Modus Ponens on Boolean Algebras Revisited

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Abstract

In a Boolean Algebra **B**, an inequality $f(x, x \to y)) \leq y$, satisfying the condition $f(1,1) = 1$, is considered for defining operations $a \to b$ among the elements of **B**. These operations are called "Conditionals" for f. In this paper, we obtain all the boolean Conditionals and Internal Conditionals, and some of their properties as, for example, monotonicity are briefly discussed.

Keywords. Modus Ponens, Intus Ponendo Ponens, Conditionals, Internal Conditionals, Strict Conjunctions, Monotonicity.

1 Introduction

Let it **B** a Boolean Algebra with complement ('), intersection ($\cdot$), disjunction (+), greatest element (**1**) and lower element (**0**). The Modus Ponendo Ponens Meta-Rule (for short, Modus Ponens or MP) is usually writen as the scheme

$$\frac{\begin{array}{l}\text{If a, then b}\\ \text{a}\end{array}}{\text{b},}$$

and it is called a Meta-Rule ([1]) for it is not essentially captured by the internal operations on **B**. In fact, if for some function $f : \mathbf{B}\times\mathbf{B} \to \mathbf{B}$ one supposes $f(a, a \to b) = b$, where $a \to b$ is an operation on **B** representig "If a, then b", then $f(\mathbf{0}, \mathbf{0} \to b) = b$ and, under the no usual hypothesis $\mathbf{0} \to b = \mathbf{1}$, it follows that $f(\mathbf{0}, \mathbf{1}) = b$, for any $b \in \mathbf{B}$. In particular, there is no boolean binary operation that allows to obtain b from a and $a \to b$. The fact that, inside **B**, starting from a and $a \to b$, b can never be reached by means of elementary boolean operations seems to indicate the lack of a rational path from a and $a \to b$ towards b.

Notwithstanding, as MP is frequently used in the sense that:

$$\frac{\begin{array}{l} a \to b \text{ is true} \\ a \text{ is true} \end{array}}{b \text{ is true,}}$$

an inequality $f(a,(a \to b)) \leq b$, provided that $f(\mathbf{1},\mathbf{1}\ ) = \mathbf{1}$, could be considered ([2]) for defining the operations $a \to b$ among the elements of $\mathbf{B}$ representing "If a, then b" and called "Conditionals", as if $a = \mathbf{1}$ (a is true) and $a \to b = \mathbf{1}$ ($a \to b$ is true), then $f(\mathbf{1},\mathbf{1}) = \mathbf{1} \leq b$, or $b = \mathbf{1}$ (b is true). Such inequality is not affected by the case $a = \mathbf{0}$, as $f(\mathbf{0},\mathbf{1}) \leq b$ for any $b \in \mathbf{B}$ only implies $f(\mathbf{0},\mathbf{1}) = \mathbf{0}$.

As it will be shown the only binary (boolean) operations $a \to b$ which, under the condition $f(a,b) = a \cdot b$, verify the inequality $a \cdot (a \to b) \leq b$ are:

$$a \to b = a',\ b,\ a \cdot b,\ a' + b,\ a' \cdot b,\ a' \cdot b',\ a \cdot b + a' \cdot b',$$

and among them only $a' + b$ and a' satisfy $\mathbf{0} \to b = \mathbf{1}$. The first, $a \to b = a' + b$ verifies

$$a \cdot (a' + b) = a \cdot b \leq b,$$

that is the inequality is reached through the equality $f(a, a \to b) = a \cdot b$, and it should be pointed out that, avoiding the property $\mathbf{0} \to b = \mathbf{1}$, it is also the same case with $a \to b = b, a \cdot b, a \cdot b + a' \cdot b'$, but it is not the case with $a \to b = a' \cdot b', a', a' \cdot b$.

It should also be noticed that with $a \to b = b$, $a \cdot b + a' \cdot b'$, there is no essential contradiction against the existence of some function f such that $f(a, a \to b) = b$. The contradiction appears with $a \to b = a \cdot b$, $a' + b$, a', as $f(a, a \cdot b) = b$ and $f(a, a' + b) = f(a, a') = b$ implies, taking $a = \mathbf{0}$, $f(\mathbf{0},\mathbf{0}) = b$ and $f(\mathbf{0},\mathbf{1}) = b$, for any $b \in \mathbf{B}$. Another contradiction appears with $a \to b = a' \cdot b'$, $a' \cdot b$, as taking $a = \mathbf{1}$, it is $f(\mathbf{1},\mathbf{1} \to b) = f(\mathbf{1},\mathbf{0}) = b$, for any $b \in \mathbf{B}$.

2 MP-functions and Conditionals

Definition. Given non-constant functions $f : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$, such that $f(\mathbf{1},\mathbf{1}) = \mathbf{1}$, and $\to : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$, the function f is called a MP-function and the function $\to$ an f-Conditional, provided that

$$f(x, x \to y) \leq y, \text{ for any } x, y \text{ in } \mathbf{B}.$$

When this inequality is reached through the equation $f(x, x \to y) = x \cdot y$, f is called a Modus Intus Ponendo Ponens-function, or for short, MIP-function and $\to$

a f-Internal Conditional.

Of course, any f-Internal Conditional is a Conditional, but the reverse is not always the case as it was shown in the Introduction's example. It is obvious that if f is a MP-function and $g \leq f$, it is also g a MP- function, and if f is a MIP-function, g is a MP-function. It is an interesting problem, given a f-Conditional $\rightarrow$ to search for the greatest MP-function g for which $\rightarrow$ is a g-Conditional, if it exists.

From now on, we will consider only MP-functions f that are boolean, that is, such that

$$f(x,y) = \alpha \cdot x \cdot y + \beta \cdot x' \cdot y + \gamma \cdot x \cdot y' + \delta \cdot x' \cdot y',$$

for $\alpha, \beta, \gamma, \delta \in \{\mathbf{0}, \mathbf{1}\} \subset \mathbf{B}$.

Example 1. If $a \rightarrow b = a' + b$, it is $f(a, a' + b) = \alpha \cdot a \cdot b + \beta \cdot a' + \gamma \cdot a \cdot b'$, and then:

• $f(a, a' + b) = a \cdot b$ iff $\alpha = \mathbf{1}, \beta = \gamma = \mathbf{0}$, or $f(x,y) = x \cdot y + \delta \cdot x' \cdot y'$

Consequently, for $a \rightarrow b = a' + b$ there are only two MIP-functions: $f(x,y) = x \cdot y$ and $g(x,y) = x \cdot y + x' \cdot y'$. As it is $f \leq g$, g is the greatest for $a \rightarrow b = a' + b$.

• $f(a, a' + b) \leq b$ iff $\alpha \cdot a \cdot b + \beta \cdot a' + \gamma \cdot a \cdot b' \leq b$ iff $\beta \cdot a' \cdot b' + \gamma \cdot a \cdot b' = \mathbf{0}$ iff $\beta = \gamma = \mathbf{0}$.

Consequently, $f(x,y) = \alpha \cdot x \cdot y + \delta \cdot x' \cdot y'$ and the possible MP-functions are $f(x,y) = x \cdot y$, $k(x,y) = x' \cdot y'$ and $g(x,y) = x \cdot y + x' \cdot y'$. But k is not valid as $k(\mathbf{1}, \mathbf{1}) = \mathbf{1}' \cdot \mathbf{1}' = \mathbf{0}$. The only MP-functions are f and g, and in this case there is no difference between MP and MIP-functions.

It should be noticed that if $a \rightarrow b = a \cdot b$, then $f(a, a \cdot b) = \alpha \cdot a \cdot b + \gamma \cdot a \cdot b' + \delta \cdot a'$, and a similar conclusion follows.

Example 2. If $a \rightarrow b = a \cdot b + a' \cdot b'$, it is

$$f(a, a \cdot b + a' \cdot b') = \alpha \cdot a \cdot b + \beta \cdot a' \cdot b' + \gamma \cdot a \cdot b' + \delta \cdot a' \cdot b$$

and then:

• $f(a, a \cdot b + a' \cdot b') = a \cdot b$ iff $\alpha = \mathbf{1}$, $\beta = \gamma = \delta = \mathbf{0}$, or $f(x,y) = x \cdot y$ is the only MIP- function for $a \rightarrow b = a \cdot b + a' \cdot b'$.

• $f(a, a \cdot b + a' \cdot b') \leq b$ iff $\beta \cdot a' \cdot b' + \gamma \cdot a \cdot b' = \mathbf{0}$ iff $\beta = \gamma = \mathbf{0}$, or $f(x,y) = \alpha \cdot x \cdot y + \delta \cdot x' \cdot y'$, and $f(x,y) = x \cdot y$ and $g(x,y) = x \cdot y + x' \cdot y'$ are the only two MP-functions for $a \rightarrow b = a \cdot b + a' \cdot b'$, and only one of them is a MIP-function.

Example 3. Let's find the internal conditionals $x \to y$ corresponding to the function $f(x,y) = x{+}y$, if any. It should be $f(a, a \to b) = a{\cdot}b$ or $a{+}(a \to b) = a{\cdot}b$, but as intersecting with a', it follows

$$a' \cdot a + a' \cdot (a \to b) = a' \cdot (a \cdot b) = \mathbf{0},$$

this last equation implies $a' \cdot (a \to b) = \mathbf{0}$, or $a \to b \leq a$. Consequently, this is a necessary but not sufficient condition for $a \to b$ being an Internal Conditional relative to $f(x,y) = x + y$.

As the boolean conditionals can be written $a \to b = \alpha \cdot a \cdot b + \beta \cdot a' \cdot b + \gamma \cdot a \cdot b' + \delta \cdot a' \cdot b'$, with $\alpha, \beta, \gamma, \delta \in \{\mathbf{0}, \mathbf{1}\} \subset E$, from the condition it follows

$$\beta \cdot a' \cdot b + \delta \cdot a' \cdot b' = \mathbf{0}, \quad \text{or} \quad \beta = \delta = \mathbf{0},$$

and the possible Internal Conditionals are among those that are $a \to b = \alpha \cdot a \cdot b + \gamma \cdot a \cdot b'$. Excluding the case $\alpha = \gamma = \mathbf{0}$, they are

$$a \to b = a \cdot b, \ a \cdot b', \ a,$$

and no one of them satisfies $a + (a \to b) = a \cdot b$. Then, relatively to $f(x,y) = x + y$ there are no boolean Internal Conditionals.

Then, are there conditionals? If there are, they should verify $a + (a \to b) \leq b$, and a necessary condition for this is $b' \cdot (a + (a \to b)) = \mathbf{0}$, or $b' \cdot (a + \alpha \cdot a \cdot b + \beta \cdot a' \cdot b + \gamma \cdot a \cdot b' + \delta \cdot a' \cdot b') = \mathbf{0}$, equivalent to $a \cdot b' + \gamma \cdot a \cdot b' + \delta \cdot a' \cdot b' = \mathbf{0}$, never verified for all the a, b in E. We can conclude that there are no boolean conditionals $x \to y$ for $f(x,y) = x + y$.

Example 4. Let's analyze the general boolean case when $\mathbf{0} \to y = \mathbf{1}$, for any y in $\mathbf{B}$. It should be

$$f(x,y) = A \cdot x \cdot y + B \cdot x' \cdot y + C \cdot x \cdot y' + D \cdot x' \cdot y'$$

$$x \to y = \alpha \cdot x \cdot y + \beta \cdot x' \cdot y + \gamma \cdot x \cdot y' + \delta \cdot x' \cdot y',$$

with A, B, C, D and $\alpha, \beta, \gamma, \delta$ in $\{\mathbf{0},\mathbf{1}\} \subset E$. But, being $f(\mathbf{1},\mathbf{1}) = A$, it should be $A = \mathbf{1}$, and $f(x,y) = x \cdot y + B \cdot x' \cdot y + C \cdot x \cdot y' + D \cdot x' \cdot y'$. Then:

$$f(a, a \to b) = a \cdot (a \to b) + B \cdot a' \cdot (a \to b) + C \cdot a \cdot (a \to b)' + D \cdot a' \cdot (a \to b)' \leq b$$

To have $\mathbf{1} = \mathbf{0} \to y = \beta \cdot y + \delta \cdot y'$ it should be $\beta = \delta = \mathbf{1}$, and

$$a \to b = \alpha \cdot a \cdot b + a' \cdot b + \gamma \cdot a \cdot b' + a' \cdot b' = \alpha \cdot a \cdot b + a' + \gamma \cdot a \cdot b',$$

$$(a \to b)' = (\alpha' + a' + b') \cdot a \cdot (\gamma' + a' + b).$$

Then:

$$f(a, a \to b) = (\alpha \cdot a \cdot b + \gamma \cdot a \cdot b') + B \cdot a' + C \cdot (\alpha' + b') \cdot a \cdot (\gamma' + b) + D \cdot \mathbf{0} =$$

$$(\alpha \cdot a \cdot b + \gamma \cdot a \cdot b') + B \cdot a' + C \cdot (\alpha' + b') \cdot (\gamma' + b) \cdot a \leq b.$$

As $x \leq y$ and $x \cdot y' = \mathbf{0}$ are equivalent,

$$(\alpha \cdot a \cdot b + \gamma \cdot a \cdot b') \cdot b' + B \cdot a' \cdot b' + C \cdot (\alpha' + b') \cdot (\gamma' + b) \cdot a \cdot b' = \mathbf{0},$$

or

$$\gamma \cdot a \cdot b' + B \cdot a' \cdot b' + C \cdot \gamma' \cdot a \cdot b' = \mathbf{0}.$$

Consequently: $\gamma = \mathbf{0}$, $B = \mathbf{0}$, $C = \mathbf{0}$, and the possible f are $f(x, y) = x \cdot y + D \cdot x' \cdot y'$, with $x \to y = \alpha \cdot x \cdot y + x' \cdot y + x' \cdot y' = \alpha \cdot x \cdot y + x'$. The solutions are:
1. $f(x, y) = x \cdot y, \quad x \to y = x'$ (not internal conditional)
2. $f(x, y) = x \cdot y, \quad x \to y = x' + y$ (the usual case, internal conditional)
3. $f(x, y) = x \cdot y + x' \cdot y', \quad x \to y = x'$ (not internal conditional)
4. $f(x, y) = x \cdot y + x' \cdot y', \quad x \to y = x' + y$ (the usual case, internal conditional).

Then, of four cases, two are not internal conditionals , two are internal conditionals and the usual conditional (the material conditional $x \to y = x' + y$) admits two different MIP- functions.

Definition. An operation $*$ on **B** is called **monotonic** ([3]) if, for any x, y and any c in E, it holds the inequality $x * y \leq (x \cdot c) * y$. For example, $\cdot$ is not monotonic.

Then a conditional $\to$ is monotonic when $x \to y \leq x \cdot c \to y$, for any x, y, c in E.

The afore mentioned four conditionals are monotonic as they have the form $x \to y = x' + h(y)$, and because of this:

$$x \to y = x' + h(y) \leq x' + c' + h(y) = (x \cdot c)' + h(y) = (x \cdot c) \to y.$$

3 The General Boolean Case

Let's now analyze the general case without the constraint $\mathbf{0} \to y = \mathbf{1}$. From

$$f(x, y) = x \cdot y + B \cdot x' \cdot y + C \cdot x \cdot y' + D \cdot x' \cdot y'$$

$$x \to y = \alpha \cdot x \cdot y + \beta \cdot x' \cdot y + \gamma \cdot x \cdot y' + \delta \cdot x' \cdot y',$$

it follows

$$f(a, a \to b) = a \cdot (\alpha \cdot a \cdot b + \beta \cdot a' \cdot b + \gamma \cdot a \cdot b' + \delta \cdot a' \cdot b') +$$

$B\cdot a'\cdot(\alpha\cdot a\cdot b+\beta\cdot a'\cdot b+\gamma\cdot a\cdot b'+\delta\cdot a'\cdot b')+C\cdot a\cdot(\alpha'+a'+b')\cdot(\beta'+a+b')\cdot(\gamma'+a'+b)\cdot(\delta'+a+b)+$

$$D \cdot a' \cdot (\alpha' + a' + b') \cdot (\beta' + a + b') \cdot (\gamma' + a' + b) \cdot (\delta' + a + b) =$$

$a\cdot(\alpha\cdot b+\gamma\cdot b')+B\cdot a'\cdot(\beta\cdot b+\delta\cdot b')+C\cdot a\cdot(\alpha'+b')\cdot(\gamma'+b)+D\cdot a'\cdot(\beta'+b')\cdot(\delta'+b) \leq b.$

Then, intersecting with b' it follows:

$$\gamma \cdot a \cdot b' + B \cdot \delta \cdot a' \cdot b' + C \cdot a \cdot b' \cdot (\alpha + b') \cdot (\gamma' + b) + D \cdot a' \cdot b' \cdot (\beta' + b') \cdot (\delta' + b) = \mathbf{0}$$

$$\gamma \cdot a \cdot b' + B \cdot \delta \cdot a' \cdot b' + C \cdot a \cdot b' \cdot \gamma' + D \cdot a' \cdot b' \cdot \delta' = \mathbf{0},$$

$$(\gamma + C \cdot \gamma') \cdot a \cdot b' + (B \cdot \delta + D \cdot \delta') \cdot a' \cdot b' = \mathbf{0},$$

that holds if and only if $\gamma = \mathbf{0}$ & $C = \mathbf{0}$ & $B \cdot \delta = \mathbf{0}$ & $D \cdot \delta' = \mathbf{0}$.

Then, starting from a boolean operation $x \to y = \alpha \cdot x \cdot y + \beta \cdot x' \cdot y + \delta \cdot x' \cdot y'$, to obtain the MP-functions for $\to$, if $\delta = \mathbf{0}$ it should be $D = \mathbf{0}$, and if $\delta = \mathbf{1}$ it should be $B = \mathbf{0}$. So, for each $\to$ there are exactly two MP-functions. For example, if $x \to y = x \cdot y$, it is $\delta = \mathbf{0}$, $D = \mathbf{0}$, and $B \in \{\mathbf{0}, \mathbf{1}\}$, and the corresponding two MP-functions are $f(x, y) = x \cdot y$, and $f(x, y) = x \cdot y + x' \cdot y = y$.

The following table lists all the possible cases, pointing out if it is $\mathbf{0} \to b = \mathbf{1}$, if monotonicity holds, and if the conditionality is internal.

α	β	δ	$x \to y$	$\mathbf{0} \to y = \mathbf{1}$	monotonic	$f(x,y)$	internal
1	**0**	**0**	$x \cdot y$	Not	Not	y	Yes
						$x \cdot y$	Yes
0	**1**	**0**	$x' \cdot y$	Not	Yes	y	Not
						$x \cdot y$	Not
0	**0**	**1**	$x' \cdot y'$	Not	Yes	$x \cdot y + x' \cdot y'$	Not
						$x \cdot y$	Not
1	**1**	**0**	$x \cdot y + x' \cdot y = y$	Not	Yes	y	Not
						$x \cdot y$	Yes
1	**0**	**1**	$x \cdot y + x' \cdot y'$	Not	Not	$x \cdot y + x' \cdot y'$	Not
						$x \cdot y$	Yes
0	**1**	**1**	$x' \cdot y + x' \cdot y' = x'$	Yes	Yes	$x \cdot y + x' \cdot y'$	Not
						$x \cdot y$	Not
1	**1**	**1**	$x' + y$	Yes	Yes	$x \cdot y + x' \cdot y'$	Yes
						$x \cdot y$	Yes

4 Strict Conjunctions in Boolean Algebras

In a Boolean Algebra $(\mathbf{B}, \cdot, +)$, an internal operation $*$ is a strict conjunction ([4]) if for all x, y in $\mathbf{B}$ it holds:

$$x \leq y \;\; if\ and\ only\ if \;\; \exists \;\; r(x,y) \;\; in\ \mathbf{B}\ such\ that\ y * r = x$$

We will try to search the MP-functions

$$f(x,y) = A \cdot x \cdot y + B \cdot x' \cdot y + C \cdot x \cdot y' + D \cdot x' \cdot y',$$

which are strict conjunctions, that is, such that

$$a \leq b \;\; iff \;\; \exists\, r \;\; such\ that \;\; (b,r) = A \cdot b \cdot r + B \cdot b' \cdot r + C \cdot b \cdot r' + D \cdot b' \cdot r' = a$$

By the sufficient condition it should be, for any b, r in $\mathbf{B}$:

$$A \cdot b \cdot r + B \cdot b' \cdot r + C \cdot b \cdot r' + D \cdot b' \cdot r' \leq b, \;\; or \;\; B \cdot b' \cdot r + D \cdot b' \cdot r' = \mathbf{0}.$$

So $B = D = \mathbf{0}$, and it will be

$$f(x,y) = A \cdot x \cdot y + C \cdot x \cdot y' \;, \;\; with \;\; A, C \in \{\mathbf{0}, \mathbf{1}\}.$$

Furthermore, if $A = C = \mathbf{1}$, $f(x,y) = x \cdot y + x \cdot y' = x$. Obviously, if $a < b$ there is not r such that $f(b,r) = b = a$. On the other hand, if $A = C = \mathbf{0}$, it is $f(x,y) = \mathbf{0}$, which also is not a strict conjunction.

So, only $f(x,y) = x \cdot y$ and $f(x,y) = x \cdot y'$ can be strict conjunctions, but the second one is not a MP-function.

Taking $f(x,y) = x \cdot y$, if $a \leq b$, there is $r(a,b) = a$ such that $f(b,r) = b \cdot r = b \cdot a = a$, and, on the other hand, if there is r such that $f(b,r) = b \cdot r = a$, it is $a \leq b$. So $f(x,y) = x \cdot y$ is in fact, a strict conjunction.

Which are the boolean $r(x,y)$ verifying the conditions? Let's see it. If $a \leq b$, we search $\alpha, \beta, \gamma, \delta$ in $\{\mathbf{0}, \mathbf{1}\}$ such that

$$f(b, r(a,b)) = f(b, \alpha \cdot a \cdot b + \beta \cdot a' \cdot b + \gamma \cdot a \cdot b' + \delta \cdot a' \cdot b') = a$$

$$b \cdot (\alpha \cdot a \cdot b + \beta \cdot a' \cdot b + \gamma \cdot a \cdot b' + \delta \cdot a' \cdot b') = a$$

$$\alpha \cdot a \cdot b + \beta \cdot a' \cdot b = a$$

Intersecting with a': $\beta \cdot a' \cdot b = \mathbf{0}$, for all $a \leq b$, and $\beta = \mathbf{0}$. So, $\alpha \cdot a \cdot b = a$ and $\alpha = \mathbf{1}$. The possible cases are:

$$r(a,b) = a \cdot b = a$$

$$r(a,b) = a \cdot b + a \cdot b' = a$$

$$r(a,b) = a \cdot b + a' \cdot b' = a + a' \cdot b' = a + b'$$

$$r(a,b) = a \cdot b + a \cdot b' + a' \cdot b' = a + a' \cdot b' = a + b'$$

Then, $r(x,y) = x$ or $r(x,y) = x + y'$.

Finally, as we know that for $f(x,y) = x \cdot y$ the boolean operations

$$x \to y = \alpha \cdot x \cdot y + \beta \cdot x' \cdot y + \gamma \cdot x \cdot y' + \delta \cdot x' \cdot y'$$

such that $f(x, x \to y) \leq y$ for any x, y in $\mathbf{B}$, are $x \to y = \alpha \cdot x \cdot y + \beta \cdot x' \cdot y + \delta \cdot x' \cdot y'$, with $\alpha, \beta, \delta \in \{\mathbf{0}, \mathbf{1}\}$, it follows:

$$x \to y = y + x';\ y;\ x \cdot y + x' \cdot y';\ x';\ x \cdot y;\ x' \cdot y;\ x' \cdot y'.$$

Conclusions

We have obtained all the boolean Conditionals and all the Internal Conditionals $\to$ with respect to some boolean functions f satisfying $f(\mathbf{1}, \mathbf{1}) = \mathbf{1}$ which are called, respectively, MP-functions and MIP-functions. There are seven Conditionals, only two of them verify $\mathbf{0} \to y = \mathbf{1}$ for any y, and each of them is a f-Conditional for two functions f.
Furthermore, the only MP-function being a strict conjunction was obtained.

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