leq x + y$ and let $a = d \wedge x$. So $d\bar{a} = d \cdot (\bar{d} \vee \bar{x}) = d\bar{x}$ by ([9], Ax.11'), which implies $d = d \vee a = d\bar{a} + a = d\bar{x} + (d \wedge x)$. Now

$d\bar{x} \le (x+y)\cdot\bar{x} = y\wedge\bar{x}$ and $d\bar{x}\le d$, then $d \le (d\wedge y\wedge\bar{x}) + (d\wedge x)$. Therefore the thesis since $d\wedge y = y\wedge z$ and $d\wedge x = x\wedge z$. □

Finally we recall some elementary facts [9]. If I is an ideal of A, the congruence ($x/I = y/I$ iff $x\bar{y} + \bar{x}y \in I$) defines the quotient MV-algebra A/I whose the operations are those naturally induced by the operations of A.

A is said to be linearly ordered if the underlying lattice is linearly ordered and obviously A/P is linearly ordered if $P \in \operatorname{Spec} A$.

An element $a \in A$, $a \neq 0$, is an atom of A if a is an atom in the underlying lattice.

3 Orthogonal decompositions

It is already known that $\operatorname{Spec} A$ has a decomposition $\operatorname{Spec} A = \cup\{T_M : M \in \operatorname{Max} A\}$, where $\operatorname{Max} A$ denotes the maximal ideals of A and $T_M = \{P \in \operatorname{Spec} A : P \subseteq M\}$ ([7], Cor. 5).

Let $C \subseteq \operatorname{Spec} A$. Now C is closed in $\operatorname{Spec} A$ iff $C = \mathbb{V}(I) = \{P \in \operatorname{Spec} A : I \subseteq P\}$ for some ideal I of A. Then, in the above decomposition, each T_M is closed since $T_M = \mathbb{V}(O_M)$ where $O_M = \cap\{P : P \in T_M\}$ ([7], Prop. 2).

Let Y be a nonempty subset of A. As usually, we denote by $\operatorname{id}(Y) = \{x \in A : x \le y_1 + \ldots + y_n, y_1, y_2, \ldots, y_n \in Y\}$ the ideal of A generated by Y. If $Y = \{y\}$, we put $\operatorname{id}(Y) = \operatorname{id}(y)$.

Further, we set $\operatorname{Rad} A = \cap\{M : M \in \operatorname{Max} A\}$; $\operatorname{Min} A$ denotes the minimal prime ideals of A.

The following result holds:

Theorem 2. *Let $M \in \operatorname{Max} A$. Then $O_M = M^2$, where $M^2 = \operatorname{id}\{x\cdot y : x, y \in M\}$.*

Proof. We note that $O_M = \cap\{P \in \operatorname{Min} A : P \in T_M\}$. Further, $\operatorname{Min} A$ coincides with the set of minimal prime ideals of the underlying lattice ([10], Lemma1.2). Hence $O_M = \{x \in A : x^\perp \not\subseteq M\}$ by ([12], Prop. 2.2).

Let $P \in T_M$, so M/P is an ideal in A/P. Moreover $M/P \neq A/P$, thus $M/P = \operatorname{Rad}(A/P)$. If x, $y \in M$, then x/P, $y/P \in \operatorname{Rad}(A/P)$, and hence $x/P \cdot y/P = xy/P = 0$ by ([2], Thm. 1). This means $xy \in P$ for any $P \in T_M$, i.e. $M^2 \subseteq O_M$.

Now let $x \in O_M$, then $x^\perp \not\subseteq M$ and choose $y \in x^\perp - M$. For some integer n, $u = \bar{y}^n \in M$. Let $P \in \operatorname{Spec} A$. If $x \in P$, then $x/P = 0 \le u^k/P$ for any integer k. If $x \notin P$, then $y \in P$ since $x \wedge y = 0$. So $y/P = 0$, i.e. $\bar{y}/P = 1$ which implies $u/P = 1$. So again $x/P \le u^k/P$ for any integer k. P is arbitrary in $\operatorname{Spec} A$, it follows that $x \le u^k$ for all integers k. In particular, $x \le u^2 \in M^2$ and therefore $O_M \subseteq M^2$, which gives the thesis. □

The subspaces T_M are, in general, not open. Were each T_M open, then, as $\operatorname{Spec} A$ is compact, we would see that A is semilocal, that is $\operatorname{Max} A$ would be finite. The converse is true as well so we have:

Theorem 3. *The decomposition* $\operatorname{Spec} A = \cup\{T_M : M \in \operatorname{Max} A\}$ *has each T_M clopen iff A is semilocal.*

We can of course have decompositions $\mathrm{Spec}\, A = \cup_{i\in I} T_i$ with each T_i clopen even if A is not semilocal. The partitioning of $\mathrm{Spec}\, A$ into clopen subsets requires an adequate supply of idempotents in A. As some MV-algebras have no non-trivial idempotents, clopen decompositions of the corresponding MV-space do not always exist.

Let us examine first the case where we have a decomposition $\mathrm{Spec}\, A = \cup_{i\in I} T_i$ where each T_i is a closed subspace. Let $M \in \mathrm{Max}\, A$, then $M \in T_i$ for some i. We claim that $T_M \subseteq T_i$. M contains certainly some $m \in \mathrm{Min}\, A$. If $m \notin T_i$, then $m \in T_j$ with $j \neq i$. But then $M \in T_j$ which is impossible. Thus $T_M \subseteq T_i$. We have $T_i = \cup\{T_M \ : \ M \in T_i\}$. Indeed, let $P \in T_i$ and $M \in \mathrm{Max}\, A$ be such that $P \subseteq M$, hence $M \in T_i$, so $T_M \subseteq T_i$ and $P \in T_M$. Therefore we have

Proposition 4. *If* $\mathrm{Spec}\, A = \cup_{i\in I} T_i$ *is a decomposition of closed sets, then* $T_i = \cup\{T_M \ : \ M \in \mathrm{Max}\, A \cap T_i\}$ *for each* $i \in I$.

Thus all "closed" decompositions refine to $\mathrm{Spec}\, A = \cup\{T_M \ : \ M \in \mathrm{Max}\, A\}$. Let us then focus on decompositions where as many components as possible come from the basis of compact open sets. That is, where as many components as possible are of the form $V(a)$, some $a \in A$. If two compact open sets $V(a)$, $V(b)$ are disjoint, then we have $V(a) \cap V(b) = V(a \wedge b) = \emptyset$ and so $a \wedge b = 0$. Let us then look for partitions of $\mathrm{Spec}\, A$ related to orthogonal subsets of A. Every orthogonal subset of A can be extended, via an application of Zorn lemma, to a maximal orthogonal subset and we shall look for decompositions correlated with such subsets. Given A call a decomposition $\mathrm{Spec}\, A = \cup_{i\in I} T_i \cup X$ an orthogonal decomposition if each $T_i = V(a_i)$ for some $a_i \in A$, where the set $S = \{a_i \ : \ i \in I\}$ is a maximal orthogonal subset and $X \cap T_i = \phi$ for any $i \in I$. We have the following:

Theorem 5. *Let A be an MV-algebra and suppose* $\mathrm{Spec}\, A = \cup_{i\in I} T_i \cup X$ *is a decomposition with each T_i compact open. Then the decomposition is orthogonal iff int* $X = \emptyset$.

Proof. Suppose first that $\mathrm{Spec}\, A = \cup_{i\in I} T_i \cup X$ is an orthogonal decomposition. There is an $a_i \in A$ such that $T_i = V(a_i)$ and $S = \{a_i \ : \ i \in I\}$ is a maximal orthogonal subset of A. Suppose $\mathrm{int}\, X \neq \emptyset$ and let $V(b) \subseteq X$ for some $0 \neq b \in A$. As $X \cap T_i = \emptyset$ for any $i \in I$, we have $V(b) \cap V(a_i) = \emptyset$ for any $i \in I$. Hence $b \in S^{\perp}$. But as S is a maximal orthogonal set, $S^{\perp} = \{0\}$ and so $b = 0$.

We infer $\mathrm{int}\, X = \emptyset$. Conversely, suppose we have a decomposition $\mathrm{Spec}\, A = \cup_{i\in I} T_i \cup X$ with each T_i compact open and $\mathrm{int}\, X = \emptyset$. Let $S = \{a_i \ : \ i \in I\}$, where $T_i = V(a_i)$. If S is not a maximal orthogonal set, then $S^{\perp} \neq 0$. Choose $b \in S^{\perp}$, $b \neq 0$, then $V(b) \neq \emptyset$. But then $V(b) = V(b) \cap \mathrm{Spec}\, A = \cup_{i\in I}(V(b) \cap V(a_i)) \cup (V(b) \cup X)$. As $V(b) \cap V(a_i) = V(b \wedge a_i) = V(0) = \emptyset$ for any $i \in I$, we have $V(b) = V(b) \cap X \subseteq X$. Thus $\mathrm{int}\, X \neq \emptyset$ and hence it follows that S is a maximal orthogonal set and so the given decomposition is orthogonal. □

Remark 6. With the same notations of the proof of Theorem 5, we note that $X = \mathbb{V}(S)$.

Thus we have a correspondence between maximal orthogonal subsets of a given MV-algebra A and the maximal decomposition of $\mathrm{Spec}\, A$. Note however that this correspondence is many-one, different maximal orthogonal sets corresponding to the same orthogonal decomposition.

Observe that not all spectral spaces admit a decomposition of this type. Consider, for example, the sublattice $\mathcal{L}$ (with 0,1) of the lattice of all open subsets of the real line generated by the open sets that contain the number 0. This sublattice has no orthogonal subsets. As the argument in Theorem 5 could be carried out in a distributive lattice, it follows that $\operatorname{Spec}\mathcal{L}$ has no orthogonal decompositions. By contrast ([8], 7.3.2),

Theorem 7. *If an MV-algebra A contains a non-linearly ordered subset L, then A contains a maximal orthogonal subset.*

Proof. By assumption, there are x, $y \in A$ with $x \not\leq y$ and $y \not\leq x$. Hence $x\bar{y}$, $\bar{x}y$ are non-zero. By ([9], Thm. 3.3), $x\bar{y} \wedge \bar{x}y = 0$, so $\{x\bar{y}, \bar{x}y\}$ is an orthogonal set. An application of Zorn lemma guarantees that $\{x\bar{y}, \bar{x}y\}$ is contained in a maximal orthogonal set. □

Corollary 8. *If A is a non-linearly ordered MV-algebra, then* $\operatorname{Spec} A$ *admits an orthogonal decomposition.*

If in a given orthogonal decomposition $T = \cup_{i\in I} T_i \cup X$, we have $X = \emptyset$, then all the T_i become clopen. We can ask if this happens even if $X \neq \emptyset$, so the same MV-space T can have two different orthogonal decompositions, one with the $X = \emptyset$ and the other with the $X \neq \emptyset$. Some examples will clarify this.

In the Examples below $\mathbb{C}$ will denote the MV-algebra $\mathbb{C} = \{0, c, 2c, \ldots, 1 - 2c, 1 - c, 1\}$ [9] and $\mathbb{N} = \{1, 2, \ldots\}$.

Example 1. Let $A = \mathbb{C}^n$ with $n \in N$ fixed and we denote by $b(i)$ the i-th component of a generic $b \in A$. Let b_i be the element of A with $b_i(m) = 0$, $i \neq m$, $b_i(i) = 1$; let c_i be the element of A with $c_i(m) = 0$, $i \neq m$, $c_i(i) = c$. Then both $S_1 = \{b_i \ : \ i \leq n\}$ and $S_2 = \{c_i \ : \ i \leq n\}$ are maximal orthogonal subsets of A. With respect to S_1, $\operatorname{Spec} A = V(b_1) \cup \ldots \cup V(b_n)$ and each $V(b_i)$ is clopen. With respect to $S_2, \operatorname{Spec} A = V(c_1) \cup \ldots \cup V(c_n) \cup \mathbb{V}(S_2)$. As $c_i \notin B(A)$, we see that $V(c_i)$ is not clopen. Now $\mathbb{V}(S_2) = \operatorname{Max} A$ since $\operatorname{id}(S_2) = \operatorname{Rad} A$.

Example 2. Let $A = \mathbb{C}^{\mathbb{N}}$, b_i, c_i, defined analogously as above. Again $S_1 = \{b_i \ : \ i \in \mathbb{N}\}$ and $S_2 = \{c_i \ : \ i \in \mathbb{N}\}$ are maximal orthogonal sets. $\operatorname{id}(S_1)$ is a proper ideal, hence $\mathbb{V}(S_1) \neq \emptyset$, thus $\operatorname{Spec} A = (\cup_{i\in\mathbb{N}} V(b_i)) \cup \mathbb{V}(S_1)$, each $V(b_i)$ is clopen. Also $\operatorname{Spec} A = (\cup_{i\in\mathbb{N}} V(c_i)) \cup \mathbb{V}(S_2)$, no $V(c_i)$ is clopen. Now $\operatorname{Max} A \not\subseteq \mathbb{V}(S_1)$ while $\mathbb{V}(S_2) = \operatorname{Max} A$ since again $\operatorname{id}(S_2) = \operatorname{Rad} A$.

Example 3. Let $A = [0,1]^{\mathbb{N}}$, b_i as before. Here $V(b_i) = \{M_i\}$ where $M_i = \{x \in A \ : \ x(i) = 0\} \in \operatorname{Max} A$. $\mathbb{V}(S)$ contains all the other primes where $S = \{b_i \ : \ i \in \mathbb{N}\}$.

Example 4. Let $A = \mathcal{C}([0,1],[0,1])$, the MV-algebra of continuous functions [6]. We have $B(A) = \{0, 1\}$, hence in any orthogonal decomposition $\operatorname{Spec} A = \cup_{i\in I} T_i \cup X$, no T_i will ever be clopen.

Given the data of these examples, we may wish to impose further conditions on the orthogonal decompositions in order to be able to distinguish their properties.

Furthermore we would like the compact open component of an orthogonal decomposition to be "minimal" in some sense. The above examples provide us with two notions of "minimal".

Consider in Examples (1), (2) above the clopen sets $V(b_i)$. If $P_i = \{x : x(i) = 0\}$, $M_i = \{x : x(i) \in \mathrm{Rad}\,\mathbb{C}\}$, then $V(b_i) = \{P_i, M_i\}, V(c_i) = \{P_i\}$, and so $V(b_i) = V(c_i) \cup \{M_i\}$. $V(c_i)$ is compact open and $\mathrm{int}\{M_i\} = \emptyset$. On the other hand the only proper compact open subset of $V(c_i)$ is $\emptyset$. Thus we distinguish the two different orthogonal decompositions by these properties.

Let A be an MV-algebra and suppose $\mathrm{Spec}\,A = \cup_{i\in I} T_i \cup X$ is an orthogonal decomposition. We shall say the decomposition is unrefinable if no $T_i = \Theta \cup Y$, where Θ is open, $\Theta \cap Y = \emptyset$, $\mathrm{int}\,Y \neq \emptyset$. We shall call the decomposition atomic if no T_i contains a nonempty compact open proper subset, i.e. T_i is an atom in the lattice of compact open subsets of $\mathrm{Spec}\,A$.

Unrefinable means of course that we cannot replace each T_i by two proper disjoint compact open subsets of T_i. Atomic means that each T_i is an atom in the locale of open subsets of $\mathrm{Spec}\,A$. We shall deal with the unrefinable case first but we need to recall some definitions. For $x \in A$ and $n \in \mathbb{N}$, we define inductively $(n+1)x = nx + x$, $x^0 = 1$, $x^{n+1} = x^n \cdot x$. We say x has finite order n if n is the least integer such that $nx = 1$. If no such n exists, we say that x has infinite order, $\mathrm{ord}\,x = \infty$. If each $x \in A$ has finite order n, A is said locally finite ([9], Def. 3.10).

Theorem 9. *Suppose $a \in A$, $\mathrm{ord}\,a = \infty$, is such that $\{x \in A : x \leq a\}$ has orthogonal elements. Then $V(a) = \Theta \cup Y$, $\Theta \cap Y = \emptyset$, int $Y \neq \emptyset$ for some compact open $\Theta \subseteq V(a)$.*

Proof. By hypothesis there are a_1, $a_2 \leq a$, a_1, $a_2 \neq 0$, $a_1 \wedge a_2 = 0$. Let $\Theta = V(a_1)$, $Y = V(a) - \Theta$. Then $V(a) = \Theta \cup Y$, $\Theta \cap Y = \emptyset$ and so $V(a_2) \subseteq Y$, $V(a) \neq \emptyset$. □

Corollary 10. *Suppose* Spec A *has an unrefinable orthogonal decomposition. Then A contains a maximal orthogonal subset S such that $\{x \in A : x \leq a\}$ is linearly ordered for each $a \in S$.*

Proof. Let $\mathrm{Spec}\,A = \cup_{i\in I} V(a_i) \cup X$ be an unrefinable orthogonal decomposition and let $S = \{a_i : i \in I\}$. We know that S is a maximal orthogonal set. Let $a_i \in S$, $L_i = \{x \in A : x \leq a_i\}$. If L_i is not linearly ordered, then, as in Theorem 7, there are x, $y \in L_i$, x, $y \neq 0$ with $x \wedge y = 0$. But then $V(a_i) = V(x) \cup Y$ with $V(x) \cap Y = \emptyset$, int $Y \neq \emptyset$. This contradiction proves the Corollary. □

Referring back to Example (4), as it is clear that for a non-zero function $f \in \mathcal{C}([0,1],[0,1])$, the set $\{g : g \leq f\}$ is not linearly ordered, we see that $\mathrm{Spec}\,\mathcal{C}([0,1],[0,1])$ admits no unrefinable orthogonal decompositions.

Converse to the preceding Corollary, we ask, if S is a maximal orthogonal subset of A such that $\{x \in A : x \leq a\}$ is linearly ordered for each $a \in S$, does Spec A admits an unrefinable decomposition?

Given such an S we know that $\mathrm{Spec}\,A = \cup_{a\in S} V(a) \cup X$, int $X = \emptyset$. Suppose for some $a \in S$ we have $V(a) = \Theta \cup Y$, $\Theta \cap Y = \emptyset$, int $Y \neq \emptyset$, Θ compact open. Then $\Theta = V(b)$ for some $b \in A$; since int $Y \neq \emptyset$, there is a $c \in A$ with $\emptyset \neq V(c) \subseteq Y$. Hence $V(b) \cap V(c) = \emptyset$, so $b \wedge c = 0$.

Now $V(b)$, $V(c) \subseteq V(a) = V(\mathrm{id}(a))$. Thus there are m, n such that $b \leq ma$, $c \leq na$. We can choose m, n minimal, hence $b \not\leq (m-1)\cdot a$, $c \not\leq (n-1)\cdot a$. Thus $b\bar{a}^{m-1} \neq 0 \neq c\bar{a}^{n-1}$.

Now $b\bar{a}^{m-1} \leq ma \cdot \bar{a}^{m-1} \leq a$; similarly $c\bar{a}^{n-1} \leq a$. By assumption $\{x \in A : x \leq a\}$ is linearly ordered, so $b\bar{a}^{m-1} \wedge c\bar{a}^{n-1} \neq 0$, but $b\bar{a}^{m-1} \wedge c\bar{a}^{n-1} \leq b \wedge c = 0$. This contradiction establishes the converse to the Corollary.

Following ([8], 7.3) or ([13], Def.19.2), we call a maximal orthogonal subset $S \subseteq A$ a basis if for all $a \in S$, $\{x \in A : x \leq a\}$ is a linearly ordered set.

Summarizing we have

Theorem 11. Spec A *admits an unrefinable orthogonal decomposition iff A contains a basis.*

Remark 12. In Example (1) we see that $\{x \in A : x \leq b_i\}$ is linearly ordered for each b_i, so that $S = \{b_1, b_2, \ldots, b_n\}$ is a basis.

As each $V(b_i) = \Theta \cup Y$, Θ compact open, int $Y = \emptyset$ we see that Theorem 11 is "best" in some sense. On the other hand, we see in Example (1) that the $V(c_i)$ contain no proper nonempty compact open sets. The corresponding decomposition is atomic. Similarly in Example (2). The $V(c_i)$ of Example (2) are also atoms in the corresponding locale. It is clear in these examples, and from the definitions, that an atomic decomposition is unrefinable. This relation is reflected in the basis.

Then we define a strong basis to be a maximal orthogonal subset $S \subseteq A$ such that $\mathrm{id}(a)$ is a minimal ideal for each $a \in S$ (here minimal ideal always means minimal non-zero ideal).

Theorem 13. *Every minimal ideal is linearly ordered.*

Proof. Suppose I is a minimal ideal and that there are x, $y \in I$, $x \not\leq y$, $y \not\leq x$. Then $\{x\bar{y}, \bar{x}y\} \subseteq I$ is an orthogonal set. Let $J = (x\bar{y})^{\perp}$, $J \neq 0$ as $\bar{x}y \in J$. But $\bar{x}y \in I$, so $I \cap J \neq \{0\}$, hence $I \cap J = I$. Thus $I \subseteq J$ and so $x\bar{y} \in J$, which is impossible. □

Corollary 14. *If $S \subseteq A$ is a strong basis, then it is a basis.*

Proposition 15. *Let $a \in A$. Then $V(a)$ is an atom in the lattice of compact open subsets of* Spec A *iff* $\mathrm{id}(a)$ *is a minimal ideal.*

Proof. Suppose there's an ideal $J \neq 0$, $J \subseteq \mathrm{id}(a)$, $J \neq \mathrm{id}(a)$. Then there is a prime ideal P, $J \subseteq P$, $a \notin P$. Choose $b \in J$, $b \neq O$. As $b \in \mathrm{id}(a)$, we have $V(b) \subseteq V(a)$. $V(b) \neq \emptyset$, hence $V(a) = V(b)$ by hypothesis. Since $a \notin P$, we see that $b \notin P$, i.e. $J \not\subseteq P$ which is absurd.

Conversely, let $\emptyset \neq \Theta \subseteq V(a)$ with $\Theta = V(b)$ for some $b \in A$. Then, from $V(b) \subseteq V(a)$, it follows $\mathrm{id}(b) \subseteq \mathrm{id}(a)$, hence $\mathrm{id}(b) = \mathrm{id}(a)$ and this means $\Theta = V(b) = V(a)$. □

For an MV-algebra A, we define the socle of A, Soc A, to be the sum of the minimal ideals, or 0 if no minimal ideals exist. Thus Soc $A = \mathrm{id}\ (\cup\{I : I \text{ minimal ideal}\})$.

Theorem 16. *The following are equivalent:*

(i) $\mathrm{Spec}\,A$ *admits an atomic orthogonal decomposition,*

(ii) *A has a strong basis,*

(iii) $(\mathrm{Soc}\,A)^{\perp} = \{0\}$.

Proof. Assume (i). Let $\mathrm{Spec}\,A = \cup_{i \in I} V(a_i) \cup X$ be an atomic partition. Let $S = \{a_i \ : \ i \in I\}$. S is a maximal orthogonal set and, by Theorem 13 and Proposition 15, we see that S is a strong basis.
Assume (ii). If S is a strong basis for A, we have $\mathrm{id}(a)$ is a minimal ideal for each $a \in S$, hence evidently $S \subseteq \mathrm{Soc}\,A$. Thus we see $(\mathrm{Soc}\,A)^{\perp} \subseteq S^{\perp} = \{0\}$.
Assume (iii). Since $(SocA)^{\perp} = \{0\}$, we see that $\mathrm{Soc}\,A \neq \{0\}$, hence A contains minimal ideals. For each minimal ideal I, choose an $a_I \in I$, $a_I \neq 0$. Then $\mathrm{id}(a_I) = I$. Let $S = \{a_I \ : \ I \text{ is a minimal ideal}\}$.

Claim 17. S is a strong basis. Clearly S is an orthogonal set. Let $b \in S^{\perp}$ and $x \in \mathrm{Soc}\,A$. Then there are minimal ideals $I_1, I_2, \ldots, I_n$ such that $x \leq a_{I_1} + \ldots + a_{I_n}$ with $a_{I_i} \in I_i$ for each $i = 1, 2 \ldots, n$. Now $x \wedge b \leq (a_{I_1} \wedge b) + \ldots + (a_{I_n} \wedge b) = 0$ by Theorem 1. Thus $b \in (\mathrm{Soc}\,A)^{\perp}$, so $b = 0$ and then S is maximal orthogonal. Now the decomposition $\mathrm{Spec}\,A = \cup_{a_I \in S} V(a_I) \cup X$, with $X = \mathbb{V}(S)$, is atomic by Proposition 15.

□

Later we will examine the notion of basis and strong basis when A is semisimple, i.e. $\mathrm{Rad}\,A = \{0\}$.

Besides the notions of unrefinable and atomic decompositions, another somewhat natural condition we may wish to impose on an orthogonal partition $\mathrm{Spec}\,A = \cup_{i \in I} T_i \cup X$ is that X be an irreducible closed subset. Since $\mathrm{Spec}\,A$ is a spectral space, this would mean that X is the closure of a point, that is $X = \mathbb{V}(P_0) = \{P \in \mathrm{Spec}\,A \ : \ P_0 \subseteq P\}$. The corresponding maximal orthogonal set S must then satisfy $\mathrm{id}(S) = P_0$.

Consider, for example, the subalgebra $A \subseteq \mathbb{C}^{\mathbb{N}}$ generated by the ideal $I = \{x \in \mathbb{C}^{\mathbb{N}} \ : \ x(n) \in \mathrm{Rad}\,\mathbb{C}$ and $x(n) = 0$ on a cofinite subset of $\mathbb{N}\}$, i.e. $A = I \cup \bar{I}$, where $\bar{I} = \{x \in A \ : \ \bar{x} \in I\}$. Letting c_i, $i \in \mathbb{N}$, be as in Example (2), we see that each $c_i \in A$. We obtain the partition $\mathrm{Spec}\,A = (\cup_{i \in \mathbb{N}} V(c_i)) \cup \mathbb{V}(S)$, where $S = \{c_i \ : \ i \in \mathbb{N}\}$. It is easy to see that $I = \mathrm{Rad}\,A$, $\mathrm{id}(S) = I$ and $\mathbb{V}(S) = I$, I being the unique maximal ideal of A and A is a perfect MV-algebra ([7], p. 342). Call an orthogonal decomposition $\mathrm{Spec}\,A = \cup_{i \in I} T_i \cup X$ irreducible if X is an irreducible closed nonempty subset of $\mathrm{Spec}\,A$. We have that

Proposition 18. *For an MV-algebra A,* $\mathrm{Spec}\,A$ *admits an irreducible orthogonal decomposition iff there is a maximal orthogonal subset $S \subseteq A$ with* $\mathrm{id}(S) \in \mathrm{Spec}\,A$.

Proof. We saw that X irreducible implies that $\mathrm{id}(S)$ is a prime ideal where S is the corresponding maximal orthogonal set for the irreducible partition. Suppose now that S is a maximal orthogonal subset of A and $\mathrm{id}(S) \in \mathrm{Spec}\,A$. We then have the orthogonal decomposition $\mathrm{Spec}\,A = \cup_{a \in S} V(a) \cup \mathbb{V}(S)$. Since $\mathbb{V}(S) = \mathbb{V}(\mathrm{id}(S))$ and $\mathrm{id}(S)$ is prime, it follows that $\mathbb{V}(S)$ is an irreducible closed subset. □

We also have:

Proposition 19. *For a maximal orthogonal set $S \subseteq A$, $\mathrm{id}\, S \in \mathrm{Spec}\, A$ iff $\mathrm{id}\, S \cap S^{'} \neq \emptyset$ for all orthogonal sets $S^{'} \subseteq A$.*

Proof. If $\mathrm{id}(S)$ is prime, then clearly $\mathrm{id}(S) \cap S^{'} \neq \emptyset$ for any orthogonal set $S^{'}$. Conversely suppose that $\mathrm{id}\, S \cap S^{'} \neq \emptyset$ for any orthogonal set $S^{'}$. Let $x, y \in A$, then $x\bar{y} \wedge \bar{x}y = 0$. If $x \not\leq y$ and $y \not\leq x$, then $S^{'} = \{x\bar{y}, \bar{x}y\}$ is an orthogonal set, so $x\bar{y} \in \mathrm{id}(S)$ or $\bar{x}y \in \mathrm{id}(S)$. If $x \leq y$, then $0 = x\bar{y} \in \mathrm{id}(S)$. Similarly if $y \leq x$. Hence $\mathrm{id}(S)$ is prime. □

Suppose then that $\mathrm{Spec}\, A$ admits an irreducible decomposition $\mathrm{Spec}\, A = \cup_{i \in I} T_i \cup X$. Since $X = \mathbb{V}(P_0)$ for some $P_0 \in \mathrm{Spec}\, A$, we see that X is a chain under inclusion and contains exactly one maximal ideal. The remaining maximal ideals are spread amongst the T_i. Now $M \in T_i$ implies $T_M \subseteq T_i$. We can ask whether or under what conditions $T_i = \cup\{T_M \ : \ M \in T_i\}$. Recalling [7] that an hypernormal MV-algebra is an MV-algebra A such that $\mathrm{Spec}\, A$ is a disjoint union of chains of prime ideals under inclusion, we shall prove that

Proposition 20. *Let A be hypernormal. Suppose $\mathrm{Spec}\, A = \cup_{i \in I}\ T_i \cup X$ is an irreducible orthogonal decomposition. Then $T_i = \cup\{T_M \ : \ M \in T_i\}$ for all $i \in I$ except at most one. The exception T_{i_o}, if it exists, satisfies $T_{i_o} = \cup\{T_M \ : \ M \in T_{i_o}\} \cup (T_{M_o} - X)$, where M_o is the unique maximal ideal in X.*

Proof. Clearly for all i, $\cup\{T_M \ : \ M \in T_i\} \subseteq T_i$. Suppose for some i_o that $\cup\{T_M \ : \ M \in T_{i_o}\} \neq P \subseteq M$ for some $M \in \mathrm{Max}\, A$ and $M \notin T_{i_o}$. Were $M \in T_i$, $i \neq i_o$, then $P \in T_i$ as well. Thus $M \in X$ and $M = M_o$ the unique maximal ideal in X. Hence $P \in T_{M_o} - X$. Let $Q \in T_{M_o} - X$. Now $Q \subseteq P$ or $P \subseteq Q$ as A is hypernormal. If $Q \subseteq P$, then $Q \in T_{i_o}$. If $P \subseteq Q$ and $Q \in T_i$, then $P \in T_i$. Thus again we have $Q \in T_{i_o}$. Thus $T_{i_o} = \cup\{T_M \ : \ M \in T_{i_o}\} \cup (T_{M_o} - X)$. As M_o is unique, we see there can be at most one such T_{i_o}. □

It would be interesting to know under what conditions we would have $T_{i_o} = T_{M_o} - X$ and whether we ever have $X = T_{M_o}$. The latter means that $X = \mathbb{V}(m_o)$, where m_o is the unique minimal prime contained in M_o.

4 Annihilators ideals and orthogonal sets

By an annihilator ideal I is meant an ideal $I \subseteq A$ such that $I = H^{\perp}$ for some subset $H \subseteq A$. We note that H can be taken to be an ideal in A as $H^{\perp} = (\mathrm{id}(H))^{\perp}$.

Lemma 21. *Suppose $a \in A$ is such that if $x \leq a$, then $\{y \in A \ : \ y \leq x\}$ is not a linearly ordered set. Then A contains an infinite ascending sequence of annihilators ideals.*

Proof. It follows from the assumptions, as in Theorem 7, that there $x_1, y_1 \leq a$, $0 \neq x_1$, y_1 and $x_1 \wedge y_1 = 0$. Hence $y_1 \in x_1^{\perp}$. Suppose we have constructed annihilator ideals $x_1^{\perp} \subseteq x_2^{\perp} \subseteq \ldots \subseteq x_n^{\perp}$ with $x_n < x_{n-1} < \ldots < x_1 \leq a$ and $x_n^{\perp} - x_{n-1}^{\perp} \neq \emptyset$.

As $x_n \leq a$, there are $x_{n+1}, y_{n+1} \leq x_n, x_{n+1}, y_{n+1} \neq 0, x_{n+1} \wedge y_{n+1} = 0$. Hence $x_n^\perp \subseteq x_{n+1}^\perp$. Now $y_{n+1} \in x_{n+1}^\perp - x_n^\perp$ as $y_{n+1} \wedge x_n = y_{n+1}$. By induction we have an infinite ascending sequence of annihilator ideals. □

We shall say that A satisfies the *ascending chain condition* (acc) on annihilator ideals if every ascending sequence of annihilator ideals is finite, where we assume A not linearly ordered, that is the order of the underlying lattice is not linear.

Proposition 22. *Suppose A satisfies acc on annihilator ideals. Thus every orthogonal subset of A is finite.*

Proof. Suppose to the contrary that $S = \{a_1, a_2, \dots\}$ is a denumerable orthogonal subset of A. For $n = 1, 2, \dots$, let $S_n = S - \{a_1, \dots, a_n\}$. Then $S \supseteq S_1 \supseteq S_2 \supseteq \dots$. Hence $S_1^\perp \subseteq S_2^\perp \subseteq \dots$. By acc, there must be an $n \in \mathbb{N}$ with $S_n^\perp = S_{n+1}^\perp$. Now $a_{n+1} \in S_{n+1}^\perp$, hence $a_{n+1} \in S_n^\perp$. But $a_{n+1} \in S_n$ and this is impossible. □

Proposition 23. *Suppose A is such that all orthogonal sets are finite. Then A satisfies acc on annihilator ideals.*

Proof. Suppose we have an infinite ascending sequence $H_1^\perp \subseteq H_2^\perp \subseteq \dots$ of annihilator ideals. For each $n = 1, 2, \dots$, let $D_n = H_n^{\perp\perp} \cap H_{n+1}^\perp$. There is an $x \in H_{n+1}^\perp - H_n^\perp$, so there is a $y \in H_n$ such that $x \wedge y \neq 0$. Now $H_n \subseteq H_n^{\perp\perp}$, so $y \in H_n^{\perp\perp}$. Hence $x \wedge y \in H_n^{\perp\perp}$. As $x \in H_{n+1}^\perp$, so is $x \wedge y$. Then $0 \neq x \wedge y$ is in D_n. For each n, therefore choose $a_n \in D_n$, $a_n \neq 0$ and let $S = \{a_1, a_2, \dots\}$. Consider $a_i \wedge a_{i+j}$, $j \geq 1$. Since $a_i \in D_i = H_i^{\perp\perp} \cap H_{i+1}^\perp$ and $H_{i+1}^\perp \subseteq H_{i+j}^\perp$, then $a_i \in H_{i+j}^\perp$. But $a_{i+j} \in D_{i+j} \subseteq H_{i+j}^{\perp\perp}$. So $a_i \wedge a_{i+j} = 0$, therefore S is an infinite orthogonal sequence and the proposition is proved. □

The next theorem is called the "Finite Basis Theorem" and it is similar to the one for lattice-ordered groups (([8], 7.4.6), ([13], Thm. 46.12)) and relatively normal lattices [18].

Theorem 24. *The following are equivalent:*

(i) $\operatorname{Min} A$ *is finite with at least two members,*

(ii) A *satisfies acc on annihilator ideals,*

(iii) A *has a finite basis,*

(iv) $\operatorname{Spec} A$ *admits an unrefinable orthogonal finite decomposition.*

Proof. (iii) and (iv) are equivalent by Theorem 11, hence we show the equivalence of (i), (ii) and (iii).

Assume (i). Let $\operatorname{Min} A = \{m_1, m_2, \dots, m_n\}$ and since $\cap_{i=1}^n m_i = 0$, we can write $A \subseteq A/m_1 \times A/m_2 \times \dots \times A/m_n$ subdirectly. Each A/m_i is linearly ordered and A is not linearly ordered as $n \geq 2$. Clearly orthogonal sets in $A/m_1 \times \dots \times A/m_n$ are finite, thus the same must be true for A. Hence A satisfies acc on annihilator ideals by Proposition 23.

Assume (ii). Let $S = \{a_1, a_2, \dots, a_n\}$ be an orthogonal set. By Lemma 21, there are $b_i \le a_i$ such that $\{x \in A : x \le b_i\}$ is linearly ordered. Hence $\{b_1, b_2, \dots, b_n\}$ is an orthogonal set as well, suppose $\{b_1, b_2, \dots, b_n\}^\perp \neq \{0\}$ and choose $a \neq 0$, $a \in \{b_1, \dots, b_n\}^\perp$. Then $\{a, b_1, \dots, b_n\}$ is an orthogonal set. Applying Lemma 21 again, there is a $b_{n+1} \le a$ such that $\{x \in A : x \le b_{n+1}\}$ is linearly ordered. So $\{b_1, \dots, b_{n+1}\}$ is orthogonal, $\{b_1, \dots, b_n\} \subseteq \{b_1, \dots, b_n, b_{n+1}\}$. By Proposition 22, all orthogonal sets are finite, so this process must end with a maximal orthogonal set $\{b_1, \dots, b_m\}$ such that $\{x \in A : x \le b_i\}$ is linearly ordered for any $i = 1, \dots, m$. Hence we have a finite basis.
Assume (iii). Let $S = \{a_1, \dots, a_n\}$ be a finite basis. Since $\{x \in A : x \le a_i\}$ is a linearly ordered lattice ideal of A, we know $a_i^\perp$ is a prime ideal. Since S is a maximal orthogonal set, we also know $\{0\} = S^\perp = a_1^\perp \cap \dots \cap a_n^\perp$. Let $m \in \operatorname{Min} A$, then $a_1^\perp \cap \dots \cap a_n^\perp \subseteq m$ and so $a_i^\perp \subseteq m$ for some $i \in \{1, \dots, n\}$; thus we infer $\operatorname{Min} A = \{a_1^\perp, \dots, a_n^\perp\}$. As S is an orthogonal set, it has at least two members, hence so does $\operatorname{Min} A$. □

An analysis of the above results would show if A has a finite basis, then the cardinality of any orthogonal set would be less than or equal to 2^n where $n = \operatorname{card}(\operatorname{Min} A)$. We would also have an upper bound on the length of any ascending sequence of annihilator ideals. A, of course, must be semilocal with $\operatorname{card}(\operatorname{Max} A) \le \operatorname{card}(\operatorname{Min} A)$.

If $S = \{a_1, a_2, \dots, a_n\}$ is a basis, then $\operatorname{Min} A = \{a_1^\perp, \dots, a_n^\perp\}$. $a_i \notin a_i^\perp$ but $a_i \in a_j^\perp$ when $i \neq j$. Hence $a_i^\perp \in V(a_i)$. Suppose M_i is the unique maximal ideal over $a_i^\perp$. If $M_i \in V(a_j)$, then $a_i^\perp \in V(a_j)$ and so $i = j$. So if for $i \neq j$, we have $a_i^\perp$, $a_j^\perp \subseteq M \in \operatorname{Max} A$, then $S \subseteq M$. More can be said in the semisimple case and we will treat this later.

The cardinality of any basis must be equal to the cardinality of $\operatorname{Min} A$, thus all bases have the same cardinality. This fact is true even when the basis are not finite. That is

Proposition 25. *Suppose an MV-algebra A has bases S_1 and S_2.*
Then $\operatorname{card} S_1 = \operatorname{card} S_2$.

Proof. Index S_1, S_2 by $\triangle_1$, $\triangle_2$ so $S_1 = \{a_\alpha : \alpha \in \triangle_1\}$ and $S_2 = \{b_\gamma : \gamma \in \triangle_2\}$. For each $\gamma \in \triangle_2$, there is an $\alpha(\gamma) \in \triangle_1$ with $b\gamma \wedge a_{\alpha(\gamma)} \neq 0$ as $S_1^\perp = \{0\}$. Moreover if $b_\gamma \wedge a_{\alpha_1} \neq 0$, $b_\gamma \wedge a_{\alpha_2} \neq 0$, then $\alpha_1 = \alpha_2$. For as $\{x \in A : x \le b_\gamma\}$ is linearly ordered, we can assume $b_\gamma \wedge a_{\alpha_1} \le b_\gamma \wedge a_{\alpha_2}$. Thus $b_\gamma \wedge a_{\alpha_1} = b_\gamma \wedge a_{\alpha_1} \wedge a_{\alpha_2} = 0$ if $\alpha_1 \neq \alpha_2$. Thus we have an injection $\gamma \to \alpha(\gamma)$ of $\triangle_2$ into $\triangle_1$. By symmetry, there is an injection $\triangle_1$ into A_2, so by the Schroeder-Bernstein theorem, $\operatorname{card} \triangle_1 = \operatorname{card} \triangle_2$, hence $\operatorname{card} S_1 = \operatorname{card} S_2$. □

Observe, in the above, that if $b_\gamma \wedge a_\alpha \neq 0$, then b_γ and a_α must be comparable. For if $b_\gamma \not\le a_\alpha$ and $a_\alpha \not\le b_\gamma$. Then $0 \neq b_\gamma \bar{a}_\alpha$, $\bar{b}_\gamma a_\alpha$ by ([9], Thm. 1.13) but $b_\gamma \bar{a}_\alpha \wedge \bar{b}_\gamma a_\alpha = 0$ by ([9], Thm. 3.3). Let $x = a_\alpha \bar{b}_\gamma$, $y = b_\gamma \bar{a}_\alpha$. So $x \le a_\alpha$, $y \le b_\gamma$, $x \wedge y = 0$, $x, y \neq 0$. Hence $x \le b_\gamma \wedge a_\alpha$ or $b_\gamma \wedge a_\alpha \le x$. Similarly $y \le b_\gamma \wedge a_\alpha$ or $b_\gamma \wedge a_\alpha \le y$. If $b_\gamma \wedge a_\alpha \le x, y$, then $b_\gamma \wedge a_\alpha \le x \wedge y = 0$. If $y \le b_\gamma \wedge a_\alpha \le x$, then $y = 0$. Similarly if $x \le b_\gamma \wedge a_\alpha \le y$, then $x = 0$. Thus we must have $x, y \le b_\gamma \wedge a_\alpha$.

Thus $x \leq y$ or $y \leq x$ which is impossible as $x \wedge y = 0$. Hence $b_\gamma \leq a_\alpha$ or $a_\alpha \leq b_\gamma$. Thus we proved ([13], Cor.19.4):

Proposition 26. *Given two basis S_1, S_2 and $a \in S_1$, there exists a unique $b \in S_2$ with $a \leq b$ or $b \leq a$.*

A linearly ordered MV-algebra A is local, that is has a unique maximal ideal which is $\operatorname{Rad} A$. By ([5], Prop. 3.1), $B(A) = \{0, 1\}$. This allows to show

Proposition 27. *If* $\operatorname{Min} A$ *is finite, then $B(A)$ is a finite Boolean algebra.*

Proof. Let $\operatorname{Min} A = \{m_1, m_2, \ldots, m_n\}$. Clearly $A \subseteq A/m_1 \times \ldots \times A/m_n$ subdirectly. Each A/m_i is linearly ordered, so $B(A) \subseteq B(A/m_1 \times \ldots \times A/m_n) = B(A/m_1) \times \ldots \times B(A/m_n) = \{0, 1\}^n$. □

Let $At(B(A))$ be the set of all atoms of $B(A)$. Then the following result, already known in lattice theory (cfr., e.g., ([19], Cor. 5.13)), holds:

Proposition 28. *Let $a \in B(A)$. Then $V(a)$ is connected in* $\operatorname{Spec} A$ *iff $a \in At(B(A))$.*

Remark 29. If $a \in At(B(A))$, $V(a)$ is connected component because $V(a)$ is clopen connected.

Proposition 30. *Let A be semilocal. For any $M \in \operatorname{Max} A$, $T_M = V(a)$ for some $a \in At(B(A))$. Further, if C is a connected component of* $\operatorname{Spec} A$*, then $C = T_M$ for some $M \in \operatorname{Max} A$.*

Proof. By Theorem 3, any T_M is clopen and hence $T_M = V(a)$ for some $a \in B(A)$. But $a \in At(B(A))$ by Proposition 28 since T_M is connected by ([7], Cor. 4). Now let C be a connected component of $\operatorname{Spec} A$ and $P \in C$. Since $P \in T_M$ for some $M \in \operatorname{Max} A$, then $T_M \subseteq C$ but T_M is clopen and hence $C = T_M$ necessarily. □

The next theorem is well known for commutative rings with unit (cfr., e.g., ([15], Thm. 2.5)).

Theorem 31. $\operatorname{Min} A$ *is finite iff* $P \not\subseteq \cup\{Q : Q \in \operatorname{Min} A - \{P\}\}$ *for any* $P \in \operatorname{Min} A$.

Proof. Let $\operatorname{Min} A = \{P_1, P_2, \ldots, P_n\}$ and, without loss of generality, assume $P_1 \subseteq P_2 \cup \ldots \cup P_n$. Clearly $P_1 \not\subseteq P_i$ for any $i = 2, \ldots, n$, hence there exists a $x_i \in P_1 - P_i$ for any $i = 2, \ldots, n$, so that $x = x_2 + \ldots + x_n \in P_1$. But $x \in P_h$ for some $h \in \{2, \ldots, n\}$ and $x_h \leq x$, thus $x_h \in P_h$ which is a contradiction.

Conversely, assume $\operatorname{Min} A$ infinite and define the set $F = \{a \in A : \{P \in \operatorname{Min} A : a \notin P\}$ is cofinite$\}$. Evidently $1 \in F$ and let $a \in F$, $a \leq b$. Since $\{P \in \operatorname{Min} A : a \notin P\} \subseteq \{P \in \operatorname{Min} A : b \notin P\}$, we have that $b \in F$.

Suppose now a, $b \in F$ and clearly $\{P \in \operatorname{Min} A : a \wedge b \notin P\} = \{P \in \operatorname{Min} A : a \notin P\} \cap \{P \in \operatorname{Min} A : b \notin P\}$. This implies that $a \wedge b \in F$, hence F is a proper lattice filter of A. Certainly there exists (cfr., e.g., ([14], Thm. 7)) a prime ideal $Q \in \operatorname{Spec} A$ such that $Q \cap F = \emptyset$. Let $N \in \operatorname{Min} A$ be such that $N \subseteq Q$ and since $N \not\subseteq \cup\{P : P \in \operatorname{Min} A - \{N\}\}$, there exists $x \in N$ such that $x \notin P$ for any $P \in \operatorname{Min} A - \{Q\}$. Now $\{P \in \operatorname{Min} A : x \notin P\} = \operatorname{Min} A - \{Q\}$, thus $x \in F$, so $x \in F \cap Q = \emptyset$, a contradiction. □

We saw that in any MV-algebra A with $\mathrm{Min}\, A$ finite (and hence compact), any orthogonal subset of A is finite by Theorem 24. However, we have a more general result:

Theorem 32. *Let X be a compact subspace of $\mathrm{Spec}\, A$ such that $\cap\{P : P \in X\} = \{0\}$. If S is an orthogonal set of A such that $\mathrm{id}(S) \nsubseteq P$ for any $P \in X$, then S is finite.*

Proof. Let $X(a) = X \cap V(a)$ for any $a \in S$. If $P \in X$, there exists $a \in S$ such that $a \notin P$, which implies $P \in V(a) \cap X = X(a)$. So $X = \cup_{a \in S} X(a)$ and since X is compact, we have $X = X(a_1) \cup \ldots \cup X(a_n)$ for some $a_1, a_2, \ldots, a_n$. Let $a \in S$, $a \notin \{a_1, a_2, \ldots, a_n\}$. By hypothesis, there exists $P \in X$ such that $a \in P$, i.e. $P \in X \cap V(a) = X(a) \neq \emptyset$. Since $a \wedge a_i = 0$ for any $i = 1, 2, \ldots, n$, we have $X(a) = X(a) \cap X = \cup_{i=1}^{n} X(a) \cap X(a_i) = \emptyset$, a contradiction. Then $S = \{a_1, \ldots, a_n\}$, i.e. S is finite. □

Since $\mathrm{Max}\, A$ is a compact subspace of $\mathrm{Spec}\, A$, the following Corollary is immediate:

Corollary 33. *Let A be semisimple and S be an orthogonal set. If $S \nsubseteq M$ for any $M \in \mathrm{Max}\, A$, then S is finite.*

Corollary 34. *Let $\mathrm{Min}\, A$ be compact and S be an orthogonal set. If $S \nsubseteq P$ for any $P \in \mathrm{Min}\, A$, then S is finite.*

5 The semisimple case

We shall show that if A is semisimple and has a basis, then A has a basis of idempotent elements. A consequence of this will be that A has a basis iff A has a strong basis. In Examples (1) and (2), which are not semisimple MV-algebras, we see that each algebra has both a basis and a strong basis. This may suggest that in all algebras the existence of a basis implies the existence of a strong basis. The following example shows this is not the case.

Let $A = {}^*[0,1] \times {}^*[0,1]$, where ${}^*[0,1]$ is a proper non-standard model of the unit interval $[0,1]$ [5]. $S = \{e_1, e_2\}$, $e_1 = (1,0)$, $e_2(0,1)$ is a basis for A as $\{x \in A : x \leq e_i\}$ is linearly ordered for each $i = 1, 2$ and S is a maximal orthogonal set. But A contains no minimal ideals. For let I be an ideal of A and let $(x, y) \in I$, $(x, y) \neq (0, 0)$. We can assume $x \neq 0$, thus there's an infinitesimal τ, $\tau \leq x$. Hence $(\tau, 0) \in I$ and consider the arithmetical square τ^2 of τ. So $\tau^2 < \tau$, let $J = \mathrm{id}(\tau^2) \times \{0\}$, where $\mathrm{id}(\tau^2)$ is the ideal generated by τ^2 in ${}^*[0,1]$. Then J is an ideal of A and $J \subseteq I$. Claim $(\tau, 0) \notin J$. For $(\tau, 0) \in J$ implies $\tau \in \mathrm{id}(\tau^2)$, so for some natural number k, $\tau \leq k\tau^2$.

As ${}^*[0,1]$ lies in a field, we can divide by τ to obtain $1 \leq k\tau$ which is absurd. Thus $J \neq I$, hence I is not minimal. Hence A has no minimal ideals, thus $(\mathrm{Soc}\, A)^{\perp} = 0^{\perp} = A$. Hence A has no strong basis by Theorem 16.

Proposition 35. *An MV-algebra A is semisimple iff for all $x \in A$, $x = \bigvee_{n \in \mathbb{N}} (x \cdot nx)$.*

Proof. Assume $x = \bigvee_{n\in\mathbb{N}}(x \cdot nx)$ for all $x \in A$. Suppose for some $x, y \in A$ that $nx \le y$ for all $n \in \mathbb{N}$. Then $x \cdot nx \le x \cdot y$. Hence $\bigvee_{n\in\mathbb{N}}(x \cdot nx) \le x \cdot y$ and we have $x \le x \cdot y$; that is $x = x \cdot y$ and so A is semisimple by ([1], Thm. 31). Conversely, suppose A is semisimple. For $x \in A$, $n \in \mathbb{N}$, we clearly have $x \cdot nx \le x$. Assume $x \cdot nx \le y$ for all $n \in \mathbb{N}$. Let $M \in \operatorname{Max} A$, so $x/M \cdot nx/M \le y/M$ for all $n \in \mathbb{N}$, so $x/M \cdot n(x|M) \le y/M$, $n \in \mathbb{N}$. Since A/M is locally finite ([9], Thm. 4.6), there is an n such that $n(x/M) = 1$, if $x \notin M$. Thus $x/M \le y/M$, $x \notin M$. If $x \in M$, then $0 = x/M \le y/M$. Hence $x\bar{y} \in M$, but M is arbitrary in $\operatorname{Max} A$, thus $x\bar{y} = 0$ because A is semisimple. Then we get $x \le y$, ergo the thesis. □

Following [3], we say an ideal $I \subseteq A$ is α-closed if $\vee_j x_j \in I$ whenever the supremum exists and each $x_j \in I$, where j runs through an index set of cardinality less than or equal to α, α an infinite cardinal. We say I is closed if $\alpha = \operatorname{card} A$ and, as usually, we set $\aleph_0 = \operatorname{card} \mathbb{N}$.

The following three theorems are well known in lattice-ordered groups theory (cfr., e.g., ([8],11.1.9), ([13], Thm.53.7); ([8], 6.1.9), ([13], Prop.21.11); ([13], Prop.19.14); respectively):

Proposition 36. *Let A be semisimple and I be an $\aleph_0$-closed ideal. Then A/I is semisimple.*

Proof. Suppose for some $x, y \in A$ that we have $n(x/I) \le y/I$ for all $n \in \mathbb{N}$. Then $x/I \cdot n(x/I) \le x/I \cdot y/I = x \cdot y/I$. Hence $(x \cdot nx) \cdot (\overline{xy}) \in I$ for all $n \in \mathbb{N}$. Now $x = \vee_{n\in\mathbb{N}}\ (x \cdot nx)$ by Proposition 35, thus $x \cdot (\overline{xy}) = \vee_{n\in\mathbb{N}}\ ((x \cdot nx) \cdot \overline{xy})$ by ([2], Thm. 5). As I is $\aleph_0$-closed, we infer $x \cdot (\overline{xy}) \in I$. Then $x/I \le xy/I$ and so $x/I = x/I \cdot y/I$, hence A/I is semisimple. □

Proposition 37. *Annihilator ideals are always closed.*

Proof. Let $I = H^\perp$ for some subset $H \subseteq A$. Suppose $x_j \in I$, $j \in \triangle$ and $x = \vee_{j\in\triangle}\ x_j$ exists. Let $h \in H$, then $h \wedge x = \vee_{j\in\triangle}(h \wedge x_j) = 0$ by ([2], Thm. 5) as $x_j \in I$. Hence $x \in H^\perp = I$. □

Proposition 38. *Let $P \in \operatorname{Spec} A$ be such that $P^\perp \neq \{0\}$. Then $P \in \operatorname{Min} A$ and for some $0 \neq a \in A$, $P = a^\perp$ and $\{x \in A\ :\ x \le a\}$ is linearly ordered. Moreover, if $P \in \operatorname{Max} A$, then $na \in At(B(A))$ for some $n \in \mathbb{N}$.*

Proof. Let $0 \neq a \in P^\perp$. Then $P \subseteq P^{\perp\perp} \subseteq a^\perp$, now $a \notin P$, hence $a^\perp \subseteq P$, thus $P = a^\perp$. Let $x, y \le a$ and $x\bar{y} \neq 0$, $\bar{x}y \neq 0$. Since $x\bar{y} \wedge \bar{x}y = 0$ by ([9], Thm. 3.3), either $x\bar{y} \in P$ or $\bar{x}y \in P$. If $x\bar{y} \in P$, then $0 \neq x\bar{y} = x\bar{y} \wedge a = 0$, a contradiction. Similarly if $\bar{x}y \in P$. Hence $x\bar{y} = 0$ or $\bar{x}y = 0$, i.e. $x \le y$ or $y \le x$ by ([9], Thm. 1.13) and this means $\{x \in A\ :\ x \le a\}$ is a linearly ordered set. Now let $M \in \operatorname{Spec} A$ be such that $M \subseteq P = a^\perp$. Then $0 \neq a \in a^{\perp\perp} \subseteq M^\perp$ and as above, $M = a^\perp$, i.e. $M = P$ which implies $P \in \operatorname{Min} A$.

Now, if $P \in \operatorname{Max} A$, since $a \notin P$, $\overline{na} = \bar{a}^n \in P$ for some $n \in \mathbb{N}$ by ([9], Thm. 4.7). But $na \in P^\perp$, hence we have $na \wedge \overline{na} \in P \cap P^\perp = \{0\}$, i.e. $na \in B(A)$ by ([9], Thm. 1.16). Let $e = na$. By Theorem 1, $P = a^\perp = e^\perp = \operatorname{id}(\bar{e})$. Now let $b \in B(A)$ with $0 < b \le e$. Since $\bar{e} \le \bar{b}$, $P \subseteq \operatorname{id}(\bar{b})$ which implies $\operatorname{id}(\bar{e}) = P = \operatorname{id}(\bar{b})$ since $P \in \operatorname{Max} A$. This means $\bar{b} = \bar{e}$, i.e. $b = e$, hence $e \in At(B(A))$. □

Suppose now that A is semisimple with a basis S. Let $a \in S$. Since $\{x \in A : x \leq a\}$ is linearly ordered, we know that $(\mathrm{id}(a))^{\perp} = a^{\perp}$ is a prime ideal by ([1], Thm. 26). By Propositions 36 and 37, $A/a^{\perp}$ is semisimple. Since $A/a^{\perp}$ is also linearly ordered, it must be locally finite by ([1], Thm. 32), thus $a^{\perp} \in \mathrm{Max}\, A$ by ([9], Thm. 4.7). Since $a \in a^{\perp\perp}$, Prop. 38 tells us that $a^{\perp} = e^{\perp}$ for some $e \in At(B(A))$, where $e = na$ for some $n \in \mathbb{N}$. Similarly, for $0 \neq b \in S$, $b \neq a$, we have $b^{\perp} = d^{\perp}$ for some $d \in At(B(A))$, where $d = mb$ for some $m \in \mathbb{N}$. Of course, $e \neq d$ otherwise $a \in b^{\perp} = d^{\perp} = e^{\perp} = a^{\perp}$, a contradiction. Thus the set $S^{'} = \{e \in At(B(A)) : e = na \text{ for some } n \in \mathbb{N}, a \in S\}$ is clearly an orthogonal set. Since $S^{'\perp} \subseteq S^{\perp} = \{0\}$, we have that $S^{'}$ is a maximal orthogonal set, hence it is a basis. As $S^{'} \subseteq At(B(A))$, we must have $S^{'} = At(B(A))$ and therefore

Proposition 39. *If A is semisimple and has a basis, then $At(B(A))$ is also a basis.*

From this, we observe that the MV-algebra of Example (4) has no basis.

Remark 40. Following [2], if $e \in B(A)$, $A_e =< \mathrm{id}(e), +, \cdot, \sim, 0, e >$ can be made into an MV-algebra, where $\tilde{x} = x \wedge e = xe$ for all $x \in \mathrm{id}(e)$. Moreover, $A/e^{\perp}$ is isomorphic to A_e. Thus, if A is semisimple with a basis $S = At(B(A))$, we have for each $e \in S$, that $e^{\perp} = \mathrm{id}(\bar{e}) \in \mathrm{Max}\, A$ and $e^{\perp\perp} = \mathrm{id}(e)$ examining the previous discussion on Proposition 39. By ([9], Thm. 4.7), we get that A_e is a locally finite MV-algebra.

Proposition 41. *If A is semisimple and has a basis, then $B(A)$ is an atomic Boolean algebra.*

Proof. By Proposition 39, $At(B(A))$ is a basis of A, hence $At(B(A))$ is a maximal orthogonal set of A. Thus if $e' \in B(A)$, $e' \neq 0$, there is an $e \in At(B(A))$ with $e' \wedge e \neq 0$. Then $0 < e' \wedge e = (e' \wedge e) \wedge e$ and therefore $e' \wedge e = e$, i.e. $e \leq e'$ and so $B(A)$ is atomic. □

If A is complete, the converse of Proposition 41, holds, that is

Proposition 42. *Let A be complete, i.e. the underlying lattice is complete. Then A has a basis iff $B(A)$ is atomic.*

Proof. Since a complete MV-algebra is semisimple by ([2], Cor. 1), one direction is clear by Proposition 41. Conversely, let $B(A)$ be atomic. Then $\vee\{e : e \in At(B(A))\} = 1$, thus, if $x \in S^{\perp}$ where $S = At(B(A))$, we have $x = x \wedge 1 = x \wedge (\vee_{e \in S} e) = \vee_{e \in S}(x \wedge e) = 0$ by ([2], Thm. 5). This means that S is a maximal orthogonal set. By ([16], Lemma 3.3), $\mathrm{id}(e)$ is linearly ordered for each $e \in S$, thus S is a basis. □

Proposition 43. *Let A be semisimple with basis $S = At(B(A))$. Then A is subdirect subalgebra of the direct product $\Pi_{e \in S} A_e$.*

Proof. By Proposition 41, $B(A)$ is atomic and then $\vee_{e \in S} e = 1$. Thus the map $A \to \pi_{e \in S} A_e$, $x \to \{xe\}_{e \in S}$, is an homomorphism. If $xe = ye$ for all $e \in S$, then $x = x \wedge 1 = x \wedge (\vee_{e \in S})e = \vee_{e \in S} xe = \vee_{e \in S} ye = y$. Thus A is isomorphic to a subalgebra of $\Pi_{e \in S} A_e$ subdirectly since $A_e \subseteq A$ for each $e \in S$. □

Remark 44. If A is complete, the above homomorphism is an isomorphism (cfr. ([16], Thm. 3.1), ([11], Lemma 4.2.8)).

Recalling [2] that an MV-algebra is hyperarchimedean if for each $x \in A$, there is $n \in \mathbb{N}$ such that $nx \in B(A)$, we now prove that

Proposition 45. *A is semisimple with a finite basis iff A is hyperarchimedean and semilocal.*

Proof. If A has a finite basis, $At(B(A)) = \{e_1, e_2, \ldots, e_n\}$ is a basis by Proposition 41. By Proposition 43, $A = A_1 \times \ldots \times A_n$, where $A_i = \mathrm{id}(e_i)$ is locally-finite by Remark 40. Let $x = (x_1, x_2, \ldots, x_n) \in A$, then $k_i x_i = e_i$ for some $k_i \in \mathbb{N}$. If $k = \max\{k_1, \ldots, k_n\}$, then $kx = (e_1, \ldots, e_n) \in B(A)$ and A is hyperarchimedean. By Theorem 24, $\mathrm{Min}\, A$ is finite and A is semilocal.

Conversely, A is semisimple by ([2], Thm. 3) and $\mathrm{Max}\, A = \mathrm{Min}\, A$ by ([2], Thm. 4). Thus has a finite basis by Theorem 24. □

Again, suppose $S = At(B(A))$ is a basis for a semisimple MV-algebra A. Consider the corresponding orthogonal decomposition $\mathrm{Spec}\, A = \cup_{e \in S} V(e) \cup X$, where $X = \mathbb{V}(S)$. Recalling the proof of Proposition 39, for $e \in S$, we have $M = e^{\perp} \in \mathrm{Max}\, A$ and $e \notin M$. Hence $M \in V(e)$ and if $P \in V(e)$, then $e \notin P$, so $e^{\perp} = M \subseteq P$. Thus $V(e) = \{e^{\perp}\}$ and clearly $V(e)$ is an atom in the lattice of compact open subsets of $\mathrm{Spec}\, A$. By Proposition 15, $\mathrm{id}(e)$ is a minimal ideal and then S is a strong basis. It is clear that $\mathrm{Soc}\, A = \mathrm{id}(S)$ and $V(e)$ is a connected component of $\mathrm{Spec}\, A$ for each $e \in S$ by Proposition 28 and Remark 29.

Summarizing and recalling Theorems 11, 16 and Proposition 39, we have that

Theorem 46. *For a semisimple MV-algebra A, the following are equivalent:*

(i) *A has a basis,*

(ii) *A has a strong basis,*

(iii) $\mathrm{Spec}\, A$ *has an unrefinable orthogonal decomposition,*

(iv) $\mathrm{Spec}\, A$ *has an atomic orthogonal decomposition in connected components.*

Following [2], we recall that an MV-algebra is divisible if for each $a \in P$ and for each $n \in \mathbb{N}$, there exists a unique $0 \neq b \in A$ such that $nb = a$ and $\bar{b} = \bar{a} + (n-1)b$. Moreover, A is said $\aleph_0$-complete if its underlying lattice is $\aleph_0$-complete. We conclude this paper showing the following theorem already known in lattice-ordered groups (cfr., e.g., ([8], 14.4.5)):

Theorem 47. *Let A be an $\aleph_0$-complete and divisible MV-algebra. Then the following are equivalent:*

(i) *all prime ideals are $\aleph_0$-closed,*

(ii) *A is hyperarchimedean,*

(iii) *A has a finite basis.*

Proof. Assume (i). By ([3], Prop. 6), all prime ideals are maximal, so A is hyperarchimedean by ([2], Thm. 4).
Assume (ii). Let $\{a_h : h \in \mathbb{N}\}$ be an orthogonal sequence of elements of A. By divisibility of A, there are $x_h \in A$ such that $x_h \neq 0$, $hx_h = a_h$, and $\bar{a}_h + (h-1)x_h = \bar{x}_h$ for each $h \in \mathbb{N}$. Let $x = \bigvee_{h\in\mathbb{N}} x_h$ and since A is hyperarchimedean, then $nx = (n+1)x$ for some $n \in \mathbb{N}$. As $x_h \wedge x_j \leq a_h \wedge a_j$, we have $x_h \wedge x_j = 0$ for $h \neq j$. Thus, by Theorem 1, $x_h \wedge (n+1)x_{n+1} \leq (n+1)(x_h \wedge x_{n+1}) = 0$ for $h \neq n+1$. By ([2], Thm. 5), $x \wedge (n+1)x_{n+1} = \bigvee_{k\in\mathbb{N}}\{x_k \wedge (n+1)x_{n+1}\} = x_{n+1} \wedge (n+1)x_{n+1} = x_{n+1}$ as well. Then, using again Theorem 1, $nx_{n+1} \leq nx \wedge (n+1)x_{n+1} \leq n\{x \wedge (n+1)x_{n+1}\} = nx_{n+1}$. Therefore $nx_{n+1} = nx \wedge (n+1)x_{n+1} = (n+1)x \wedge (n+1)x_{n+1} = (n+1)x_{n+1}$, but $\bar{x}_{n+1} = \bar{a}_{n+1} + nx_{n+1} = \bar{a}_{n+1} + (n+1)x_{n+1} = \bar{a}_{n+1} + a_{n+1} = 1$, hence $x_{n+1} = 0$, a contradiction. Thus all orthogonal sets are finite and A has a finite basis by Theorem 24.
Assume (iii). By ([2], Cor. 1), A is semisimple. Thus $At(B(A)) = \{e_1, \dots, e_n\}$ is a finite basis by Proposition 39 and thus $\mathrm{Max}\, A = \{e_1^{\perp}, \dots, e_n^{\perp}\}$, i.e. all maximal ideals are closed by Proposition 37, hence $\aleph_0$-closed. A is quasi locally-finite by Proposition 45 and hence all prime ideals are $\aleph_0$-closed since $\mathrm{Spec}\, A = \mathrm{Max}\, A$. □

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