Fuzzy Quantum Logics

M.L. Dalla Chiara & R. Giuntini
Dipartimento di Filosofia,
via Bolognese 52, 50139, Firenze, Italy
e-mail: dachiara@mailserver.idg.fi.cnr.it
e-mail: giuntini@philos.unifi.it

Abstract

Łukasiewicz quantum logic (ŁQL) is a particular example of a fuzzy quantum logic. ŁQL is semantically characterized by the class of all quantum MV algebras. The standard quantum MV algebra is based on the set of all *effects* in a Hilbert space. From the physical point of view, effects represent physical properties that may be noisy and ambiguous.

1 Introduction

The research on fuzzy quantum logics can be described as a kind of "crossing point" between different fields. The first field is represented by Łukasiewicz many valued logics and Zadeh fuzzy logics. The second one corresponds to the unsharp approaches to quantum mechanics, first proposed by Ludwig and further developed by a number of physicists, mathematicians and logicians (Mittelstaedt, Kraus, Davies, Busch, Cattaneo, ...). In a sense, one might say that fuzzy logics are "a place where Ludwig has met Łukasiewicz and Zadeh".

In this paper we will be concerned with a particular form of fuzzy quantum logic, called Łukasiewicz quantum logic (that has been first studied in [?] and [?]).

2 Effects and the standard quantum MV algebra

Łukasiewicz many-valued logics ([?], [?]) represent generalizations of classical logic, where the basic connectives *and* and *or* have been split into two forms of conjunction and disjunction. The first kind of conjunction is non idempotent: generally, a repeated assertion "A and A" is not equivalent to a simple "A" (*Repetita iuvant!*). Similarly for the *or*. The second kind of conjunction and disjunction have a lattice behaviour. Hence they are idempotent, commutative and associative. The standard semantic model of Łukasiewicz infinite many valued logic is based on the [0,1]-interval:

$$\mathcal{M}_{[0,1]} = \langle [0,1] \, , \oplus \, ,^{*} \, , \mathbf{0} \, , \mathbf{1} \rangle \, , \tag{2.1}$$

where

(i) $\mathbf{1} = 1; \;\; \mathbf{0} = 0.$

(ii) $a^* = \mathbf{1} - a.$

(∗ corresponds to the negation).

(iii) $$a \oplus b = \begin{cases} a+b, & \text{if } a+b \leq 1 \\ \mathbf{1}, & \text{otherwise} \end{cases}$$

(in other words, ⊕ represents the *truncated sum*).

On this basis the following operations and relation are defined:

$$a \odot b = (a^* \oplus b^*)^* \tag{2.2}$$

$$ab = (a \oplus b^*) \odot b \tag{2.3}$$

$$ab = (a \odot b^*) \oplus b \tag{2.4}$$

$$a \sqsubseteq b \text{ iff } ab = a \tag{2.5}$$

One can easily verify that ⊕, ⊙ are not idempotent, whereas , represent respectively the *inf* and the *sup* operations in the linearly ordered lattice $\langle [0,1], \sqsubseteq \rangle$ (where $\sqsubseteq$ turns out to coincide with the natural order for the reals).

Our $[0,1]$-structure represents a particular example of an MV algebra: it is also called the *standard MV algebra*. Differently from the standard case, a generic MV algebra $\mathcal{M} = \langle M \, , \oplus \, ,^{*} \, , \mathbf{0} \, , \mathbf{1} \rangle$ is not necessarily linear: generally, the relation $\sqsubseteq$ is only a partial order. A Boolean algebra turns out to be a particular case of an MV algebra, where ⊕ is idempotent. As a consequence, the two disjunctions ⊕ and collapse into one and the same operation (similarly, the two conjunctions).

An interesting intuitive interpretation of the Łukasiewicz connectives has been proposed by Mundici [**?**]. The basic idea is the use of *Ulam games*, where players are supposed to lie a certain number of times (the number may be either determined or undetermined). Instead of lying players, one may also think of information sources that are disturbed by a certain noise.

Quantum MV algebras (introduced in Giuntini [**?**]) represent weakenings of MV algebras, where the second kind of conjunction and disjunction (and) do not have a lattice behaviour: in particular, and are, generally, non commutative.

The standard quantum MV algebra (shortly QMV algebra) is based on the set of all *effects* in a Hilbert space. From the intuitive point of view, effects represent a kind of maximal mathematical representative of the notion of physical property that is compatible with the statistical rules of quantum theory. Suppose a quantum physical system $\mathbf{S}$, to which a Hilbert space $\mathcal{H}$ has been associated. Following the standard formalism of the theory, the possible states that our system $\mathbf{S}$ may assume correspond to statistical operators W. What about the mathematical representation of the physical properties that may hold for $\mathbf{S}$? According to von Neumann's axiomatization, a good mathematical counterpart for the intuitive notion of quantum property is the concept of *projection* operator in $\mathcal{H}$ (or, equivalently, the concept of closed subspace). In the unsharp approach, instead, quantum properties are represented as effects. An effect in $\mathcal{H}$ is any linear bounded operator E for which a Born probability can be defined. In other words, for any state W, the trace of the composed operator WE is a number that belongs to the interval $[0, 1]$:

$$\mathrm{Tr}(WE) \in [0,1] \tag{2.6}$$

From the physical point of view, this number represents the probability that our system $\mathbf{S}$ in state W satisfies the property represented by E.

Effects turn out to be generalizations of projections: any projection is an effect, but not the other way around. In the following $E(\mathcal{H})$ and $P(\mathcal{H})$ will represent the set of all effects and the set of all projections in $\mathcal{H}$ respectively. In the intended interpretation, effects correspond to quantum properties that may be disturbed by a certain noise; whereas projections are always sharp and non ambiguous. A characteristic difference between effects and projections is the following: a non null projection is always satisfied with certainty (i.e. with probability 1) by at least one state; on the contrary, there are non null effects that are semantically undetermined for any state (in other words, any state assign to them a probability value different from 0 and from 1). An extreme case is represented by the semitransparent effect $\frac{1}{2}\mathbb{I}$, to which any state assigns probability $\frac{1}{2}$. From the physical point of view, this kind of fuzziness may be regarded as depending on the accuracy of the measurement (which triggers the property), and also on the accuracy involved in the operational definitions for the physical quantities to which our property refers.

Following the $[0, 1]$ analogy, a similar algebraic structure can be naturally induced on the set $E(\mathcal{H})$ of all effects of $\mathcal{H}$. Let us consider the following structure that will represent our standard QMV algebra:

$$\mathcal{E}(\mathcal{H}) = \langle E(\mathcal{H}) \,, \oplus \,,^{*} \,, \mathbf{1} \,, \mathbf{0} \rangle \tag{2.7}$$

Here the operations $\oplus \,,^{*} \,, \mathbf{1} \,, \mathbf{0}$ of $\mathcal{E}(\mathcal{H})$ are defined as follows:

(i) $\mathbf{1} = \mathbb{I}$

(where $\mathbb{I}$ is the identity operator).

(ii) $\mathbf{0} = \mathbb{O}$

(where $\mathbb{O}$ is the null operator)

(iii) $\quad E \oplus F = \begin{cases} E+F, & \text{if } E+F \in E(\mathcal{H}); \\ \mathbb{1}, & \text{otherwise} \end{cases}$

(where $+$ is the usual operator-sum).

(iv) $\quad E^* = \mathbb{1} - E.$

The operations $\odot$, , and the relation $\sqsubseteq$ are defined like in the $[0,1]$-case. The relation $\sqsubseteq$ coincides with the usual effect-order relation $\leq$ (where, $E \leq F$ iff for any density operator W: $\mathrm{Tr}(WE) \leq \mathrm{Tr}(WF)$).

Our effect structure turns out to be "very close" to an MV algebra. However one of the MV axioms (the so called Łukasiewicz axiom: $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a)$ is here violated. As a consequence, the conjunction and the disjunction are no more commutative.

A quantum MV structure can be similarly induced also on the set $P(\mathcal{H})$ of all the orthogonal projections of $\mathcal{H}$. It is sufficient to put:

$$P \oplus Q := \begin{cases} P+Q, & \text{if } P+Q \in P(\mathcal{H}); \\ \mathbb{1}, & \text{otherwise.} \end{cases} \tag{2.8}$$

The operations *, $\mathbf{1}$, $\mathbf{0}$ will be defined like in the effect-case. Also here, the relation $\sqsubseteq$ turns out to coincide with the natural projection-order.

As is well known, differently from effects, projections have a lattice structure (with respect to the natural order). However, the *infimum* and the *supremum* operations do not coincide with and , which generally preserve their non commutativity.

A physical interpretation of the QMV operations in $\mathcal{E}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ has been studied in [**?**].

3 Axiomatizing QMV algebras

Similarly to MV algebras, also the notion of QMV algebra can be axiomatized ([**?**]).

Let us first consider a group of axioms (S1)-(S6) that concern both the QMV and the MV structures:

(S1) *Associativity*
$(a \oplus b) \oplus c = a \oplus (b \oplus c)$

(S2) *Commutativity*
$a \oplus b = b \oplus a$

(S3) *Excluded middle*
$a \oplus a^* = \mathbf{1}$

(S4) *Neutral element*
$a \oplus \mathbf{0} = a$

(S5) *Boundness*

$a \oplus \mathbf{1} = \mathbf{1}$

(S6) *Double negation*

$a^{**} = a$

The structures satisfying (S1)-(S6) have been called by Gudder ([**?**], *supplement algebras*.

Definition 3.1 *An MV-algebra is a supplement algebra* $\mathcal{M} = (M, \oplus, ^*, \mathbf{1}, \mathbf{0})$ *that further satisfies the Lukasiewicz axiom (MV):*

$$\textit{(MV)} \quad (a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$$

Definition 3.2 *A QMV-algebra is a supplement algebra* $\mathcal{M} = \langle M, \oplus, ^*, \mathbf{1}, \mathbf{0} \rangle$ *that further satisfies the following weakenings of the Lukasiewicz axiom:*

(QMV1) *Weak absorption*

$a(ba) = a$

(QMV2) *Weak associativity*

$(ab)c = (ab)(bc)$

(QMV3) *First weak distributivity*

$a \oplus (b(a \oplus c)^*) = (a \oplus b)\,(a \oplus (a \oplus c)^*)$

(QMV4) *Second weak distributivity*

$a \oplus (a^* b) = a \oplus b$

(QMV5) *Pseudo Dummett law*

$(a^* \oplus b)\,(b^* \oplus a) = \mathbf{1}$

Every MV algebra turns out to be a QMV algebra, but not the other way around.

Some examples

The standard MV algebra, based on the $[0, 1]$ interval is an example of a *linear* MV algebra.

Zadeh fuzzy sets give rise to concrete examples of MV algebras that are not, generally, linear. In a fuzzy set theoretical MV algebra the support $F(X)$ consists of all fuzzy subsets f of a set X ($f \in [0, 1]^X$), while the operations are defined as follows, $\forall f, g \in [0, 1]^X$ and $\forall x \in X$:

$$(f \oplus g)(x) = \text{Min}\{1, f(x) + g(x)\} \quad (3.1)$$

$$(f^*)(x) = 1 - f(x) \quad (3.2)$$

$$\mathbf{1}(x) = 1 \text{ and } \mathbf{0}(x) = 0 \quad (3.3)$$

According to an important theorem proved by Belluce ([?]), an MV algebra $\mathcal{M}$ is embeddable into an MV algebra of fuzzy sets iff $\mathcal{M}$ is *semisimple* (where an MV algebra is said to be semisimple iff the intersection of all its maximal ideals is $\{\mathbf{0}\}$).

One can easily check that our effect algebra $\mathcal{E}(\mathcal{H})$ is a QMV algebra. As we have seen, $\mathcal{E}(\mathcal{H})$ is not an MV algebra, since it violates (MV). However, $\mathcal{E}(\mathcal{H})$ contains a substructure that is isomorphic to the standard MV algebra: consider the restriction of $E(\mathcal{H})$ to the class of the multiples $\lambda\mathbb{I}$ of the identity operator $\mathbb{I}$, with $\lambda \in [0,1]$. Differently from the standard MV algebra, the standard QMV algebra $\mathcal{E}(\mathcal{H})$ is not linear. However, it satisfies the following weaker condition, which is called *quasi-linearity*:

$$\text{(QL)} \qquad a \not\sqsubseteq b \implies ab = b$$

Supplement algebras that satisfy the quasi linearity condition have been called *quantum involution algebras* (shortly *QI algebra*). One can easily show that the the class of all quantum involution algebras coincides with the class of all quasi linear QMV algebras.

Both the variety of MV and of QMV algebras determine, in a natural way, a corresponding logic, via an algebraic semantic characterization. The logic determined by the MV algebras is the well known Łukasiewicz infinite many valued logic L^{∞}, whose tautologies are all and only the sentences that receive truth value $\mathbf{1}$ in any semantic model based on a MV algebra. *Lukasiewicz quantum logic* is the logic that is similarly characterizd by the class of all QMV algebras.

4 Unsharp orthoalgebras

Quantum involution algebras turn out to have a strong relation with a class of partial structures that have been recently studied by many authors and that are called in the literature with different names. The interest in partial algebraic structures and in partial logics (which are generally not closed under the basic logical connectives) appears to be quite natural in the framework of quantum theory. In fact, one has often noticed that the structure of the *quantum events* cannot be adequately represented as closed under conjunction and disjunction. This is a natural consequence of the non existence of *joint distributions of strongly incompatible observables*. Suppose that α and β describe two strongly incompatible events (for instance: "the value for the spin in the x- direction is up"; "the value for the spin in the y- direction is down"). It is quite natural to regard the conjunction of α and β as meaningless, since it represents an experimentally non conceivable event.

Unharp ortholagebras are examples of partial algebraic structures where the basic operation is not always defined. When is defined for two alements a and b, we will write $\exists(ab)$.

Definition 4.1 *An unsharp orthoalgebra is a partial algebraic structure* $\mathcal{A} = \langle A\,,\,,\mathbf{1}\,,\mathbf{0}\rangle$*, where is a partial binary operation. The following conditions* *hold:*

(UOA1) *Weak commutativity*
$\exists(ab) \implies \exists(ba)$ *and* $ab = ba$.

(UOA2) *Weak associativity*
$[\exists(bc)$ *and* $\exists(a(bc))] \implies [\exists(ab)$ *and* $\exists((ab)c)$ *and* $a(bc) = (ab)c]$.

(UOA3) *Strong excluded middle*
For any a*, there is a unique* x *s.t.* $ax = \mathbf{1}$.

(UOA4) *Weak consistency*
$\exists(a\mathbf{1}) \implies a = \mathbf{0}$.

Unsharp orthoalgebras have been also called *effect algebras* (see [?]) or *weak orthoalgebras* (see [?]). One can prove that the concept of unsharp orthoalgebra is equivalent to the notion of *difference poset* (or D-poset), investigated by Kôpka and Chovanec [?], Dvurečenskij and Pulmannová ([?]).

A *sharp orthoalgebra* (or simply an *orthoalgebra*) satisfies besides conditions (UOA1)-(UOA3), the strong *noncontradiction principle*:

(UOA4) $\exists(aa) \implies a = \mathbf{0}$.

The set of all effects $E(\mathcal{H})$ can be naturally structured as an unsharp orthoalgebra $\langle E(\mathcal{H})\,,\,,\mathbf{1}\,,\mathbf{0}\rangle$, where

(i) $\exists(EF) \iff E + F \in E(\mathcal{H})$.
$\exists(EF) \implies EF = E + F$;

(ii) $\mathbf{0}$ and $\mathbf{1}$ are the null and the identity operators, respectively;

This structure is not a sharp orthoalgebra, since the strong contradiction principle is violated. For instance: the semitransparent operator $\frac{1}{2}\mathbb{I}$ is an effect (to which any state assignes probability $\frac{1}{2}$); further, $\exists(\frac{1}{2}\mathbb{I}\frac{1}{2}\mathbb{I})$ and $\frac{1}{2}\mathbb{I} \neq \mathbb{O}$.

Differently from $E(\mathcal{H})$, the set of the projections $P(\mathcal{H})$ gives rise to a sharp orthoalgebra. Hence, effects may be generally regarded as a kind of unsharp generalizations of projections.

The logic that can be associated to the class of all unsharp orthoalgebras is an example of a *partial quantum logic* (see [?]).

One can show that any unsharp ortholagebra can be extended to a total structure that is a quantum involution algebra.

Theorem 4.1 *Let* $\mathcal{A} = \langle A\,,\,,\mathbf{1}\,,\mathbf{0}\rangle$ *be any unsharp ortholagebra. Consider the following total structure* $\mathcal{A}^\circ := \langle A\,,\oplus\,,\mathbf{1}\,,\mathbf{0}\rangle$*, where*

$$a \oplus b = \begin{cases} ab, & \text{if } \exists(ab); \\ \mathbf{1}, & \text{otherwise} \end{cases} \tag{4.1}$$

Then $\mathcal{A}^\circ$ *is a quantum involution algebra* ([?]).

Gudder ([?]) has proved that $\mathcal{A}^\circ$ is the unique quantum involution algebra such that:

1) $\oplus$ extends ;

2) the extension preserves the order.

From the intuitive point of view, all this confirms that the definition of the truncated sum in $\mathcal{E}(\mathcal{H})$ represents an interesting choice.

5 Some strange properties of quantum MV algebras and some open problems

The variety of MV algebras satisfies some general properties, which render quite tractable the corresponding logic L^∞. The most important properties (proved by Chang [?], [?]) are the following:

I) Every MV algebra can be represented as a subdirect product of linear MV algebras.

II) Any equation that holds in the standard MV algebra holds in any linear MV algebra.

As a consequence one obtains that the logic L^∞ is semantically characterized by the standard MV algebra.
All this breaks down in the case of QMV algebras:

Theorem 5.1 *Not every QMV algebra can be represented as a subdirect product of quasi linear QMV algebras* ([?]).

As a consequence, Łukasiewicz quantum logic is not characterized by the class of all quasilinear QMV algebras.
Some open problems are the following:

- Is the logic of all quasilinear QMV algebras axiomatizable ?
- Is the logic of $\mathcal{E}(\mathcal{H})$ axiomatizable ?
- Are the logic of $\mathcal{E}(\mathcal{H})$ and the logic of all quasilinear QMV algebras the same logic ?

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