How to Make your Logic Fuzzy

(PRELIMINARY VERSION)

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1 Introduction and Motivating Examples

The aim of this paper is to provide a methodology for turning a known crisp logic into a fuzzy system. We require of the methodology that it be meaningful in general terms, using processes which are independent of the notion of fuzziness, and that it yield a considerable number of known fuzzy systems.

To appreciate the need for such a methodology, consider for example the modal propositional logic **K**, with one modality , and let us examine our options for turning it into a fuzzy system. This logic is complete for the crisp Kripke semantics. Kripke models have the form $= (S, R, a, h)$, where $S \neq$ is a set of possible worlds, $R \subseteq S \times S$ is a crisp binary relation, (of the form $R : S \times S \mapsto \{0, 1\}$), $a \in S$ is the actual world, and h is a binary function assigning to each $t \in S$ and each atomic q a crisp value $h(t, q) \in \{0, 1\}$.

h can be extended to all wffs in the usual way with the inductive evaluation of $h(t, A)$ being

$h(t, A) = 1$ iff for all y such that tRy we have $h(y, A) = 1$.

or

$h(t, A) =$ Inf $\{h(y, A) \mid tRy\}$.

We say A if $h(a, A) = 1$.

Let us try and turn this logic fuzzy!

Working intuitively, one may turn modal logic into a fuzzy modal logic in several ways [**?**]):

1. changing the function $h(t, q)$ into a fuzzy function $h^\sharp(t, q) \in [0, 1]$ (obtaining real number values);

2. changing the crisp relation R into a fuzzy one $R^\sharp : S^2 \mapsto [0, 1]$;

3. making $a \in S$ fuzzy.

4. any combination of the above.

Is there a methodology involved to the above or do we just go from logic to logic and make fuzzy whatever semantical component we find?

What if we use a different semantics for **K** and make fuzzy the functions involved in that semantics? Do we get yet another batch of fuzzy modal logics?

Clearly we could use some general principles.

The methodology we use is that of combining two logics together through the fibring of their semantics. There are general ways of doing that independent of the logics themselves, provided they satisfy some simple assumptions [**?**, **?**].

Let $_1$ and $_2$ be two logics, say for example that $_1$ is modal logic and $_2$ is intuitionistic logic. Our methodology allows for several methods of combinations.

1. One layer $_1(_2)$ allowing for substituting intuitionistic formulas for the atoms of .

2. One layer $_2(_1)$ allowing the substitution of modal formulas for the atoms of intuitionistic logic.

3. More fibred layers say $_1(_2(_1))$ and so on.

4. Full fibring $_{1,2} =_1 \otimes_2$.

Methods for obtaining semantics for such fibred logics from the semantics of the components were developed in [**?**] and [**?**] and [**?**, **?**]. The results are independent of the semantics employed, are logic independent, and yield a huge number of known combined logics, existing in the literature in the past thirty years.

In our particular example, let $Ł_\infty$ be Łukasiewicz infinite valued logic (with values in $[0, 1]$) and let us apply our fibring machinery to and $Ł_\infty$. We get the following:

1. The fibred semantics for $(Ł_\infty)$ is the fuzzy semantics with h fuzzy, R crisp.

2. The fibred semantics for $Ł_\infty()$ is the semantics with R fuzzy and h crisp.

3. The semantics for $Ł_\infty((Ł_\infty))$ is the semantics where both R and h are fuzzy.

So in short, when you ask me how to make your logic $_1$ fuzzy, I would answer—take a pure fuzzy logic $_2$ (e.g. $Ł_\infty$ or any other) and fibre it to $_1$ in different ways.

The rest of this section explains intuitively, via examples, how *fuzzling* (making fuzzy by fibring) is done and how it relates to the general theory of combining logics. Later sections will develop the formal machinery.

We begin by quickly motivating the notion of fibring. In many application areas there arises the formal need of combining two languages together. The most well known in applied logic is the use of temporal logic to describe and verify the temporal behaviour of systems. Here $_2$ is the language for describing a static system and $_1$ is a temporal language. By substituting sentences of $_2$ as 'atoms' within $_1$ we get to express temporal properties of the system. This is called 'temporalising' of $_2$, see [**?**, **?**]. Another well known family of examples are multimodal logics, such as logics of knowledge and belief, logics of action and dynamic logics.

Example 1. **[Fibring two modalities]** Let $_{1,2}$ be two modal propositional languages built on the same atoms, with modalities $_1$ and $_2$ respectively. Assume $_i$ is complete for the class of models $\mathcal{K}_i$. The models are of the form $= (S, R, a, h)$, where S is the set of possible worlds, $R \subseteq S^2$ is the accessibility relation, $A \in S$ is the actual world and h is the assignment function, associating with each $t \in S$ and atomic q a crisp value $h(t, q) \in \{0, 1\}$. We can assume the model satisfies the following:

$$S = \{x \mid aR^n x, \text{ for some } n\}$$

where xR^0y iff $x = y$ and, $xR^{n+1}y$ iff $\exists z(xRz \wedge zR^n y)$.

Satisfaction $tA, t \in S, A$ a wff, is defined in the traditional manner:

- tq iff $h(t, q) = 1$, for q atomic;
- $tA \wedge B$ iff tA and tB;
- $t \sim A$ iff $t \not A$;
- tA iff for all s such that tRs we have sA;
- tA iff for some s, tRs and sA;
- A iff aA;
- $\mathcal{K}A$ iff A for all $\in \mathcal{K}$.

The *level (1, 2)* fibred language $_{(1,2)} =_1 (_2)$ allows for wffs of the form $B = B_1(x_1/C_1, \ldots, x_n/C_n, y_1, \ldots, y_m)$ where $B_1(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a wff of $_1$. $C_1, \ldots, C_n$ are wffs of $_2$ and B is obtained by the simultaneous substitution in B_1 of C_i for x_i respectively.

Let $\alpha = (e_1, \ldots, e_n)$ be a sequence of alternating numbers from $\{1, 2\}$. We define the fibred language $_\alpha$ by induction.

$$_{(e)} =_e, e = 1, 2.$$
$$_{(e_1,\ldots,e_n)} =_{e_1} (_{(e_2,\ldots,e_n)}).$$

Let $_\infty = \bigcup_\alpha {}_\alpha$ denote the full fibred language.

We now define fibred semantics for $_\alpha$. A level (1, 2) fibred model has the form $= (S, R, a, h,)$ where $= (S, R, a, h)$ is a Kripke model of $\mathcal{K}_1$ and is a function on S giving for each $t \in S$ a model $(t) = (S_t, R_t, a_t, h_t)$ in $\mathcal{K}_2$.

We can assume that S and all the S_t are pairwise disjoint.

Satisfaction is defined in the usual manner, with the crucial fibring clause being the following:

$$w_i A \text{ iff}$$

1. $w \in S_t$ for some t, and $i = 2$ and for all $s \in S_t(wR_t s \to sA)$.
2. $w \in S$ and $i = 1$ and for all $s(wRs \to sA)$.
3. $w \in S$ and $i = 2$ and $a_{ti}A$.
4. Undefined,, when $w \in S_t$ and $i = 1$.

In other words, when, at a point $w \in S$, we want to evaluate ${}_2A$, we go to (t) and continue the evaluation.

A level $\alpha = (e_1, \ldots, e_n)$ fibred model is defined by induction. It has the form $(S, R, a, h,)$ where (S, R, a, h) is a model in $\mathcal{K}_{e_1}$ and for each $t \in S$, (t) is a model in $\mathcal{K}_{(e_2,\ldots,e_n)}$, i.e. a model of level $(e_2, \ldots, e_n)$.

Let $\mathcal{K}_\infty = \bigcup_\alpha \mathcal{K}_\alpha$.

The models of $\mathcal{K}_\alpha$ can provide semantics for ${}_\alpha$.

Example 2. [**Simplified fibred models**] Consider a model in $\mathcal{K}_{(1,2)}$. There is another way of looking at this fibred model. Since S and $S_t, t \in S$ are all pairwise disjoint, let ${}^* = (S^*, W_a, R^*, a, h^*, {}^*)$ be the model with:

$$\begin{array}{l} S^* = S \cup \bigcup_{t\in S} S_t \\ R^* = R \cup \bigcup_t R_t \\ W_a = \{a\} \cup \{a_t \mid t \in S\} \\ h^* = h \cup \bigcup_t h_t \\ {}^* : S^* \mapsto S^* \text{ be the function with} \\ {}^*(x) = \begin{cases} a_x, & x \in S \\ x & \text{otherwise} \end{cases} \end{array}$$

Consider a language * with the modalities , and a jump operator . Satisfaction is defined by

$$\begin{array}{l} xA \text{ iff } \forall y(xRy \to yA) \\ x \vdash A \text{ iff } \exists y(xRy \wedge yA) \\ xA \text{ iff } {}^*(x)A. \end{array}$$

Let A be a wff of the mixed language. Translate ${}_1$ as and ${}_2$ as . Let A^* be the translation then

$$aA \text{ iff } a^* A^*$$

where is satisfaction in $(S, R, a, h,)$ and * s satisfaction in $(S^*, W_a, R^*, a, h^*, {}^*)$.

The sets S and S_t can be retrieved by

$$\begin{array}{l} S = \{x \in S^* \mid aR^{*n}x, \text{ for some } n\} \\ S_t = \{x \in S^* \mid a_t R^{*n}x, \text{ for some } n\}. \end{array}$$

and the models (t) and can be retrieved by restricting R^* and h^* to S and S_t.

In fact, let $(S^*, R^*, a, h^*, ^*)$ be a Kripke model with * a function from S^* to S^*. We call this model a *simplified fibred model* (*SFM-models*) iff the following holds:

Let $W_a = \{a\} \cup \{y \mid \exists x \in S^*(y =^* (x)\}$

For $y \in W_a$, let $S_y = \{t \mid yR^{*n}t$, for some $n\}$.

Then

1. $y_1 \neq y_2 \rightarrow S_{y_1} \cap S_{y_2} =$
2. $^*(x) \neq x \wedge^* (y) \neq y \rightarrow^* (x) \neq^* (y)$.
3. $S^* = \bigcup_{y \in W_a} S_y$
4. An *SFM*-model is said to be of *depth* $n+1$ if for all x, $^*(^{*n}(x)) = F^{*n}(x)$, where $^{*0}(x) = x$.

Example 3. [**Motivating fuzzy values**] We now give a concrete example of an *SFM*-model of level 1. Figure **??** shows a $_1$ Kripke model.

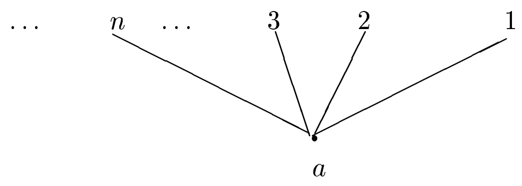

Figure 1:

Here $S = \{a\} \cup \{1, 2, 3, \ldots\}$ with aRn holding, for $n = 1, 2, \ldots$.

Assume $h(a, q) = 0$ and $h(n, q) = 1$ for $n = 1, 2, \ldots$. Try to evaluate $_{12}q$.

$a_{12}q$ iff for some n, n_2q. Since $_2$ is in the $_2$ language, we cannot continue to evaluate. We need an $_2$ model to get a value at n. The fibring function (n) gives an $_2$ model (S_n, R_n, a_n, h_n). Let $S_m = \{a_m\} \cup \{(m, n) \mid n = 1, 2, 3, \ldots\}$. Let R_m be defined by

$$xR_my \text{ iff } \begin{cases} x = a_m \\ \text{or} \\ x = (m, n_1) \text{ and } y = (m, n_2) \text{ and } n_2 \leq n_1 \end{cases}$$

and let

$$h_m(a_m, q) = 0$$

and

$$h_m((m, n), q) = 1 \text{ iff } m \leq n.$$

To complete the picture, let $(a) = (1)$. Thus ${}_2q$ is false at a_n in all the models (n), but we have

$$(m, n)_2 q \text{ iff } m \leq n.$$

This particular fibred model has a sepecial feature which is important. All the models (n), have isomorphic *frames*; they are isomorphic to $(T = \{0, 1, , , \ldots\}, \leq, 0)$, through π_m, where $\pi_m(a_m) = 0$, and $\pi_m(m, -n) = \frac{1}{n}$, and they differ only in the assignment $h_m(q)$. The image of the truth set $h_m({}_2q) = \{y \mid y_2 q\}$ is projected on $\{0, 1, \ldots, \frac{1}{n} \mid n = 1, 2, 3, \ldots\}$ gets larger and larger as m incrases. In the limit we have

$$\bigcup_m \pi_m h_m({}_2q) = \{\frac{1}{n} \mid n = 1, 2, \ldots\}.$$

Since we are interested in $a({}_1{}_2q)$, where the table for ${}_1$ is existential, we can say that ${}_1$ *almost* holds; it *approaches the* 'fuzzy' (or 'modal-${}_2$') truth set $\{\frac{1}{m} \mid m = 1, 2, \ldots\}$.

This is quite a conceptual jump. The model (S, R, a, h) is a model of ${}_1$ and has no business getting set values from the set T via the mappings π_m of the models of ${}_2$. However, since all the fibred models $(t), t \in S$ are based on isomorphic frames, we can extend the evaluation from the fibred models back into the ${}_1$ language.

It is important to note that the way we extended the evaluation from the fibred model to ${}_1$ of ${}_1$ was arbitrary. We chose a way of doing it which was reasonable, but nevertheless it was a choice. We could have said let us take as value for $a_{12}q$, not the union of $\pi_m h_m({}_2q)$ but the maximum or some other reasonable definition.

Having adopted a good definition, we now consider the expanded model $(S, R, a, h, , T, \pi)$.

We can define an ${}_2$-fuzzy value $\mu_t(A)$, for $t \in S$ and any A as follows:

- $\mu_t(A) = \pi_t h_t(A) = \{\pi_t(s) \mid s \in S_t \text{ and } sA\}$ for A in ${}_2$ or A atomic.
- $\mu_t(A \wedge B) = \mu_t(A) \cap \mu_t(B)$
- $\mu_t(\sim A) = T - \mu_t(A)$
- $\mu_t({}_1A) = \bigcup_{\{s \mid tRs\}} \mu_s(A)$
- $\mu_t({}_1A) = \bigcap_{\{s \mid tRs\}} \mu_s(A)$

What we have done can be best understood in algebraic terms. Let be the boolean algebra of the set $T = \{0, 1, , , \ldots\}$ with the interior operation Q^2, for $Q \subseteq T$ begin

$$Q^2 = \{x \in T \mid \text{ for all } yx, y \in Q\}.$$

Assign to each atom q and $t \in S$ the 'fuzzy' algebraic subset $\mu_t(q) \subseteq T$.

In our particular model we assign

$$\mu_n(q) = \{1, , \ldots, \frac{1}{n}\}$$
$$\mu_a(q) = \mu_1(q)$$

We extend the assignment by

- $\mu_t(_2A) = (\mu_t(A))^2$
- $\mu_t(_1A) = \bigcap_{\{s|tRs\}} \mu_s(A)$

The next example brings the idea forward even more clearly.

Example 4. [**Many valued modal logic**] This is an example of fibring semantical models (modal logic) with algebraic models (Łukasiewicz many-valued logic). We consider the modal language $_1$ with and the many valued language $_2$, with $\{\wedge, \vee, \to, \neg\}$ and with truth values at the real interval $[0,1]$. We study $_1(_2)$. The albebraic models of $_2$ are linearly ordered abelian groups which are embeddable in $[0,1]$. So it is sufficient to consider assignments μ of values and truth table for values in $[0,1]$. The following are the algebraic functions:

- The domain is $[0,1]$
- $\leq$ is numerical $\leq$.
- $\top = \{0\}$ (0 is truth).
- $\perp$ is 1 (1 is falsity).
- $f_\wedge(x,y) = \max(x,y)$.
- $f_\vee(x,y) = \min(x,y)$
- $f_\neg(x) = 1 - x$
- $f_\to(x,y) = \max(0, y - x)$.

We now turn to fibring.

Let $= (S, R, a, h)$ be a Kripke model for . The fibring function associates with each $t \in S$ an algebraic model $_t = (A_t, \leq, f_\wedge, f_\vee, f_\to, f_\neg, \{0\}, \mu_t)$. Since $A_t = [0,1]$, fibring algebras $_t$ to t is nothing more than associating with each t an arbitrary many-valued assignment μ_t to the atoms of the modal language.

Let us now evaluate $(q \to p), q, p$ atomic, at the model .

- $a(q \to p)$ iff for all $t \in S$ such that $aRt, tq \to p$.
- Since the main connective of $q \to p$ is many-valued, we have $tq \to p$ iff $_tq \to p$ iff $\mu_t(q \to p) = 0$ iff $\max(0, \mu_t(p) - \mu_t(q)) = 0$ iff $\mu_t(p) \leq \mu_t(q)$.

We would like to highlight a point which will be of importance later. Consider the above fibring. We start with $= (S, R, a, h)$. Then with each $t \in S$, we fibre an algebra $_t$. Since all the algebras have the same domain, the fibring reduces to μ_t, the assignment. Let us pause at this stage and consider the entity (S, R, a, h, μ) and let us try to evaluate tq. Since q contains no many-valued connectives, we get

tq holds iff $\forall s$ (tRs implies sq) iff $\forall s(tRs$ implies $h(t,q)=1)$. Consider the wff $Iq=$ def $(q \to q) \to q$. Really Iq is q but it is formally a many-valued wff. So we have to evalaute it at the algebra $_t$. We have $_tI(q)$ iff $\mu_t(q)=0$. $tI(q)$ iff for all $s(tRs$ implies $\mu_t(q)=0)$.

To summarise, consider tq; we have two ways of looking at it.

1. Regard 'q' as an atom of the modal language, in which case

$$tq \text{ iff for all } S, tRs \text{ implies } h(s,q)=1$$

2. Regard 'q' as an atom of the many-valued language, in which case

$$tq \text{ iff for all } s, tRs \text{ implies } \mu_s(q)=0.$$

The two evaluations need not give the same result.

We now have the opportunity to make tq fuzzy (i.e. 'fuzzle' the satisfaction , or in other words, 'fuzzle' the modal logic) by extending μ_t to q: ($\sharp$)

$\mu_t(q) = \mathrm{Sup}_{\{s|tRs\}}\mu_s(q)$. The reader should note that this definition is a chosen one and we could have chosen some other 'averaging' function.

Using ($\sharp$) we can now fuzzle any wff of the modal logic and extend μ_t to all wffs, by taking the many-valued table for $\wedge, \vee, \neg$ and $\to$. We have thus by understable intuitive definition, through ($\sharp$), turned (S,R,a,μ) into a sort of modal many-valued logic by changing the crisp {0,1} assignment h into a fuzzy μ. Note that what we are getting is not fibring, it is something new.

Example 5. [**Persistence**] This example will fibre modal logic to the intermediate logic Dummett's **LC**. It will serve to prepare the ground for fibring in the presence of persistence. Let be intuitionistic implication. **LC** is the extension of intuitionistic logic with the axiom schema

$$(pq) \vee (qp)$$

or if disjunction is not available, we can write an implicational axiom schema

$$(pq)(((qp)r)r).$$

Let $_1$ be the language with $\{,\wedge,\vee,\bot\}$ and let $_2$ be modal logic with . Consider the intuitionistic **LC** model with $U=[0,1]$ (unit real numbers interval) of the form $(U,\leq,0,h)$. Since we are dealing with intuitionistic model, we must have persistence, i.e. for all atomic q and any $t,s \in U$. (**)

$t \leq s$ and $h(t,q)=1$ imply $h(s,q)=1$.

We also require, for technical reasons, that for all $q, h(1,q)=1$.

Satisfaction is defined as follows:

- $tA \wedge B$ iff tA and tB

- $tA \vee B$ iff tA or tB
- tAB iff $\forall s(t \leq s \wedge sA$ imply $sB)$.
- $t\perp$ iff $t = 1$

The reader familiar with -conorms, (see [?]), can view the above as follows: For each atomic q let

$$\mu(q) = \text{ Inf } \{t \mid h(t,q) = 1\}$$

We have (beause of persistence) that μ can be extended to all wffs as follows:

- $\mu(A \wedge B) = \max(\mu(A), \mu(B))$
- $\mu(A \vee B) = \min(\mu(A), \mu(B))$
- $\mu(A \rightarrow B) = \text{ Inf } \{t \mid \max\ (t, \mu(A)) \geq \mu(B)\}$

For each $t \in U$, let $(t) = (S_t, R_t, a_t, h_t)$ be a modal model of . Note that is not intuitionistically definable from and so we have to explicitly include if we want. Here we assume we have only.

By general fibring principles, we must have persistence for modal formulas as well, for example, for ${}^k A$. [*]
$(t)^k A$ and $t \leq s$ imply $(s)^k A$

This means that

$$t \leq s \rightarrow [\forall y[a_t R_t^k y \rightarrow yA] \rightarrow \forall y[a_s R_s^k y \rightarrow yA]]$$

It is possible to show that we can assume without loss of generality (i.e. without changing the semantic consequence relation) that: (†)
$t \leq s \wedge xR_s y \rightarrow xR_t y$.[1] In fact if we let $S = \bigcup_t S_t$ we can assume that the fibred models are

$$(t) = (s, R_t, a_t, h_t).$$

We are going to assume the following additional properties: $a_t = a$ for some fixed a and $t \leq s$ and $h_t(x,q) = 1$ imply $h_s(x,q) = 1$ for all $x \in S$ and atomic q. We believe one can show that such assumptions can be made without loss of generality.

So the models differ only in their accessibility relation R_t which satisfies (†) above, and the assignment h_t.

Define functions $h^\sharp(x,q) \in U, q$ atomic, $x \in S$ and $R^\sharp : S^2 \mapsto U$ by letting

$$h^\sharp(x,q) = \text{ Inf } \{t \mid h_t(x,q) = 1\}.$$
$$R^\sharp(x,y) = \text{ Sup } \{t \mid xR_t y\}.$$

(Let us assume the Sup is attained.)

Consider the system $(U, \leq, 0, \mu, S, R^\sharp, a, h^\sharp)$. We can view this system in two ways:

[1] This condition is for . For we need $t \leq s \wedge xR_t y \rightarrow xR_s y$.

1. An **LC** model $(U, \leq, 0, \mu)$ with a fibring of modal models (S, R_t, a, h_t), where xR_ty holds iff $R^\sharp(x,y) \geq t$, and $h_t(x,q) = 1$ iff $t \geq h^\sharp(x,q)$.

2. A fuzzy model $(S, R^\sharp, a, h^\sharp)$ where the accessibility relation $R^\sharp$ and the assignment $h^\sharp$ are fuzzy and where the fuzzy truth set is $(U, \leq, 0, \mu)$ and evaluation is done using the -conorm max, as indicated above.[2]

Let us explore further the fuzzy model $(S, R^\sharp, a, h^\sharp)$. Consider, for $x \in S$, the statement x_tA, i.e. xA in the model (t). Because of persisitence, we can define

$$\mu^\sharp(x, A) = \text{ Inf } \{t \mid x_tA\}.$$

Consider $\mu^\sharp(x, A)$

$$\begin{array}{rcl} \mu^\sharp(x, A) & = & \text{Inf}\{t \mid x_tA\} \\ & = & \text{Inf } \{t \mid \forall y(xR_ty \text{ implies } y_tA)\} \end{array}$$

but xR_ty holds iff $t \leq R^\sharp(x,y)$ and y_tA holds iff $\mu^\sharp(y, A) \leq t$.

Hence

$$\mu^\sharp(x, A) = \text{ Inf } \{t \mid \forall y(t \leq R^\sharp(x,y) \text{ implies } \mu^\sharp(y, A) \leq t)\}$$

The previous two examples show that modal and many valued logic can be put together in two different ways. If we start with a modal model (S, R, a, h) then we can fuzzle (make fuzzy) h by changing it into a many valued assignment μ and extend to the entire modal language. If we start with a many valued model μ then we can fuzzle μ by changing it into a function into elements of a modal algebra. This turned out to be equivalnet to looking at modal models where the possible world relation is fuzzy but the assignment is crisp. I.e. models of the form $(S, R^\sharp, a, \mu)$ where $R^\sharp(x,y) \in [0,1]$, while μ is a $\{0,1\}$ assignment. μ can be exteded to all wffs, in which case it bcomes a $[0,1]$ valued function.

Thge obvious combination of the two approaches is to make both $R^\sharp$ and $\mu^\sharp$ fuzzy. This leads us to the following definition.

Definition 1 *An algebraic fuzzled many valued modal model has the form* $(S, R^\sharp, a, \mu^\sharp)$, *where* $R^\sharp; S^2 \mapsto [0,1]$ *is a fuzzy possible world relation and for each* $s \in S$ *and atomic* $q, \mu^\sharp(q) \in [0,1]$.

$\mu^\sharp_s$ *can be exteded to arbitrary formulas as follows:*

$$\mu^\sharp_s(A * B) = f_*(\mu^\sharp_s(A), \mu^\sharp_s(B))$$

[2] If we choose a different -conorm, say

$$\begin{array}{l} \mu(A \wedge B) = \text{ min } (1, \mu(A) + \mu(B)) \\ \mu(A \to B) = \text{ max } (0, \mu(B) - \mu(A)) \\ \mu(A \vee B) = \text{ max } (0, \mu(A) + \mu(B) - 1) \end{array}$$

we get evalutaion which makes the accessibility relation Łukasiewicz fuzzy.

where $* \in \{\wedge, \vee, \rightarrow, \neg\}$ *and* f_* *is the many valued truth table for* $*$.

$$\mu^\sharp_x(A) = \mathrm{Inf}_t[\textit{for all } y, R^\sharp(x,y) \geq x \textit{ implies } \mu^\sharp_y(A) \leq t].$$

Summary *We summarise the ideas of this section.*

- *Making fuzzy is identical with fibring in a special way.*
- *Any logic* $_1$ *can be 'made fuzzy' by fibring it with* $_2$ *as* $_1(_2)$.
- *If* $_1$ *is the Lukasiewicz infinite valued logic and* $_2$ *is modal logic then* $_1(_2)$ *can be understood as modal logic with fuzzy accessibility but crisp assignment to atoms while* $_2(_1)$ *is modal logic with fuzzy assignment to atoms (but crisp accessibility).*

Query: *What about* $_1(_2(_1))$ *and* $_2(_1(_2))$*? Do we get fuzzy accessibility and fuzzy assignment?*

The complete version will deal also with the case study of fuzzy modal logic in detail.

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