(Pure) Logic out of Probability

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Abstract

Today, Logic and Probability are mostly seen as independent fields with a separate history and set of foundations. Against this dominating perception, only a very few people (Laplace, Boole, Peirce) have suspected there was some affinity or relation between them. The truth is they have a considerable common ground which underlies the historical foundation of both disciplines and, in this century, has prompted notable thinkers as Reichenbach [14], Carnap [2] [3] or Popper [12] [13] (and Gaifman [5], Scott & Krauss [21], Fenstad [4], Miller [10] [11], David Lewis [9], Stalnaker [22], Hintikka [7] or Suppes [23]) to consider connection-building treatments of Logic and Probability as desirable. Indeed such a line of thinking can be pursued (this author, for one, attempted it in [15-19]). In so doing, one straightforwardly obtains a logic based on —as the simple unifying concept— an *additive non-functional truth valuation* which, though technically indistinguishable from (axiomatic) Probability, can however be totally "decontaminated" from parasitical probabilistic interpretations (such as the usual readings of "event", "probability" or "conditioning") and be given instead a strictly logical reading and justification (in terms of "sentence", "truth" or "relativity"). Once some deeply-ingrained reading habits are overcome, the required concepts and formulas flow easily, and the resulting assertion-based *sentential calculus* becomes a very natural extension of ordinary two-valued reasoning. Furthermore, in the process we get: (a) intuitive geometrical and information-related interpretations of the concepts, (b) a simple theoretical explanation for some poorly justified formulas (intermittently advanced by various authors, some mentioned above), and (c) a semantics —and a proof theory— for general assertions that is unproblematically derived and also fully consistent with empirical or *ad hoc* approximate-reasoning "Bayesian" formulas found by Artificial Intelligence researchers.
Keywords: Sentential logic, boolean algebra, logical semantics, probabilistic semantics, probability logic, many-valued logics, supervaluations, uncertainty, rational belief, proof theory.

The present paper is a summary of [18] and [19] as delivered orally (in Italian) in the Naples Workshop on Nov. 2nd, 1995, under the announced title *Between Logic and Probability*. (That it should be delivered in Naples is just appropriate: the ideas for [18] came effortlessly out of daily uphill trips —shuttling for Arco Felice— on the Mergellina cable train in the mild December mornings of 1981.) The full summary is reference [0].

Introduction

When we argue, we do not always fully assert what we say. We often make half-hearted assertions of sentences we are not sure about, or we even use as assertions sentences we hardly believe to be the case. And yet we proceed by reasoning from such weak premises. If we admit we do, and want to treat this inside Logic, we apparently need to qualify assertions, or rather quantify their strength, and try to follow and control what effects weak assertions may have in the reasoning process, whether and how they affect its logical validity and how we can tell the strength of the conclusion. All this seems to be indeed a proper *logical* subject that, however, very few logicians (Reichenbach [14], Carnap [2] and Popper [12] are the conspicuous exceptions) did ever set out to confront.

Just to fix ideas, take as a first example a reasoning in Physics wherein one of the premises is the positive result of an experiment. Suppose we may even quantify the error ε of the experiment —meaning that the truth of the assertion 'result is positive' is, say, "$1-\varepsilon$". We then perform the formal reasoning —assumed logically valid— and obtain the conclusion. We now want to know what confidence we may have in it, given ε. That, everybody would agree, is a legitimate logician's concern. (It is what we develop in our *proof theory*, see [19])

For a second example, take the well-known sorites about bald men: "If a man with i hairs is not bald then a man with $i-1$ hairs is still not bald. Suppose a man has n hairs. Therefore, a man with 0 hairs is still not bald". Formally:

$$\begin{array}{l} A_i \to A_{i-1} \quad (i:1,\ldots,n) \\ A_n \\ \hline A_0 \end{array}$$

This is a paradox because the reasoning is formally correct (it consists of merely n applications of the Modus Ponens rule), the $n+1$ premises are deemed flawless, but the conclusion is outright false (or, more precisely, a *contradictio in terminis*). Usually, it is the length of the argument that is put to blame. There is, however, a more concrete and satisfactory answer we can offer. The n premises $A_i \to A_{i-1}$ cannot obviously be asserted with the same assurance whatever the index value. That's why the argument fails: for low values of i the premises simply cannot be asserted, even if the rest can, so we can *never* have all premises asserted, and the reasoning is formally valid but vacuously so. As it will be later seen, we propose instead to provide every premise A with a value $v(A)$ in [0,1] —computed in an unspecified way (statistically, by opinion survey, or whatever)— with the unique requirement that a zero value means that the premise is to be taken as false, 1 means a true —and therefore fully assertable— premise, and $v(A) = 1 - \varepsilon$ ($\varepsilon > 0$) means that we can assert A but with some apprehension or risk ε. Obviously, the value $v(A_i \rightarrow A_{i-1})$ decreases with i, so that when i is n (or even, say, around $n/2$ or $n/3$) it is 1 or very near 1, but when i approaches, say, $n/10$ —and surely when it becomes zero— the value of $A_i \rightarrow A_{i-1}$ (= the predisposition we have to assert it —or the willingness to assume the risk) comes down to an exceedingly low number. According to our proof theory (omitted here for space reasons), the conclusion A_0 has the same truth value, at best, as that lowest of numbers (and, thus, the reasoner would be willing to assert the conclusion just no more than he or she would willing to assert $A_1 \rightarrow A_0$).

That we normally need to qualify/quantify what we assert is implied by all mainstream logicians when they treat assertions as true, believed or merely hypothesized sentences, leaving it to the reasoner who uses them to count actively on these factors, as part of the reasoning process itself. After all, following Tarski's introduction of the (T) schema, it is the user who is supposed to be, through the full mastery of the (T) translation process, the sole referee who can validate the truth of the assertions, their aptness to describe an actual situation, their strength (as beliefs) or the relevance of their use in the current logical context. As the reasoner should normally be capable, when asked to do so, to assign consistently relative strengths to the assertions used, it is just natural to assume as we do, first, that assertions have each an associated, measurable strength, and that, second, this strength has significant and measurable effects on the truth of the sentences, the validity of the conclusion and the soundness of the reasoning. That this strength is computable and that it behaves as an *additive valuation* is accounted for by the condition generally assumed to apply: the reasoner behaves in a rational fashion, as classically characterized by Ramsey and De Finetti (though non-rational behavior can be also modelized [17]).

1 "Truth" valuations over sentences

We only need to assume that we have a set $\mathcal{L}$ of sentences that form a Boolean algebra (with respect to the three $\wedge$, $\vee$ and $\neg$ connectives and two special sentences $\bot$ and $\top$). We then have a complete Proof Theory by identifying the " $\vdash$ " order defined by the Boolean algebra with the deductive consequence relation. So the algebra of sentences we started with automatically becomes the Lindenbaum-Tarski algebra of all sentences modulo the interderivability relation " $\dashv\vdash$ " given by the $\vdash$ order (i.e. $A \dashv\vdash B$ iff $A = B$). We then only need to assume, further, that all sentences are valued in $[0, 1]$, which can be done in the standard way of a *normalized measure* $v : \mathcal{L} \longrightarrow [0, 1] : A \mapsto [\![A]\!]$, by just requiring that $\top$ gets a value of 1 and that the valuation v is *additive* (i.e. $[\![A \vee B]\!] = [\![A]\!] + [\![B]\!] - [\![A \wedge B]\!]$). We then have also the whole Model Theory of Sentential Logic.

This "truth" valuation is no more nor less than a (finitely aditive) *probability* in all technical senses, but we definitely want to avoid the usual probabilistic connotations and not be distracted away from pure, plain Logic. Thus, for instance, A will be plainly a *sentence* (in a language $\mathcal{L}$), not an *event* (in a sample space Ω), and $[\![A]\!] = 1$, 0 and $1/2$ will respectively mean just *truth* (or, more precisely, that "A is taken as truth"), *falsity* and *undecided belief* when expressly *asserting* the sentence A, clearly not the same as (respectively) probabilistic "certainty", zero-probability or balanced odds when evaluating the *uncertain* outcome of A. The presumed Booleanity of the sentences is either assumed (which is undemanding) or it just arises naturally from a "minimal algebra" of sentences as suggested by Popper [12] or from provably equivalent simple assumptions (e.g. by Cox). Also, the additive character of the valuation amounts to having a 'rational' (Ramsey) or 'coherent' (De Finetti) *belief*, a concept so akin to 'strength of assertion' in Logic as to be all but exchangeable.

From the Booleanity of $\mathcal{L}$ and the above properties of the v valuation we immediately obtain:

$$[\![\neg A]\!] = 1 - [\![A]\!] \qquad (1)$$

$$[\![A \wedge B]\!] \leq \min([\![A]\!], [\![B]\!]) \qquad (2)$$

$$[\![A \vee B]\!] \geq \max([\![A]\!], [\![B]\!])$$

$$[\![A \rightarrow B]\!] = 1 - [\![A]\!] + [\![A \wedge B]\!]$$

$$[\![A \leftrightarrow B]\!] = 1 - [\![A \vee B]\!] + [\![A \wedge B]\!]$$

2 Sentences as set extensions, and truth as measure

Any Boolean Algebra has a *representation* on a set structure (a field of sets) as Stone proved long ago in a famous theorem (see e.g.[8]). Thus, given the Boolean sentence algebra $\mathcal{L}$, there exist both a set Θ (whatever the meaning we give to its elements θ) *and* a 'representation' function that can be characterized as an isomorphism of $\mathcal{L}$ into the Boolean subalgebra $\mathcal{B}$ of clopens in $\mathcal{P}(\Theta)$, i.e.

$$\rho : \mathcal{L} \longleftrightarrow \mathcal{B} : A \mapsto \mathbf{A} \quad (\mathcal{B} \subset \mathcal{P}(\Theta), \mathbf{A} \subset \Theta)$$

We may indulge in calling the members of Θ *possible worlds*, or *cases* (as did Laplace or Boole [1]) or *possibilities* (as Shafer) or even *observers*, *states*, etc. Θ is thus the real *universe of discourse* or *reference frame* (the set of *possible worlds*). It also coincides with Fenstad's [4] *model space*.