Generalized Version of the Compatibility Theorem. Two Examples

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Abstract

In a previous work ([3]) we proved that the Nguyen's condition for $[f(\widetilde{A})]_\alpha$ to be equal to $f(A_\alpha)$ also holds for the most general class of the L-fuzzy subsets, where L is an arbitrary lattice. Here we recall the main points of the proof ad present some examples ralated to non-linear lattices.
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1 The Compatibility Result

Let $(L, \preceq)$ be a complete lattice with minimum and maximum elements denoted respectively by m and M, and let $\widetilde{\mathcal{P}}_L(X)$ be the family of the L-fuzzy subsets of the space X, that is the family of the maps $(\widetilde{A})$ from X to L.

Let f be a map from X to another space Y and let $\widetilde{B} = f(\widetilde{A})$ the image of the fuzzy subet $\widetilde{A}$ obtained by means of the Zadeh extension priciple, that is let $\widetilde{B}$ be the fuzzy subset of Y defined by

$$(1) \qquad \widetilde{B}(y) = \begin{cases} \sup\{\widetilde{A}(x) | f(x) = y\} & \text{if } y \in f(X) \\ m & \text{otherwise} \end{cases}$$

The question we analyse here is: "are the $\alpha-$cuts of $\widetilde{B}$ the images by f of the $\alpha-$cuts of $\widetilde{A}$?" H.T. Nguyen ([1]) gave an answer to this question in the case where L has a linear ordering. We extended his result to any lattice, and also to any sup-semilattice provided that some conditions hold ([3]). We recall here the results we obtained and we present some examples regarding lattices with non-linear orderings.

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Definition We say that a family $\{A^*_\alpha \mid \alpha \in L\}$ of crisp subsets of X is a fuzzy generator if the following condition hold:

$$\alpha', \alpha'' \in L,\ \alpha' \prec \alpha'' \Rightarrow A^*_{\alpha'} \supseteq A^*_{\alpha''}$$

The fuzzy subset $\widetilde{A}$ generated by $\{A^*_\alpha\}$ is $\widetilde{A}(x) \ = \ \sup\{\alpha \mid x \in A^*_\alpha\}$.

The following results hold:

Proposition 1 The class $\{A_\alpha\}$ of the $\alpha-$cuts is a (canonical) generator of $\widetilde{A}$.

Proposition 2 If $\{A^*_l\}$ is a generator of $\widetilde{A}$ then $A^*_\alpha \subseteq A_\alpha$ (A_α is the $\alpha-$cut).

Proposition 3 A necessary and sufficient condition for $A^*_\alpha = A_\alpha$ is $\sup\{\alpha|x \in A^*\alpha\} \ = \ \max\{\alpha|x \in A^*_\alpha\}$, that is if $\sup\{\alpha|x \in A^*_\alpha\} = \beta$, then $x \in A^*_\beta$. Now let us consider a map f from X to Y and let $f(\widetilde{A})$ be the $L-$fuzzy set induced on Y by $\widetilde{A}$ by means of (1)

Proposition 4 The family $\{f(A_\alpha)\}$ of the images of the $\alpha-$cuts is a generator of $f(\widetilde{A})$.

Proposition 5 $\sup\{\widetilde{A}(x)|f(x) = y\} = \max\{\widetilde{A}(x)|f(x) = y\}$ is a necessary and sufficient condition in order to have $[f(\widetilde{A})]_\alpha \ = \ f(A_\alpha)\ \forall\alpha$.

Proposition 6 (the Nguyen's result) If $f(u,v)$ is a function of two variables defined on $U \times V$ and $\widetilde{R}, \widetilde{S}$ are two fuzzy subsets of U and V, then we have

$$[f(\widetilde{R},\widetilde{S})]_l \ = \ f(R_l, S_l) \iff \sup\{\min[\widetilde{R}(u),\widetilde{S}(v)]|f(u,v) = y\} \ = \max\{\min[\widetilde{R}(u),\widetilde{S}(v)]|f(u,v) = y\}$$

Proposition 7 The same result also holds if we apply the extension principle to a function of several variables, i.e. if $X = U_1 \times U_2 \times \ldots U_n$.

Proposition 8 The compatibility resut also holds for the second order fuzzy sets, that is for the fuzzy sets whose membership function is a map from $[0,1]$ to $[0,1]$.

Proposition 9 The results we exposed in the points 4,5,6,7,8 also hold if $(L, \preceq)$ is a sup-semi-lattice providet that function f is surjective.

2 Examples

Example 1 Let $(L, \preceq)$ be the lattice defined by

$$\begin{array}{l} L \ = \ \{i, a, b, c, d, s\} \\ i \preceq a \preceq c \preceq s \qquad i \preceq b \preceq d \preceq s \end{array}$$

and let us consider the following fuzzy subset of the space $X = [0, 1]$;

$$\widetilde{A}(x) \ = \ \begin{cases} i & \text{if } x = 0 \\ a & \text{if } x \in]0, \frac{1}{2}] \\ b & \text{if } x \in]\frac{1}{2}, \frac{3}{4}] \\ c & \text{if } x \in]\frac{3}{4}, \frac{7}{8}] \\ d & \text{if } x \in]\frac{7}{8}, 1] \\ s & \text{if } x = 1 \end{cases}$$

Among the $\alpha-$cuts of $\widetilde{A}$ we consider in particular

$$A_b \ = \ ]\frac{1}{2}, \frac{3}{4}] \cup]\frac{7}{8}, 1]$$

$$A_c \ = \ ]\frac{3}{4}, \frac{7}{8}] \cup \{1\}$$

Now let $Y = X$ and let $f : X \to Y$ be the function defined by

$$f(x) \ = \ \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

It is easy to recognize that

$$\widetilde{B}(y) \ = \ [f(\widetilde{A})](y) \ = \ \sup\{\widetilde{A}(\frac{y}{2}), \widetilde{A}(1 - \frac{y}{2})\} \tag{2}$$

Since $\frac{y}{2} \in [0, \frac{1}{2}]$ we have

$$\widetilde{A}(\frac{y}{2}) \ = \ \begin{cases} i & \text{if } y = 0 \\ a & \text{if } y > 0 \end{cases}$$

$$\widetilde{A}(1 - \frac{y}{2}) \ = \ \begin{cases} a & \text{if } y = 1 \\ b & \text{if } \frac{1}{2} \le y < 1 \\ c & \text{if } \frac{1}{4} \le y < \frac{1}{2} \\ d & \text{if } o < y < \frac{1}{4} \\ s & \text{if } y = 0 \end{cases}$$

Using the formula (2) we can easily calculate the subset $\widetilde{B}$. We obtain

$$\widetilde{B}(y) = \begin{cases} \sup(i,s) = s & \text{if } y = 0 \\ \sup(a,d) = s & \text{if } 0 < y < \frac{1}{4} \\ \sup(a,c) = c & \text{if } \frac{1}{4} \leq y < \frac{1}{2} \\ \sup(a,b) = s & \text{if } \frac{1}{2} \leq y < 1 \\ \sup(a,a) = a & \text{if } y = 1 \end{cases}$$

So we recognize that the condition of proposition 5 does not hold and therefore some of the subsets B_θ $(\theta \in \{i,a,b,c,d.s\})$ are different from the corresponding $f(A_\theta)$. In particular $B_c = [0,1[$ is different from $f(A_c) = \{0\} \cup [\frac{1}{4}, \frac{1}{2}[$. Note that B_θ may be equal to $f(A_\theta)$ for some particular θ. In our case we have, for example $B_b = f(A_b)$.

Example 2 The range $(L, \preceq)$ of the membership functions is the structure given by

$$\begin{aligned} L &= \{a,b,g,d,e,f,g,s\} \\ a \preceq e,& \quad b \preceq e, \quad e \preceq s \\ c \preceq g,& \quad d \preceq g, \quad g \preceq s \end{aligned}$$

The spaces X and Y are the interval $[0,1]$ anf function f is

$$f(x) = 4x^2 - 4x + 1$$

Note that the structure $(L, \preceq)$ is not a lattice, but only a sup-semilattice. Nevertheless we can apply the propsitions of paragraph 1 because the function f is surjective. Now let us analyse the fuzzy subset of the X space given by

$$\widetilde{A}(x) = \begin{cases} a & \text{if } 0 \leq x < \frac{1}{8} \\ e & \text{if } \frac{1}{8} \leq x < \frac{3}{8} \\ b & \text{if } \frac{3}{8} \leq x < \frac{1}{2} \\ f & \text{if } x = \frac{1}{2} \\ c & \text{if } \frac{1}{2} < x \leq \frac{5}{8} \\ g & \text{if } \frac{5}{8} < x \leq \frac{7}{8} \\ d & \text{if } \frac{7}{8} < x \leq 1 \end{cases}$$

By means of the extension principle we can easily calculate $\widetilde{B}(y) = \sup\{\widetilde{A}(\frac{1-\sqrt{y}}{2}), \widetilde{A}(\frac{1+\sqrt{y}}{2})$. It is the fuzzy subset

$$\widetilde{B}(y) = \begin{cases} \sup(f,f) = f & \text{if } y = 0 \\ \sup(b,c) = s & \text{if } 0 < y \leq \frac{1}{16} \\ \sup(e,g) = s & \text{if } \frac{1}{16} < y \leq \frac{9}{16} \\ \sup(a,d) = s & \text{if } \frac{9}{16} < y \leq 1 \end{cases}$$

It is easy to recognize that B_θ and $f(A_\theta)$ are respectively

$$\begin{aligned} B_i &= B_f = [0,1] \\ B_a &= B_b = B_c = B_d = B_e = B_g =]0,1] \end{aligned}$$

$$\begin{aligned} f(A_a) &= f(A_d) =]\frac{1}{16},1] \\ f(A_b) &= f(A_c) =]0,\frac{9}{16}] \\ f(A_e) &= f(A_g) =]\frac{1}{16},\frac{9}{16}] \\ f(A_f) &= \{0\} \end{aligned}$$

In this case all the $\alpha-$cuts of $\widetilde{B}$ are different from the images of the corresponding $\alpha-$cuts of $\widetilde{A}$.

References

[1] H.T. Nguyen: "A Note on the Extension Principle for Fuzzy Sets", J. Math. Anal. and Appl. 64 (1978) 369-380.

[2] L.A. Zadeh: "The Concept of a Linguistic Variable and its Application to Approximate Reasoning", Inform. Sci. 8 (1975) 199-249.

[3] C. Bertoluzza, A. Bodini: "A new proof of the Nguten compatibility theorem in a more general context", (1995) Presented to Fuzzy Set and Systems.