

# Using Fuzzy Similarity Relations to Revise and Update a Knowledge Base

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## Abstract

Similarity-based models were first used by Ruspini to give semantics to fuzzy logic ([7]). In these models, incomplete information is represented by an *evidential set*, i.e. a set of possible worlds that are compatible with the evidence, together with a *fuzzy similarity relation* on the set of possible worlds that allows to describe the resemblance of arbitrary subsets of worlds to those belonging to the evidential set. On the other hand, the question addressed by theory change formalisms is which kind of modifications have to be performed to a knowledge base when adding (retracting) a new proposition to (from) that knowledge base. Revision and update are the two theory change operations which have received more attention in the literature. In this paper we study a connection between similarity-based models and theory change, coming from the fact that revision and updating operators have been characterized by Katsuno and Mendelson in terms of pre-order relations in a set of possible worlds. Pre-order relations arise in a natural way in the similarity-based models.

**Keywords:** Fuzzy Similarity Relation, Implication and Consistency Measures, Updating and Revision Operators.

## 1 Introduction

The aim of this paper is to explore some relationships between similarity-based models and theory change operators. Similarity-based models were first used by Ruspini to give semantics to fuzzy logic ([7]). These models, which combine logic and metric concepts, consist of an *evidential set*, i.e., a set of possible worlds that

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are compatible with evidence, together with a fuzzy similarity relation on the set of possible worlds that allows to describe the resemblance of arbitrary subsets of worlds to those belonging to the evidencial set. On the other hand, theory change formalisms deal with mechanisms that allow to give amount of the tasks of adding (retracting) a proposition to (from) an existing knowledge base. The natural question addressed by these formalisms is what should the resulting theory be. In particular, one of the basic problems is when the new information to be added is inconsistent with the given knowledge base. Concerning this problem, a growing body of work takes as a departure point the postulates proposed by Alchourrón, Gärdenfors and Makinson ([1]) for the so-called revision operators and the Katsuno and Mendelson's postulates ([5]) for the so-called updating operators. Both kind of operators have been also characterized by Katsuno and Mendelson in terms of pre-order relations among the set of interpretations (worlds) of the logical language considered.

There is a straightforward way of relating theory change (revision and update) and similarity-based models. Namely, some measures of proximity definable in similarity models naturally induce pre-ordering relations between worlds that, in turn, are required by Katsuno and Mendelson ([4]and [5]) to characterize theory change operators that satisfy the postulates of revision and updating operations. In this paper, we investigate this relationship and study which are the properties of the theory change operators arising from such connection.

This paper is organized as follows. Section 2 and Section 3 contain some basic background knowledge about the notion of similarity-based models and about revision and updating operators including the set of corresponding postulates and representation theorems. Section 4 is devoted to present our approach to connect theory change and similarity-based models together. Finally, section 5 is devoted to related and future works.

## 2 Similarity-based Models

As already noticed, Ruspini's approach to possibilistic reasoning is based on the introduction of a fuzzy binary relation  $\mathbf{S}$ , called similarity relation, that maps pairs of possible worlds into numbers between 0 and 1 ( $\mathbf{S} : \mathcal{W} \times \mathcal{W} \mapsto [0, 1]$ ). This function captures a notion of proximity between possible worlds, with a value of 1 corresponding to the identity of possible worlds and a value 0 indicating that knowledge of propositions that are true in one possible world does not provide any indication about propositions that are true in the other. Some requirements must be satisfied by similarity functions. For example, the similarity degree of any world with itself has the highest value (i.e. 1), it is symmetric and satisfies a relaxed form of transitivity. This is expressed as:  $S(w, w'') \geq S(w, w') \otimes S(w', w'')$ . In general,  $\otimes$  is considered to be a triangular norm (or T-norm). Its quasi-inverse is represented by  $\otimes \longrightarrow$  defined as:

$$a \otimes \longrightarrow b = \sup\{c \in [0, 1] \mid a \otimes c \leq b\}.$$

We say that a similarity function is separating if, and only if, it holds:

$$S(w_1, w_2) = 1 \text{ if, and only if, } w_1 = w_2.$$

From now on, we shall consider a propositional logical language  $L$  and  $I$  will stand for the set of its interpretations or set of worlds. Having defined a similarity function  $S$  on  $I$  and an evidence set  $E \subseteq I$ , Ruspini introduces a pair of unconditioned implication and consistency measures on  $L$  defined respectively as:

$$\begin{aligned} I_{S,E}(p) &= \inf_{w_1 \in E} \sup_{w_2 \models p} S(w_1, w_2) \\ C_{S,E}(p) &= \sup_{w_1 \in E} \sup_{w_2 \models p} S(w_1, w_2). \end{aligned}$$

These measures, as Ruspini points out, are lower and upper bounds of the resemblance degrees between  $p$ -worlds and worlds in  $E$ , from the point of view of  $E$ . The value of  $I_{S,E}(p)$  provides the measure of what extent  $p$  can be considered true given the incomplete knowledge represented by  $S$  and  $E$ . In particular, if the set of worlds is finite then,  $I_{S,E}(p) = 1$  if, and only if,  $E \models p$ . On the other hand, the value of  $C_{S,E}(p)$  provides the measure of what extent  $p$  can be considered compatible with the available knowledge. In particular, in the finite case,  $C_{S,E}(p) = 1$  if, and only if,  $E \not\models \neg p$ . Observe that, when the evidential set is a singleton  $w$  both measures coincide, i.e.,  $I_{S,w}(p) = C_{S,w}(p)$  for any world  $w$  and proposition  $p$ .

Since  $C_{S,E}$  is a possibility measure, it holds that  $C_{S,E}(p \vee q) = \max\{C_{S,E}(p), C_{S,E}(q)\}$  for any propositions  $p$  and  $q$ . Therefore, we also have  $C_{S,E}(p) = \max\{C_{S,E}(p \wedge q), C_{S,E}(p \wedge \neg q)\}$ . In particular, when  $C_{S,E}(p \wedge q) > C_{S,E}(p \wedge \neg q)$ , it results that  $C_{S,E}(p) = C_{S,E}(p \wedge q)$ . This can be interpreted as: the  $(p \wedge q)$ -worlds are closer (consistent) to some world of the evidential set  $E$  than the  $(p \wedge \neg q)$ -worlds are. In this context, the term ‘‘closer’’ is used in the sense of ‘‘more similar’’.

Furthermore, Ruspini generalizes the semantical entailment relationship between propositions in terms of a measure of their neighborhood to the evidential set  $E$ , by defining the so-called degree of conditioned implication and consistency measures as follows:

$$\begin{aligned} I_{S,E}(q | p) &= \inf_{w \in E} (C_{S,w}(p) \otimes \rightarrow C_{S,w}(q)) \\ C_{S,E}(q | p) &= \sup_{w \in E} (C_{S,w}(p) \otimes \rightarrow C_{S,w}(q)). \end{aligned}$$

In particular, when  $E \models p$  then we recover the previous unconditioned measures. Moreover, if  $S$  reduces to the classical equality relation, i. e.  $S(w, w') = 1$  if  $w = w'$ , 0 otherwise, we have that  $I_{S,E}(q | p) = 1$  whenever  $E \models p \rightarrow q$  and  $C_{S,E}(q | p) = 0$  whenever  $E \models \neg(p \rightarrow q)$ .

However, these conditioned measures may have some difficulties in a general setting since, for instance, it may happen that  $I_{S,E}(q | p) > 0$  or  $C_{S,E}(q | p) = 1$  being  $p$  and  $q$  mutually exclusive. In [3] a modified version of these conditional measures are proposed by defining:

$$\begin{aligned} I_{S,E}^2(q | p) &= \inf_{w \in E} (C_{S,w}(p) \otimes \rightarrow C_{S,w}(q \wedge p)) \\ C_{S,E}^2(q | p) &= \sup_{w \in E} (C_{S,w}(p) \otimes \rightarrow C_{S,w}(q \wedge p)). \end{aligned}$$

Observe that when  $p$  and  $q$  are mutually exclusive,  $I_{S,E}^2(q | p) = 0$  and  $C_{S,E}^2(q | p) = 0$ . Moreover,  $C_{S,E}^2(q | p) = 1$  if there exists one world  $w \in E$  such that  $C_{S,w}(p) = C_{S,w}(p \wedge q)$  or equivalently  $C_{S,w}(p \wedge q) \geq C_{S,w}(p \wedge \neg q)$ . That is, the  $(p \wedge q)$ -worlds are almost as closer (consistent) to some world of the evidential set  $E$  as

the  $(p \wedge \neg q)$ -worlds are. Reciprocally, if  $C_{S,E}^2(q | p) < 1$  then, for any world  $w \in \mathbf{E}$   $C_{S,w}(p) > C_{S,w}(p \wedge \neg q)$  or equivalently  $C_{S,w}(p) = C_{S,w}(p \wedge q) > C_{S,w}(p \wedge \neg q)$ . In particular, if the set of worlds is finite then the equivalence holds, i.e.,  $C_{S,E}^2(q | p) = 1$  if, and only if, there exists one world  $w \in \mathbf{E}$  such that  $C_{S,w}(p) = C_{S,w}(p \wedge q)$ .

### 3 Theory Change

Given a knowledge base  $KB$  (represented by a set of sentences in a language  $L$ ) and a sentence  $\phi$ ,  $KB_\phi^*$  denotes the revision of  $KB$  by  $\phi$ , that is, the new knowledge base resulting from the addition of  $\phi$  to the old knowledge base  $KB$ .

In [1], Alchourrón, Gärdenfors and Makinson proposed eight postulates which must be satisfied by any reasonable revision operator. These postulates are<sup>1</sup>:

- ★1.  $KB_\phi^* = Cn(KB_\phi^*)$ .
- ★2.  $\phi \in KB_\phi^*$ .
- ★3.  $KB_\phi^* \subseteq Cn(KB \cup \{\phi\})$ .
- ★4. if  $\neg\phi \notin KB$  then  $Cn(KB \cup \{\phi\}) \subseteq KB_\phi^*$ .
- ★5. if  $Cn(\phi) \neq L$  then  $KB_\phi^* \neq L$ .
- ★6. if  $\phi \equiv \nu$  then  $KB_\phi^* \equiv KB_\nu^*$ .
- ★7.  $KB_{\phi \wedge \nu}^* \subseteq Cn(KB_\phi^* \cup \{\nu\})$ .
- ★8. if  $\neg\phi \notin KB_\nu^*$  then  $Cn(KB_\phi^* \cup \{\nu\}) \subseteq KB_{\phi \wedge \nu}^*$ .

In [4] Katsuno and Mendelzon gave a model theoretic characterization of these postulates by means of pre-orders on interpretations when the language  $L$  is finitary. A pre-order  $\leq$  on  $\mathbf{I}$  is a reflexive and transitive relation. A pre-order is total if for all  $I$  and  $J$  interpretations, either  $I \leq J$  or  $J \leq I$ . They consider a function that assigns to each knowledge base  $KB$  a pre-order  $\leq_{KB}$  over  $\mathbf{I}$ . This assignment is *faithful*<sup>2</sup> if the following three conditions hold:

1. if  $I, I' \in \text{Mod}(KB)$ <sup>3</sup> then  $I <_{KB} I'$  does not hold,
2. if  $I \in \text{Mod}(KB)$  and  $I' \notin \text{Mod}(KB)$  then  $I <_{KB} I'$  holds
3. if  $KB \equiv \phi$  then  $\leq_{KB} \equiv \leq_\phi$ .

Let  $M$  be a subset of  $\mathbf{I}$ . An interpretation  $I$  is minimal in  $M$  with respect to  $\leq_{KB}$  if  $I \in M$  and there is no  $I' \in M$  such that  $I' <_{KB} I$ . Let  $\text{Min}(M, \leq_{KB})$  be the set of all  $I \in M$  such that  $I$  is minimal in  $M$  with respect to  $\leq_{KB}$ . Katsuno and Mendelzon established the following characterization of the revision operators.

<sup>1</sup> $Cn$  is any consequence operator which includes classical propositional logic, is compact and satisfies the deduction theorem.

<sup>2</sup>The term *persistent* was used in [4], but it was replaced by *faithful* in [5]

<sup>3</sup> $\text{Mod}(KB)$  stands for the set of models of the knowledge base  $KB$ .

**Theorem 1:** A revision operator  $\star$  satisfies the conditions  $(\star 1) \sim (\star 8)$  if, and only if, there exists a *faithful* assignment that maps each knowledge base  $KB$  to a total pre-order  $\leq_{KB}$  such that  $\text{Mod}(KB_\phi^\star) = \text{Min}(\text{Mod}(\phi), \leq_{KB})$ .

Moreover, Alchourrón, Gärdenfors and Makinson proposed a new kind of modifications for a knowledge base, the so-called *update operators*. These operators consist of bringing the knowledge base up to date when the world described by it changes. They showed that postulates  $(\star 1) \sim (\star 8)$  only apply to revision not to update. They provided a new set of postulates that apply to update and characterized all operators satisfying these postulates in terms of a set of partial orders defined on possible worlds. They use  $KB_\phi^\diamond$  to denote the result of updating a knowledge base  $KB$  with the sentence  $\phi$ . The postulates for an update operator are:

- ◇ 1.  $KB_\phi^\diamond \models \phi$ .
- ◇ 2. If  $KB \models \phi$  then  $KB_\phi^\diamond \equiv KB$ .
- ◇ 3. If both  $KB$  and  $\phi$  are satisfiable then,  $KB_\phi^\diamond$  is satisfiable.
- ◇ 4. If  $KB_1 \equiv KB_2$  and  $\phi_1 \equiv \phi_2$  then,  $(KB_1)_{\phi_1}^\diamond \equiv (KB_2)_{\phi_2}^\diamond$ .
- ◇ 5.  $KB_\phi^\diamond \cup \{\nu\} \models KB_{\phi \wedge \nu}^\diamond$ .
- ◇ 6. If  $KB_{\phi_1}^\diamond \models \phi_2$  and  $KB_{\phi_2}^\diamond \models \phi_1$  then,  $KB_{\phi_1}^\diamond \equiv KB_{\phi_2}^\diamond$ .
- ◇ 7. If  $KB$  is complete then,  $KB_{\phi_1}^\diamond \cup KB_{\phi_2}^\diamond \models KB_{\phi_1 \vee \phi_2}^\diamond$ .
- ◇ 8.  $(KB_1 \cap KB_2)_\phi^\diamond \equiv (KB_1)_\phi^\diamond \cap (KB_2)_\phi^\diamond$ .

The following representation theorem is established for update operators by Katsuno and Mendelzon in [5] for finite knowledge bases.

**Theorem 2:** An update operator  $\diamond$  satisfies conditions  $(\diamond 1) \sim (\diamond 8)$  if, and only if, there exists a *faithful* assignment that maps each interpretation  $I$  to a partial pre-order  $\leq_I$  such that

$$\text{Mod}(KB_\phi^\diamond) = \bigcup_{I \in \text{Mod}(KB)} \text{Min}(\text{Mod}(\phi), \leq_I).$$

There are two basic differences between revision and update from a model theoretic point of view. One is that revision is characterized by total pre-orders while update is defined from a family of partial pre-orders. However, it is also possible to build a class of update operators based on a family of total pre-orders, namely, when the postulates  $(\diamond 6)$  and  $(\diamond 7)$  are replaced by the new postulate:

- ◇ 9. If  $KB$  is complete and  $KB_\phi^\diamond \cup \{\nu\}$  is satisfiable then,  $KB_{\phi \wedge \nu}^\diamond \models KB_\phi^\diamond \cup \{\nu\}$ .

The second difference is that, in the case of update more than one order can be induced by each model of  $KB$ , while for revision only one order can be induced. This “local” behaviour of update contrasts with the “global” behaviour of revision.

## 4 Similarities and Revision/Update Operators

Now, we shall show a relationship between the similarity models and the revision/update operations to be performed on a knowledge base when new information arrives. First, we shall propose a definition for revision and update from a similarity relation, and then we will prove that they verify some of the corresponding eight postulates. In the following, for the sake of brevity we will also use the notation  $\mathcal{KB}$  instead of  $\text{Mod}(KB)$ .

**Definition 1:**  $\psi \in KB_\phi^*$  if either  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  or  $C_{S,\mathcal{KB}}(\neg\psi \wedge \phi) < C_{S,\mathcal{KB}}(\psi \wedge \phi)$ .

**Definition 2:**  $\psi \in KB_\phi^\diamond$  if either  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  or  $C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi) < 1$ .

In the first definition, a formula  $\psi$  belongs to the revision of  $KB$  by  $\phi$  if, and only if,  $\psi$  is either a logical consequence of  $\phi$  or the  $(\phi \wedge \psi)$ -worlds are closer (more consistent) to the set of  $KB$ -worlds than the  $(\phi \wedge \neg\psi)$ -worlds are. In the second definition, a formula  $\psi$  belongs to the update of  $KB$  by  $\phi$  if, and only if,  $\psi$  is either a logical consequence of  $\phi$  or for any  $KB$ -world  $w$  it is closer (more consistent) to the  $(\phi \wedge \psi)$ -worlds, than to the  $(\phi \wedge \neg\psi)$ -worlds. This interpretation is explored in [6] to relate counterfactuals and theory change via similarity functions.

The theory change operators above defined verify most of the postulates for revision and update operators, as it is proved by the following two theorems within a non necessary finitary language  $L$ .

First of all we give a previous lemma that gives a partial description of models of  $KB_\phi^*$ . In the finite case this models will be fully characterized.

**Lemma 1:** The following conditions hold:

1. If  $w$  is a  $\phi$ -world such that  $C_{S,\mathcal{KB}}(w) = 1$  then  $w \in \text{Mod}(KB_\phi^*)$ .
2. If  $w$  is a  $\phi$ -world such that  $C_{S,\mathcal{KB}}^2(w \mid \phi) = 1$  then  $w \in \text{Mod}(KB_\phi^\diamond)$ .

**Proof:**

1. Suppose,  $w \notin \text{Mod}(KB_\phi^*)$  or equivalently there exists  $\psi \in KB_\phi^*$  such that  $w \models \neg\psi$  and  $C_{S,\mathcal{KB}}(w) = 1 = C_{S,\mathcal{KB}}(\phi \wedge \neg\psi)$ . Since  $\psi \in KB_\phi^*$ , then  $1 = C_{S,\mathcal{KB}}(\phi \wedge \neg\psi) < C_{S,\mathcal{KB}}(\phi \wedge \psi)$  in contradiction with the definition of  $C_{S,\mathcal{KB}}$ .
2. Suppose  $w \notin \text{Mod}(KB_\phi^\diamond)$ . Then, there exists  $\psi \in KB_\phi^\diamond$  and  $w \models \neg\psi$ . Therefore,  $1 = C_{S,\mathcal{KB}}^2(w \mid \phi) \leq C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi) < C_{S,\mathcal{KB}}^2(\psi \mid \phi)$  in contradiction with the definition of  $C_{S,\mathcal{KB}}^2$ . ■

**Theorem 3:** Let  $Cn$  be the consequence operator defined by the classical propositional logic. The operator  $\star$  defined in Definition 1 verifies the postulates  $(\star 1) \sim (\star 3)$  and  $(\star 5) \sim (\star 8)$ , but not  $(\star 4)$ .

**Proof:**

★1.  $KB_\phi^* = Cn(KB_\phi^*)$ .

It is sufficient to prove that  $Cn(KB_\phi^*) \subseteq KB_\phi^*$ . Suppose  $\psi \in Cn(KB_\phi^*)$  then, by compactness, there exist  $\phi_1, \dots, \phi_n \in KB_\phi^*$  such that  $\phi_1 \wedge \dots \wedge \phi_n \models \psi$  and  $\phi_1 \wedge \dots \wedge \phi_n \in KB_\phi^*$ .

If  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  then  $\psi \in KB_\phi^*$ .

If  $\text{Mod}(\phi \wedge \neg\psi) \neq \emptyset$  then, since  $\text{Mod}(\phi_1 \wedge \dots \wedge \phi_n) \subseteq \text{Mod}(\psi)$  and  $\text{Mod}(\neg(\phi_1 \wedge \dots \wedge \phi_n)) \supseteq \text{Mod}(\neg\psi)$ ,  $C_{S, \mathcal{KB}}(\phi \wedge \psi) \geq C_{S, \mathcal{KB}}(\phi \wedge (\phi_1 \wedge \dots \wedge \phi_n)) > C_{S, \mathcal{KB}}(\phi \wedge \neg(\phi_1 \wedge \dots \wedge \phi_n)) \geq C_{S, \mathcal{KB}}(\phi \wedge \neg\psi)$ . Therefore,  $\psi \in KB_\phi^*$ . Thus the postulate is proved.

★2.  $\phi \in KB_\phi^*$ . Immediate from the definition.

★3.  $KB_\phi^* \subseteq Cn(KB \cup \{\phi\})$ .

It is sufficient to prove that  $\text{Mod}(KB_\phi^*) \supseteq \text{Mod}(KB \cup \{\phi\})$  and this is immediate from lemma 1 taking into account that if  $w \in \text{Mod}(KB \cup \{\phi\})$  then,  $C_{S, \mathcal{KB}}(w) = 1$  and  $w$  is a  $\phi$ -world.

★5. If  $Cn(\phi) \neq L$  then  $KB_\phi^* \neq L$ .

Suppose  $KB_\phi^* = L$ . We can assume that there is  $\chi \in L$  such that  $\phi \models \chi$  and thus  $\text{Mod}(\phi \wedge \chi) \neq \emptyset$ . Now, since  $\neg\chi \in KB_\phi^*$  then,  $C_{S, \mathcal{KB}}(\neg\chi \wedge \phi) > C_{S, \mathcal{KB}}(\chi \wedge \phi) = C_{S, \mathcal{KB}}(\phi)$  in contradiction to the definition of  $C_{S, \mathcal{KB}}$ . Thus,  $KB_\phi^* \neq L$ .

★6. If  $\phi \equiv \nu$  then  $KB_\phi^* \equiv KB_\nu^*$ .

If  $\phi \equiv \nu$  then for all  $\chi \in L$ ,  $\text{Mod}(\phi \wedge \chi) = \emptyset$  if and only if  $\text{Mod}(\nu \wedge \chi) = \emptyset$  and  $C_{S, \mathcal{KB}}(\neg\chi \wedge \phi) < C_{S, \mathcal{KB}}(\chi \wedge \phi)$  if, and only if,  $C_{S, \mathcal{KB}}(\neg\chi \wedge \nu) < C_{S, \mathcal{KB}}(\chi \wedge \nu)$ . Thus this postulate is proved.

★7.  $KB_{\phi \wedge \nu}^* \subseteq Cn((KB_\phi^*) \cup \{\nu\})$ .

It is sufficient to prove that  $\text{Mod}(KB_\phi^* \cup \{\nu\}) \subseteq \text{Mod}(KB_{\phi \wedge \nu}^*)$ . Suppose  $w \in \text{Mod}(KB_\phi^* \cup \{\nu\})$ . For any  $\psi \in KB_{\phi \wedge \nu}^*$ , we consider two cases:

- $\text{Mod}(\neg\psi \wedge \phi \wedge \nu) = \emptyset$ . Since  $w \models \nu \wedge \phi$  then,  $w \models \psi$ .
- $C_{S, \mathcal{KB}}(\neg\psi \wedge \phi \wedge \nu) < C_{S, \mathcal{KB}}(\psi \wedge \phi \wedge \nu)$ . Notice that  $C_{S, \mathcal{KB}}(\psi \wedge \phi \wedge \nu) \leq C_{S, \mathcal{KB}}((\psi \vee \neg\nu) \wedge \phi)$  by definition, since  $\text{Mod}(\psi \wedge \phi) \subseteq \text{Mod}((\psi \vee \neg\nu) \wedge \phi)$ . Therefore  $C_{S, \mathcal{KB}}(\neg\psi \wedge \phi \wedge \nu) < C_{S, \mathcal{KB}}((\psi \vee \neg\nu) \wedge \phi)$  and  $(\psi \vee \neg\nu) \in KB_\phi^*$  taking into account that  $w \in \text{Mod}(KB_\phi^*)$ , we have  $w \models (\psi \vee \neg\nu)$  but by hypothesis  $w \models \nu$  thus,  $w \models \psi$ .

We have proved that, for any  $\psi \in KB_{\phi \wedge \nu}^*$   $w \models \psi$ . Thus  $w \in \text{Mod}(KB_{\phi \wedge \nu}^*)$  and the postulate is proved.

★8. If  $\neg\nu \notin KB_\phi^*$  then  $Cn(KB_\phi^* \cup \{\nu\}) \subseteq KB_{\phi \wedge \nu}^*$ .

Let  $\psi$  be a formula in the language  $L$ , suppose that  $\psi \in Cn(KB_\phi^* \cup \{\nu\}) = Cn(KB_\phi^* \cup Cn(\{\nu\}))$ . Consider the following cases:

- If  $\psi \in Cn(\{\nu\})$  then,  $\text{Mod}(\phi \wedge \nu \wedge \neg\psi) = \emptyset$ . Thus,  $\psi \in KB_{\phi \wedge \nu}^*$ .

- If  $\psi \in KB_\phi^*$  then,  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  or  $C_{S,KB}(\phi \wedge \neg\psi) < C_{S,KB}(\phi \wedge \psi)$ .  
 If  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  then,  $\text{Mod}(\phi \wedge \nu \wedge \neg\psi) = \emptyset$  and hence  $\psi \in KB_{\phi \wedge \nu}^*$ .  
 If  $C_{S,KB}(\phi \wedge \neg\psi) < C_{S,KB}(\phi \wedge \psi)$  then assume that,  $C_{S,KB}(\phi \wedge \psi \wedge \nu) \geq C_{S,KB}(\phi \wedge \psi \wedge \neg\nu)$ . Therefore, taking into account that  $C_{S,KB}(\phi \wedge \psi) = \max\{C_{S,KB}(\phi \wedge \psi \wedge \nu), C_{S,KB}(\phi \wedge \psi \wedge \neg\nu)\}$  we have,  $C_{S,KB}(\phi \wedge \psi \wedge \nu) = C_{S,KB}(\phi \wedge \psi)$ . Since  $\psi \in KB_\phi^*$ , we have  $C_{S,KB}(\phi \wedge \psi) > C_{S,KB}(\phi \wedge \neg\psi) \geq C_{S,KB}(\phi \wedge \neg\psi \wedge \nu)$ . Thus,  $\psi$  is in  $KB_{\phi \wedge \nu}^*$ .  
 In order to prove  $C_{S,KB}(\phi \wedge \psi \wedge \nu) \geq C_{S,KB}(\phi \wedge \psi \wedge \neg\nu)$ , suppose  $C_{S,KB}(\phi \wedge \psi \wedge \nu) < C_{S,KB}(\phi \wedge \psi \wedge \neg\nu)$ . Therefore,  $C_{S,KB}(\phi \wedge \psi \wedge \neg\nu) = \max\{C_{S,KB}(\phi \wedge \psi \wedge \nu), C_{S,KB}(\phi \wedge \psi \wedge \neg\nu)\} = C_{S,KB}(\phi \wedge \psi) >$  (by hypothesis)  $C_{S,KB}(\phi \wedge \neg\psi) = \max\{C_{S,KB}(\phi \wedge \neg\psi \wedge \nu), C_{S,KB}(\phi \wedge \neg\psi \wedge \neg\nu)\} > C_{S,KB}(\phi \wedge \neg\psi \wedge \nu)$ . Finally, we have  $C_{S,KB}(\phi \wedge \neg\nu) \geq C_{S,KB}(\phi \wedge \psi \wedge \neg\nu) > \max\{C_{S,KB}(\phi \wedge \psi \wedge \nu), C_{S,KB}(\phi \wedge \neg\psi \wedge \nu)\} = C_{S,KB}(\phi \wedge \nu)$ , in contradiction with  $\neg\nu \notin KB_\phi^*$ .
- If  $\psi \in Cn(KB_\phi^* \cup \{\nu\})$ ,  $\psi \notin KB_\phi^*$  and  $\psi \notin Cn(\{\nu\})$  then, there exist  $\phi_1, \dots, \phi_n \in KB_\phi^* \cup \{\nu\}$  such that  $\phi_1 \wedge \dots \wedge \phi_n \models \psi$ . Therefore, if  $\phi_i \in KB_\phi^* \cup \{\nu\}$  then,  $\phi_i \in Cn(KB_\phi^*) \cup Cn(\{\nu\})$  and using above results  $\phi_i \in KB_{\phi \wedge \nu}^*$ . Since  $\phi_1 \wedge \dots \wedge \phi_n \models \psi$  and  $\phi_i \in KB_{\phi \wedge \nu}^*$ , using property  $\star 1$ , we have  $\psi \in KB_{\phi \wedge \nu}^*$ . Thus, postulate  $\star 8$  is proved.

Finally, to prove that  $[\star 4]$ , i.e. if  $\neg\phi \notin KB$  then  $Cn(KB \cup \{\phi\}) \subseteq KB_\phi^*$ , may not hold we give the following counterexample: Let  $w$  be a  $\phi$ -world such that  $w' \notin \text{Mod}(KB \cup \{\phi\})$ . Since  $\neg\phi \notin KB$  we have  $\text{Mod}(KB) \cap \text{Mod}(\phi) \neq \emptyset$ . Let  $w_1, \dots, w_n, \dots$  be a sequence of  $\phi$ -worlds such that, for any  $i \geq 1$   $w_i \in KB$  and  $\lim_{n \rightarrow \infty} S(w, w_n) = 1$ . In this case,  $C_{S,KB}(w) = 1$  and  $w$  is a  $\phi$ -world then, from lemma 1  $w \in \text{Mod}(KB_\phi^*)$ . ■

Before going into the proof of the postulates that our update operator defined in Definition 2 satisfies, it is worth noticing that indeed our update operator is deductively closed by the classical logical consequence, i.e. it holds

$$KB_\phi^\diamond = Cn(KB_\phi^\diamond).$$

Since  $Cn$  is compact, to prove this it suffices to show:

- (i) if  $\psi \in KB_\phi^\diamond$  and  $\psi \models \chi$  then  $\chi \in KB_\phi^\diamond$ , and
- (ii) if  $\psi, \chi \in KB_\phi^\diamond$  then  $\psi \wedge \chi \in KB_\phi^\diamond$ .

But (i) is straightforward since if  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  then it also holds that  $\text{Mod}(\phi \wedge \neg\chi) = \emptyset$ , and if  $C_{S,KB}^2(\neg\psi \mid \phi) < 1$ , by monotonicity, then it also holds  $C_{S,KB}^2(\neg\chi \mid \phi) \leq C_{S,KB}^2(\neg\psi \mid \phi) < 1$ . So anyway,  $\chi \in KB_\phi^\diamond$  too. On the other hand, (ii) comes from the fact that  $C_{S,KB}^2(\cdot \mid \phi)$  is a possibility measure, and thus,  $C_{S,KB}^2(\neg(\psi \wedge \chi) \mid \phi) = \max\{C_{S,KB}^2(\neg\psi \mid \phi), C_{S,KB}^2(\neg\chi \mid \phi)\}$ .

**Theorem 4:** The operator  $\diamond$  defined in Definition 2 verifies the postulates  $(\diamond 1)$ ,  $(\diamond 2')$ , and  $(\diamond 3) \sim (\diamond 8)$ , where the new postulate  $(\diamond 2')$  is:

- $\diamond 2'$ . If  $KB \models \phi$  then  $KB_\phi^\diamond \models KB$ .



**Proof:**

- ◇ 1.  $KB_\phi^\diamond \models \phi$ .  
Since  $\text{Mod}(\phi \wedge \neg\phi) = \emptyset$ , we have  $\phi \in KB_\phi^\diamond$ . Thus  $KB_\phi^\diamond \models \phi$ .
- ◇ 2'. If  $KB \models \phi$  then  $KB \models KB_\phi^\diamond$ .  
Suppose  $\nu \in KB_\phi^\diamond$ , we shall prove that  $KB \models \nu$ . Consider the following cases:
  - $\text{Mod}(\phi \wedge \neg\nu) = \emptyset$ . Then,  $\phi \models \nu$  and thus  $KB \models \nu$ .
  - $C_{S, \mathcal{KB}}^2(\neg\nu \mid \phi) < 1$  then, for any  $w \in \mathcal{KB}$ ,  $(C_{S, w}(\phi) > C_{S, w}(\phi \wedge \neg\nu))$ , since for any  $w \in \mathcal{KB}$ ,  $C_{S, w}(\phi) = 1$  we have  $C_{S, w}(\phi \wedge \neg\nu) < 1$ . Therefore for any  $w \in \mathcal{KB}$ ,  $w \not\models \phi \wedge \neg\nu$  and taking into account that  $w \models \phi$  we have necessarily that  $w \models \nu$  for any  $w \in \mathcal{KB}$ . Thus  $KB \models \nu$ .
- ◇ 3. If both  $KB$  and  $\phi$  are satisfiable then  $KB_\phi^\diamond$  is also satisfiable.  
Suppose  $KB_\phi^\diamond$  is not satisfiable. If  $\phi$  is satisfiable then,  $\text{Mod}(\phi) \neq \emptyset$ . On the one hand  $\neg\phi \in KB_\phi^\diamond$ , therefore  $C_{S, \mathcal{KB}}^2(\phi \mid \phi) < 1$ . On the other hand, since  $KB$  is satisfiable, by definition  $C_{S, \mathcal{KB}}^2(\phi \mid \phi) = 1$ . Thus the postulate has been proved.
- ◇ 4. If  $KB_1 \equiv KB_2$  and  $\phi_1 \equiv \phi_2$  then,  $(KB_1)_{\phi_1}^\diamond \equiv (KB_2)_{\phi_2}^\diamond$ .  
It is immediate, taking into account that  $\text{Mod}(\phi_1 \wedge \neg\psi) = \emptyset$  if, and only if,  $\text{Mod}(\phi_2 \wedge \neg\psi) = \emptyset$ , and  $C_{S, \mathcal{KB}_\infty}^2(\neg\psi \mid \phi_1) < 1$  if, and only if,  $C_{S, \mathcal{KB}_\infty}^2(\neg\psi \mid \phi_2) < 1$ .
- ◇ 5.  $KB_\phi^\diamond \cup \{\nu\} \models KB_{\phi \wedge \nu}^\diamond$ .  
This condition is equivalent to  $KB_{\phi \wedge \nu}^\diamond \subseteq \text{Cn}(KB_\phi^\diamond \cup \{\nu\})$ . Suppose  $\chi \in KB_{\phi \wedge \nu}^\diamond$ .  
If  $\text{Mod}(\phi \wedge \nu \wedge \neg\chi) = \emptyset$  then,  $\phi \wedge \nu \models \chi$ . Thus  $\chi \in \text{Cn}((KB_\phi^\diamond) \cup \{\nu\})$ .  
If  $C_{S, \mathcal{KB}}^2(\neg\psi \mid \phi \wedge \nu) < 1$  then, by definition,  $\sup_{w \in \mathcal{KB}} (C_{S, w}(\phi \wedge \nu) \otimes \rightarrow C_{S, w}(\neg\psi \wedge \phi \wedge \nu)) < 1$ . Since  $\otimes \rightarrow$  is a decreasing function respect to the first variable and  $C_{S, w}(\phi) \geq C_{S, w}(\phi \wedge \nu)$  we have  $C_{S, w}(\phi) \otimes \rightarrow C_{S, w}(\neg\psi \wedge \phi \wedge \nu) \leq C_{S, w}(\phi \wedge \nu) \otimes \rightarrow C_{S, w}(\neg\psi \wedge \phi \wedge \nu)$  for any  $w \in \mathcal{KB}$ . Therefore,  $C_{S, \mathcal{KB}}^2(\neg\nu \vee \psi \mid \phi) = \sup_{w \in \mathcal{KB}} (C_{S, w}(\phi) \otimes \rightarrow C_{S, w}(\neg\psi \wedge \phi \wedge \nu)) \leq \sup_{w \in \mathcal{KB}} (C_{S, w}(\phi \wedge \nu) \otimes \rightarrow C_{S, w}(\neg\psi \wedge \phi \wedge \nu)) < 1$  and by definition,  $\neg\nu \vee \psi \in KB_\phi^\diamond$ . Thus  $\psi \in \text{Cn}(KB_\phi^\diamond \cup \{\nu\})$ .
- ◇ 6. If  $KB_{\phi_1}^\diamond \models \phi_2$  and  $KB_{\phi_2}^\diamond \models \phi_1$  then  $KB_{\phi_1}^\diamond \equiv KB_{\phi_2}^\diamond$ .  
Since the update operator is deductively closed, we will equivalently prove that if  $\phi_2 \in KB_{\phi_1}^\diamond$  and  $\phi_1 \in KB_{\phi_2}^\diamond$  then  $KB_{\phi_1}^\diamond = KB_{\phi_2}^\diamond$ . The proof will result as a consequence of these two intermediary results:
  - If  $\phi_1 \in KB_{\phi_2}^\diamond$  and  $\phi_2 \in KB_{\phi_1}^\diamond$  then, for any  $w \in \mathcal{KB}$  it holds that
 
$$C_{S, w}(\phi_1) = C_{S, w}(\phi_1 \wedge \phi_2) = C_{S, w}(\phi_2).$$

**Proof:**

Consider the case  $C_{S, \mathcal{KB}}^2(\neg\phi_2 \mid \phi_1) < 1$  and  $C_{S, \mathcal{KB}}^2(\neg\phi_1 \mid \phi_2) < 1$ . The

remaining cases are easier. Taking into account that  $u \otimes \rightarrow v = 1$  iff  $u \leq v$ , these two inequalities imply that, for any  $w \in \mathcal{KB}$ , we have

$$C_{S,w}(\phi_1) > C_{S,w}(\phi_1 \wedge \neg\phi_2), \text{ and } C_{S,w}(\phi_2) > C_{S,w}(\neg\phi_1 \wedge \phi_2)$$

therefore this leads to  $C_{S,w}(\phi_1) = C_{S,w}(\phi_1 \wedge \phi_2)$  and  $C_{S,w}(\phi_2) = C_{S,w}(\phi_1 \wedge \phi_2)$ .

- If  $\phi_1 \in KB_{\phi_2}^\circ$  and  $C_{S,w}(\phi_1) = C_{S,w}(\phi_2)$  for any  $w \in \mathcal{KB}$ , then  $KB_{\phi_1}^\circ \subseteq KB_{\phi_2}^\circ$ . Analogously, if  $\phi_2 \in KB_{\phi_1}^\circ$  and  $C_{S,w}(\phi_1) = C_{S,w}(\phi_2)$  for any  $w \in \mathcal{KB}$ , then  $KB_{\phi_2}^\circ \subseteq KB_{\phi_1}^\circ$ .

**Proof:**

We will only prove the first statement. The second is proved in the same way. Let  $\phi_1 \in KB_{\phi_2}^\circ$ . Then, either  $\text{Mod}(\phi_2 \wedge \neg\phi_1) = \emptyset$  or  $C_{S,\mathcal{KB}}^2(\neg\phi_1 \mid \phi_2) < 1$ .

- \* Suppose  $\text{Mod}(\phi_2 \wedge \neg\phi_1) = \emptyset$  and  $\psi \in KB_{\phi_1}^\circ$ . Then we consider the cases  $\text{Mod}(\phi_1 \wedge \neg\psi) = \emptyset$  or  $C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_1) < 1$ .

1. If  $\text{Mod}(\phi_1 \wedge \neg\psi) = \emptyset$  then  $\text{Mod}(\phi_2 \wedge \neg\psi) = \emptyset$  as well. Thus  $\psi \in KB_{\phi_2}^\circ$ .
2. Suppose  $C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_1) < 1$ .

We want to prove that  $C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_2) < 1$ . To do so, notice that, since  $\text{Mod}(\phi_2) \subseteq \text{Mod}(\phi_2)$ , we have that

$$C_{S,w}(\phi_2 \wedge \neg\psi) \leq C_{S,w}(\phi_1 \wedge \neg\psi).$$

Therefore, for each  $w \in \mathcal{KB}$ , it is

$$\begin{aligned} C_{S,w}(\phi_2) \otimes \rightarrow C_{S,w}(\phi_2 \wedge \neg\psi) &= \\ C_{S,w}(\phi_1) \otimes \rightarrow C_{S,w}(\phi_2 \wedge \neg\psi) &\leq \\ C_{S,w}(\phi_1) \otimes \rightarrow C_{S,w}(\phi_1 \wedge \neg\psi). & \end{aligned}$$

Thus, taking supremums, we have  $C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_2) \leq C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_1) < 1$ , that is,  $\psi \in KB_{\phi_2}^\circ$ .

- \* Suppose  $C_{S,\mathcal{KB}}^2(\neg\phi_1 \mid \phi_2) < 1$  and  $\psi \in KB_{\phi_1}^\circ$ . Again, either  $\text{Mod}(\phi_1 \wedge \neg\psi) = \emptyset$  or  $C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_1) < 1$ .

1. If  $\text{Mod}(\phi_1 \wedge \neg\psi) = \emptyset$  then  $\text{Mod}(\neg\psi) \subseteq \text{Mod}(\neg\phi_1)$ . Thus,  $C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_2) \leq C_{S,\mathcal{KB}}^2(\neg\phi_1 \mid \phi_2) < 1$ , and thus  $\psi \in KB_{\phi_2}^\circ$ .

2. Suppose now that  $C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_1) < 1$ . First, notice the following inequality

$$\begin{aligned} C_{S,w}(\phi_2 \wedge \neg\psi) &= \max\{C_{S,w}(\phi_2 \wedge \neg\psi \wedge \phi_1), \\ &\quad C_{S,w}(\phi_2 \wedge \neg\psi \wedge \neg\phi_1)\} \\ &\leq \max\{C_{S,w}(\phi_1 \wedge \neg\psi), C_{S,w}(\phi_2 \wedge \neg\phi_1)\}. \end{aligned}$$

Therefore, taking into account that for any  $w \in \mathcal{KB}$  it holds  $C_{S,w}(\phi_1) = C_{S,w}(\phi_2)$ , we have:

$$\begin{aligned} C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_2) &= \sup_{w \in \mathcal{KB}} C_{S,w}(\phi_2) \otimes \rightarrow C_{S,w}(\phi_2 \wedge \neg\psi) \\ &\leq \max\{\sup_{w \in \mathcal{KB}} C_{S,w}(\phi_2) \otimes \rightarrow C_{S,w}(\phi_1 \wedge \neg\psi), \\ &\quad \sup_{w \in \mathcal{KB}} C_{S,w}(\phi_2) \otimes \rightarrow C_{S,w}(\phi_2 \wedge \neg\phi_1)\} \\ &= \max\{C_{S,\mathcal{KB}}^2(\neg\psi \mid \phi_1), C_{S,\mathcal{KB}}^2(\neg\phi_1 \mid \phi_2)\} \\ &< 1 \end{aligned}$$

Thus  $\psi \in KB_{\phi_2}^\circ$ .

This ends the proof of postulate  $\diamond 6$ .

- $\diamond 7$ . If  $KB$  is complete then  $KB_{\phi_1}^\diamond \cup KB_{\phi_2}^\diamond \models KB_{\phi_1 \vee \phi_2}^\diamond$ .  
This is equivalent to prove  $KB_{\phi_1 \vee \phi_2}^\diamond \subseteq Cn(KB_{\phi_1}^\diamond \cup KB_{\phi_2}^\diamond)$ . Suppose  $\psi \in KB_{\phi_1 \vee \phi_2}^\diamond$  then,  $\text{Mod}((\phi_1 \vee \phi_2) \wedge \neg\psi)$  or  $C_{S, \mathcal{KB}}^2(\neg\psi \mid (\phi_1 \vee \phi_2)) < 1$ .  
If  $\text{Mod}((\phi_1 \vee \phi_2) \wedge \neg\psi)$  then,  $\text{Mod}(\phi_1 \wedge \neg\psi)$  or  $\text{Mod}(\phi_2 \wedge \neg\psi)$ . Therefore,  $\psi \in KB_{\phi_1}^\diamond$  or  $\psi \in KB_{\phi_2}^\diamond$ .  
If  $C_{S, \mathcal{KB}}^2(\neg\psi \mid (\phi_1 \vee \phi_2)) < 1$  then, for any world  $w \in \text{Mod}(KB)$  we have  $C_{S, w}((\phi_1 \vee \phi_2) \wedge \psi) > C_{S, w}((\phi_1 \vee \phi_2) \wedge \neg\psi)$ . Therefore, for any  $w \in \text{Mod}(KB)$   $\max\{C_{S, w}(\phi_1 \wedge \psi), C_{S, w}(\phi_2 \wedge \psi)\} > \max\{C_{S, w}(\phi_1 \wedge \neg\psi), C_{S, w}(\phi_2 \wedge \neg\psi)\}$ . Since  $KB$  is complete, there exists a unique world  $w \in \mathcal{KB}$ . Therefore,  $C_{S, w}(\phi_1 \wedge \psi) > C_{S, w}(\phi_1 \wedge \neg\psi)$  or  $C_{S, w}(\phi_2 \wedge \psi) > C_{S, w}(\phi_2 \wedge \neg\psi)$ . Thus, either  $\psi$  is in  $KB_{\phi_1}^\diamond$  or  $\psi$  is in  $KB_{\phi_2}^\diamond$ .
- $\diamond 8$ .  $(KB_1 \cap KB_2)_\phi^\diamond \equiv ((KB_1)_\phi^\diamond \cap (KB_2)_\phi^\diamond)$ .  
Suppose,  $\psi$  is a formula such that  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$ . Then, the equivalence is satisfied. Suppose,  $\psi$  is a formula such that  $\text{Mod}(\phi \wedge \neg\psi) \neq \emptyset$  and  $\psi \in (KB_1 \cap KB_2)_\phi^\diamond$ . Then,  $C_{S, KB_1 \cap KB_2}^2(\neg\psi \mid \phi) < 1$  and for any world  $w \in \text{Mod}(KB)$ ,  $C_{S, w}(\phi \wedge \psi) > C_{S, w}(\phi \wedge \neg\psi)$ . Since,  $\text{Mod}(KB_1 \cap KB_2) = \text{Mod}(KB_1) \cup \text{Mod}(KB_2)$ , we have  $C_{S, KB_1}^2(\neg\psi \mid \phi) < 1$  and  $C_{S, KB_2}^2(\neg\psi \mid \phi) < 1$ . Now suppose  $\psi$  is a formula such that  $\text{Mod}(\phi \wedge \neg\psi) \neq \emptyset$  and  $\psi \in ((KB_1)_\phi^\diamond \cap (KB_2)_\phi^\diamond)$ . Then,  $C_{S, KB_1}^2(\neg\psi \mid \phi) < 1$  and  $C_{S, KB_2}^2(\neg\psi \mid \phi) < 1$ . Obviously, by definition  $C_{S, KB_1 \cap KB_2}^2(\neg\psi \mid \phi) < 1$ .
- $\diamond 9$ . If  $KB$  is complete and  $KB_\phi^\diamond \cup \{\nu\}$  is satisfiable then,  $KB_{\phi \wedge \nu}^\diamond \models KB_\phi^\diamond \cup \{\nu\}$ .  
Suppose,  $\psi$  is a formula such that  $\psi \in KB_\phi^\diamond \cup \{\nu\}$ . If  $\psi$  is  $\nu$ , then by  $(\diamond 1)$   $\psi \in KB_{\phi \wedge \nu}^\diamond$ . If  $\psi \in KB_\phi^\diamond$  then,  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  or  $C_{S, \mathcal{KB}}^2(\neg\psi \mid \phi) < 1$ . If  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  then,  $\psi \in KB_{\phi \wedge \nu}^\diamond$ .  
Finally, if  $C_{S, \mathcal{KB}}^2(\neg\psi \mid \phi) < 1$ . Since  $KB$  is complete, there exists a unique world  $w \in \text{Mod}(KB)$  such that  $C_{S, w}(\phi \wedge \psi) > C_{S, w}(\phi \wedge \neg\psi)$ . Since  $KB_\phi^\diamond \cup \{\nu\}$  is satisfiable then  $C_{S, w}(\phi \wedge \nu \wedge \psi) > C_{S, w}(\phi \wedge \nu \wedge \neg\psi)$ . Thus,  $\psi \in KB_{\phi \wedge \nu}^\diamond$ . ■

These results are not surprising. Properties  $(\star 4)$  and  $(\diamond 2)$  may not hold only in the infinitary case. In the finite case these properties are recovered, as we see below. For this, we will use the consistency measures introduced in section 2 to define the following pre-orders.

**Definition 3:** Given a similarity function  $\mathbf{S}$ , we define for each knowledge base  $KB$  a total pre-order  $\leq_{KB}$  on the set of possible worlds, as follows:

$$w_1 \leq_{KB} w_2 \stackrel{\text{def}}{\iff} C_{S, \mathcal{KB}}(w_1) \geq C_{S, \mathcal{KB}}(w_2).$$

**Definition 4:** Given a similarity function  $\mathbf{S}$ , for each world  $w$  we define the total pre-order  $\leq_w$  on the set of possible worlds as follows:

$$w_1 \leq_w w_2 \stackrel{\text{def}}{\iff} C_{S, w}(w_1) \geq C_{S, w}(w_2).$$

In the particular case of  $KB$  being finite, theorems 1 and 2 allow us to define a revision and an update operators,  $\star_f$  and  $\diamond_f$  respectively, using the above pre-orders. Namely, the set of models of the revised and updated knowledge base  $KB$  by  $\phi$  are respectively:

$$\begin{aligned} \text{Mod}(KB_\phi^{\star_f}) &= \{w' \in \text{Mod}(\phi) \mid \exists w \in \mathcal{KB}, S(w, w') = C_{S, \mathcal{KB}}(\phi)\}, \\ \text{Mod}(KB_\phi^{\diamond_f}) &= \{w' \in \text{Mod}(\phi) \mid \exists w \in \mathcal{KB}, S(w, w') = C_{S, w}(\phi)\} \end{aligned}$$

Now we can present the main result of this paper: in the finite case, the operators  $\star_f$  and  $\diamond_f$ , obtained via the Katsuno Mendelzon's representation theorem from the pre-orders of definitions 3 and 4, are exactly those given by the above definitions 1 and 2, respectively. This is formally expressed in the next two theorems.

**Theorem 5:**  $\psi \in KB_\phi^{\star_f}$  if, and only if, either  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  or  $C_{S, \mathcal{KB}}(\neg\psi \wedge \phi) < C_{S, \mathcal{KB}}(\psi \wedge \phi)$ .

**Proof:**  $\psi \in KB_\phi^{\star_f}$  if, and only if,  $\text{Mod}(KB_\phi^{\star_f}) \subseteq \text{Mod}(\psi)$  if, and only if,  $\{w \in \text{Mod}(\phi) \mid C_{S, \mathcal{KB}}(w) = C_{S, \mathcal{KB}}(\phi)\} \subseteq \text{Mod}(\psi)$ . Suppose,  $C_{S, \mathcal{KB}}(\phi) = 0$  then,  $\{w \in \text{Mod}(\phi) \mid C_{S, \mathcal{KB}}(w) = C_{S, \mathcal{KB}}(\phi)\} \subseteq \text{Mod}(\psi)$  if, and only if,  $\text{Mod}(\phi) \subseteq \text{Mod}(\psi)$  if, and only if,  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$ . Now suppose  $C_{S, \mathcal{KB}}(\phi) > 0$ .  $\{w \in \text{Mod}(\phi) \mid C_{S, \mathcal{KB}}(w) = C_{S, \mathcal{KB}}(\phi)\} \subseteq \text{Mod}(\psi)$  if, and only if,  $C_{S, \mathcal{KB}}(\phi) = C_{S, \mathcal{KB}}(\phi \wedge \psi) \neq C_{S, \mathcal{KB}}(\phi \wedge \neg\psi)$  and if, and only if,  $C_{S, \mathcal{KB}}(\phi \wedge \psi) > C_{S, \mathcal{KB}}(\phi \wedge \neg\psi)$ . ■

**Theorem 6:**  $\psi \in KB_\phi^{\diamond_f}$  if, and only if, either  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$  or  $C_{S, \mathcal{KB}}^2(\neg\psi \mid \phi) < 1$ .

**Proof:**  $\psi \in KB_\phi^{\diamond_f}$  if, and only if,  $\text{Mod}(KB_\phi^{\diamond_f}) \subseteq \text{Mod}(\psi)$  if, and only if,  $\{w' \in \text{Mod}(\phi) \mid \text{there exists } w \in \text{Mod}(KB) \text{ and } S(w, w') = C_{S, w}(\phi)\} \subseteq \text{Mod}(\psi)$ . Suppose that, there exists  $w \in \text{Mod}(KB)$  such that  $C_{S, w'}(\phi) = 0$ . We have,  $\{w' \in \text{Mod}(\phi) \mid \text{there exist } w \in \text{Mod}(KB) \text{ and } S(w, w') = C_{S, w}(\phi)\} \subseteq \text{Mod}(\psi)$  if, and only if,  $\text{Mod}(\phi \wedge \neg\psi) = \emptyset$ .

Now suppose that  $C_{S, w'}(\phi) > 0$  for any  $w \in \text{Mod}(KB)$ . We have,  $\{w' \in \text{Mod}(\phi) \mid \text{there exists } w \in \text{Mod}(KB) \text{ and } S(w, w') = C_{S, w}(\phi)\} \subseteq \text{Mod}(\psi)$  if, and only if,  $C_{S, w'}(w) = C_{S, w'}(\phi) = C_{S, w'}(\phi \wedge \psi) \neq C_{S, w'}(\phi \wedge \neg\psi)$  for any  $w' \in \text{Mod}(KB)$ , if, and only if,  $C_{S, w'}(\phi \wedge \psi) > C_{S, w'}(\phi \wedge \neg\psi)$ , for any  $w' \in \text{Mod}(KB)$ , if, and only if,  $C_{S, \mathcal{KB}}^2(\neg\psi \mid \phi) = \max_{w' \in \mathcal{KB}}(C_{S, w'}(\phi) \otimes \rightarrow C_{S, w'}(\phi \wedge \neg\psi)) < 1$ . ■

## 5 Related and Future Work

We have redefined the concepts of update and revision via the similarity-based models for possibilistic and fuzzy reasoning. The revision operator is similar to that given by Dubois and Prade ([2]) to characterize revision in Possibilistic Logic. However, there are two significative differences. The first one is that our revision

operator is not commutative, i.e., the ordering of change is significative. The second one is about the worlds which are contradictory with the new information. In the Dubois and Prade's approach, those worlds are considered as impossible in the new state of affairs. However, this does not occur in our approach, because we do not change the similarity relation.

The update operator is something different from the one obtained by Dubois and Prade via possibilistic imaging [2], since in our approach, we do not need any external "closest function". Moreover, the characterization result of theorem 6 has not an analogous counterpart in Possibilistic Logic.

As a further relationship to be studied between the possibilistic and similarity based approach is the following one. In [3] Esteva, Garcia-Calves and Godo define possibility distributions via similarity functions and subsets of worlds. On the other hand, Dubois and Prade propose an update operator based on a possibility distribution on the set of possible worlds ([2]). An idea is to relate the update operation of our proposal with that of Dubois and Prade defined from the possibility distribution resulting from the mapping proposed by Esteva et al.

As a final remark, let us notice that the symmetry and transitivity properties of the fuzzy similarity relations are not needed to generate the pre-orders for defining revision and update operators. Therefore it seems interesting to generalize our approach allowing more general models than those based on fuzzy similarity relations. Namely we could allow models with a fuzzy binary relation, representing some general notion of "closeness", for which only the reflexivity and separating properties would be required.

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