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Maximal and essential ideals of MV-algebras

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Abstract

We show that an atom free ideal is densely ordered. It is shown that if I is a maximal ideal of an MV-algebra A, then $A = I^{\perp} \oplus I^{\perp\perp}$ where $I^{\perp} = \{x|x \leq e\}$ and $I^{\perp\perp} = \{x|x \leq \bar{e}\}$ for a unique idempotent e. The socle, radical and implicative radical of A are computed in certain cases. It is shown that if A is not atom free but I is a maximal ideal which is atom free, then I is densely ordered, and $I = \langle At(A)\rangle^{\perp} = \langle a\rangle^{\perp}$ where $At(A)$ is the set of atoms of A and $a \in At(A)$. Then $A = I^{\perp} \oplus I^{\perp\perp}$ where $I^{\perp}$ is atomic and $I^{\perp\perp}$ is atom free.
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1 Introduction

One of the objects studied in commutative algebra is the socle. Recall that a non-zero ideal I of an MV-algebra A is essential if $I^{\perp} = \{0\}$, where $I^{\perp} = \{x|x \wedge y = 0 \text{ for all } y \in I\}$ (see [8], [12]). Then the socle of A, denoted by Soc (A), is the intersection of all the essential ideals of A. On the other hand, the intersection of all the maximal ideals of A is the radical of A, denoted by Rad (A). One of the goals here is to obtain a relation between Soc (A) and Rad (A) similar to that of commutative algebra. In this paper, we obtain some results, namely we show that a maximal ideal is essential under some conditions. Relations are also obtained among maximal, essential and implicative ideals.

We also consider the question of splitting the algebra A in terms of ideals I and J, namely if I and J are ideals of A, we say that $A = I \oplus J$ if $I \cap J = \{0\}$ and if for each $a \in A$ we can find a unique $x \in I$ and a unique $y \in J$ such that $a = x + y$. We obtain some results in this direction, in particular where I is linearly ordered and J is prime. We also show that if A has atoms and has a maximal ideal which is atom free, then $A = I \oplus J$ where I is atom free and J is atomic. An atom free ideal is shown to be densely ordered.

We refer the reader to the references for the theory of BCK and MV-algebras, in particular to [1], [3], [7], [14], [15], [16] and [17]. We shall follow the notation and terminology of [7] and shall assume the results there without further reference, as well as the results in [1], [3] and [8]. We shall use freely the BCK-algebra operation found in [7] and [16] as we feel that computations involving the BCK-algebra operation are often more transparent than those using just the MV-algebra operations. A shall denote a general MV-algebra, $B(A)$ its Boolean subalgebra of idempotents, and $At(A)$ its set of atoms. Let $B_1(A) = B(A) - \{1\}$. Then the only idempotent in $A - B_1(A)$ is 1. I shall denote a general ideal of A which may have different additional properties attributed to it in each case.

Some frequent results we shall use in the computations involve the distributivity of multiplication over addition under some conditions. In [13] we showed the following results. We have $x = xa + x\bar{a}$ if and only if $x \wedge \bar{x} \wedge a \wedge \bar{a} = 0$. Also if $x \wedge y \wedge z \wedge \bar{z} = 0$, then $(x+y)z = xz + yz$. We shall assume these results without further mention.

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2 Direct Sum of Ideals

Recall that I is implicative if whenever $x^n \in I$ for some $n \geq 1$, then $x \in I$ (see [7] and [10]). These are precisely the ideals which give quotients A/I which are Boolean algebras. Let Inf $(A) = \{x | x^2 =$

$0\}$, $N(A) = \{x|x^n = 0$ for some integer $n \geq 1\}$ and I Rad $(A) =$ intersection of all implicative ideals of A. Observe that I is implicative if and only if Inf $(A) \subset I$ (see [10]). It is observed in [10] that we have Rad $(A) \subset$ Inf $(A) \subset N(A) \subset I$ Rad (A). If $X \subset A$ is a non-empty subset, let $\langle X \rangle$ denote the ideal of A generated by X (see [7]). Then it is shown in [10] that $\langle N(A) \rangle = I$ Rad (A), and that $N(A)$ is an ideal of A if and only if Rad $(A) = I$ Rad (A), or equivalently, if and only if every maximal ideal of A is implicative. Observe also that Inf $(A) = \{x \wedge \bar{x}$ for all $x \in A\}$ (see [10]).

We first obtain some preliminary results.

Lemma 2.1 *If I is implicative, then $I^{\perp \subset B(A)}$.*

Proof. Let $x \in I^{\perp}$. Then $x \wedge \bar{x} \in I^{\perp} \cap$ Inf $(A) \subset I^{\perp} \cap I = \{0\}$, that is, $x \in B(A)$.

Recall that it is shown in [7] Theorems 3.7 and 3.8 that I is prime and implicative if and only if it is maximal and implicative, and that I is prime and implicative if and only if for each $x \in A$ either $x \in I$ or $\bar{x} \in I$.

Theorem 2.2 *A is a Boolean algebra if and only if every prime ideal is implicative.*

Proof. Suppose that A is a Boolean algebra and that I is prime. Let $x \in A$. Then $x \wedge \bar{x} = 0 \in I$ and hence either $x \in I$ or $\bar{x} \in I$. By [7] Theorem 3.8, it follows that I is implicative. Conversely suppose that every prime ideal of A is implicative. Let $x \in A$. Suppose that I is prime and hence implicative. Then $x \wedge \bar{x} \in$ Inf $(A) \subset I$. Thus $x \wedge \bar{x} \in \{$intersection of all prime ideals of $A\} = \{0\}$. Thus $x \in B(A)$.

Remark. This removes the additional hypothesis of [10] Corollary 3.3 that A be quasi-locally finite.

Theorem 2.3 *If I is prime then $I^{\perp}$ is linearly ordered.*

Proof. Let $x, y \in I^{\perp}$. Now, $(x * y) \wedge (y * x) = 0 \in I$ and hence either $x * y \in I$ or $y * x \in I$. Thus, either $x * y \in I \cap I^{\perp} = \{0\}$ or $y * x \in I \cap I^{\perp} = \{0\}$, that is, either $x \leq y$ or $y \leq x$.

Remark. Compare this result with [1] Theorem 2.6 and [7] Theorem 4.14.

Theorem 2.4 *Suppose that I is prime and implicative. If I is not essential, then there exists $x \in B(A) \cap At(A) \subset At(B(A))$ such that $I^{\perp} = \{0, x\}$ and*
$I = \{y | y \leq \bar{x}\}$.

Proof. By Lemma 2.1, $I^{\perp} \subset B(A)$, and by Theorem 2.3, $I^{\perp}$ is linearly ordered. Then by [7] Lemma 5.1, $I^{\perp}$ is either $\{0\}$ or $\{0, x\}$ for some $x \in B(A)$. Since I is not essential, it follows that $I^{\perp} = \{0, x\}$. Now let $y \in I$. Then $yx \in I \cap I^{\perp} = \{0\}$ and hence $y \leq \bar{x}$, that is, $I \subset \{y | y \leq \bar{x}\}$. Obviously, since $x \wedge \bar{x} = 0 \in I$ and $x \neq 0$, we have $\bar{x} \in I$. Thus $I = \{y | y \leq \bar{x}\}$. Now, if $0 < b \leq x$ for some $b \in A$, then $b \wedge \bar{x} \leq x \wedge \bar{x} = 0$, that is, $b \in I^{\perp}$. This means that $b = x$, that is, $x \in B(A) \cap At(A) \subset At(B(A))$.

If $X \subset A$, let $\overline{X} = \{\bar{x} | x \in X\}$. Then the subalgebra A_I of A generated by I is $A_I = I \cup \overline{I}$ (see [1] and [7]). A is bipartite if and only if $A = A_I$ for some maximal ideal I (see [10]). The following result is easily verified.

Lemma 2.5 *If I is prime, then $B(A) \subset A_I$.*

Theorem 2.6 *Suppose that $I \neq \{0\}$. If $I \cap B(A) = \{0\}$, then $\varphi : A_I \to A/I^{\perp}$ given by $\varphi(x) = x/I^{\perp}$ is an imbedding.*

Proof. We have $I^{\perp} \cap A_I = I^{\perp} \cap (I \cup \overline{I}) = (I^{\perp} \cap I) \cup (I^{\perp} \cap \overline{I})$. Here $I^{\perp} \cap I = \{0\}$. If there exists an element x in $I^{\perp} \cap \overline{I}$, then $x \in I^{\perp}$ and $\bar{x} \in I$. Hence $x \wedge \bar{x} \in I^{\perp} \cap I = \{0\}$. Thus $\bar{x} \in I \cap B(A) = \{0\}$, that is, $1 = x \in I^{\perp}$. This means that $I = \{0\}$, a contradiction. Thus $I^{\perp} \cap \overline{I} = \emptyset$, and $I^{\perp} \cap A_I = \{0\}$. The result now follows from [7] Theorem 5.3.

Theorem 2.7 *Suppose that I is linearly ordered and $I \cap$ Rad $A = \{0\}$. Then I contains a largest element, which is of course idempotent.*

Proof. We need only consider the case $I \neq \{0\}$, A. Suppose that $I \cap B(A) = \{0\}$. Then by Theorem 2.6, $\varphi : A_I \to A/I^{\perp}$ given by $\varphi(x) = x/I^{\perp}$ is an imbedding. Since $I^{\perp}$ is prime, we have that $A/I^{\perp}$ is linearly ordered, and hence so is A_I. We now claim that A_I is locally finite. If not, then there exists $x \neq 0$ in A_I such that ord $x = \infty$ (recall from [3] that ord x is the smallest positive integer n such that $nx = 1$).

Now, for each positive integer n we cannot have $\bar{x} \le nx$ because if $\bar{x} \le nx$ then $\bar{x}\bar{x}^n = 0$, that is, $(n+1)x = 1$, contradicting the fact that ord $x = \infty$. Since A_I is linearly ordered, we must then have $nx \le \bar{x}$ for each positive integer n. This means that $x \in$ Rad (A) (see [5], [6] and [10] for this characterization of Rad (A)). But $x \in A_I = I \cup \overline{I}$. If $x \in \overline{I}$ then $\bar{x} \in I$, and since $nx \le \bar{x}$ we have $nx \in I$ for all positive integers n. Thus x, $\bar{x} \in I$ and hence $1 = x + \bar{x} \in I$, a contradiction. Thus $x \in I$, that is, $x \in I \cap$ Rad $(A) = \{0\}$, again a contradiction. Thus A_I is locally finite. Since $I \ne \{0\}$, we may take any element $x \ne 0$ in I. Then $nx = 1$ for some positive integer n, and hence $1 \in I$, a contradiction. This proves that $I \cap B(A) \ne \{0\}$. This means that there exists a non-zero idempotent in I, and by [7] Lemma 5.1, this is the largest element of I.

Corollary 2.8 *Suppose that I is linearly ordered and $I \cap$ Rad $(A) = \{0\}$. Then there exists a unique $e \in B(A)$ such that $I = \{x | x \le e\}$ and $I^\perp = \{x | x \le \bar{e}\}$.*

Proof. By Theorem 2.7, $I = \{x | x \le e\}$ for a unique $e \in B(A)$. Then for all $x \in I$ we have $x \wedge \bar{e} \le e \wedge \bar{e} = 0$, that is, $\bar{e} \in I^\perp$. If $y \in I^\perp$ then $y \wedge e = 0$, that is, $y \le \bar{e}$. Thus $I^\perp = \{x | x \le \bar{e}\}$.

Corollary 2.9 *Suppose that I is prime and* Rad $(A) \subset I$ *then there exists a unique $e \in B(A)$ such that $I^\perp = \{x | x \le e\}$ and $I^{\perp\perp} = \{x | x \le \bar{e}\}$.*

Proof. We have that $I^\perp$ is linearly ordered and $I^\perp \cap$ Rad $(A) = \{0\}$. Now apply Corollary 2.8.

Theorem 2.10 *Suppose that I is maximal. Then either I is essential or there exists a unique $e \in B(A)$ such that $e \ne 1$ and $I = I^{\perp\perp} = \{x | x \le e\}$.*

Proof. If $I^\perp \ne \{0\}$, then $I^{\perp\perp} \ne A$ and hence $I = I^{\perp\perp}$. Since I is prime an Rad $(A) \subset I$, it follows by Corollary 2.9 that there exists a unique $e \in B(A)$ such that $I = I^{\perp\perp} = \{x | x \le e\}$. Obviously $e \ne 1$.

We observe that if $a \in At(A)$, then either $a^2 = 0$ or $a^2 = a$. Hence $At(A) \subset$ Inf $(A) \cup B(A)$. Obviously, since Inf $(A) \cap B(A) = \{0\}$, we have that $At(A) \cap B(A) = \emptyset$ if and only if $At(A) \subset \text{Inf}(A)$. Similarly, let $a \in At(A)$. If $a \notin I^{\perp}$, then we can find $x \in I$ such that $a \wedge x \neq 0$. Since $a \wedge x \leq a$ and $a \in At(A)$, we have that $a \wedge x = a$. Thus $a \leq x$ and hence $a \in I$. Thus we have $At(A) \subset I \cup I^{\perp}$. It follows trivially then that if I is essential, we have $At(A) \subset I$.

Theorem 2.11 *Suppose that $At(A) \cap B(A) = \emptyset$ and suppose that $I \neq \{0\}$ is maximal. If I is implicative, then it is essential.*

Proof. Suppose that I is not essential. Then by Theorem 2.10, there exists a unique $e \in B(A)$ such that $I = \{x | x \leq e\}$ and $e \neq 0, 1$. Since I is implicative and maximal, it follows that A is bipartite, that is, $A = A_I$. Now, $\overline{I} = \{x | \bar{e} \leq x\}$. Since $e \neq 0, 1$, it follows that $0 < \bar{e}$. Because $\bar{e} \in B(A)$, we have $\bar{e} \notin At(A)$. This means that there exists $x \in A$ such that $0 < x < \bar{e}$. But then $x \notin \overline{I}$, which means that $x \in I$. Hence $x \leq e$, and it follows that $x = 0$, a contradiction. Thus I is essential.

If we remove the hypothesis that I is maximal, we can obtain a weaker result as follows.

Definition 2.12 *$I \neq \{0\}$ is weakly essential if for all ideals J such that*
$J \cap \{A - B_1(A)\} \neq \emptyset$, we have $I \cap J \neq \{0\}$ (see [9]).

Observe that if J is a proper ideal, then $J \cap \{A - B_1(A)\}$ is idempotent free. Thus $J \cap \{A - B_1(A)\} \neq \emptyset$ means that $J \cap \{A - B_1(A)\}$ contains a non-zero non-idempotent. It is shown in [9] Theorem 3.15 that if J is a proper ideal, then
$J \cap \{A - B_1(A)\} \neq \emptyset$ if and only if $J \cap \text{Inf}(A) \neq \{0\}$.

Theorem 2.13 *Suppose that $I \neq \{0\}$. If I is implicative, then it is weakly essential.*

Proof. By Lemma 2.1, $I^{\perp} \subset B(A)$. Suppose that J is an ideal satisfying $J \cap \{A - B_1(A)\} \neq \emptyset$. If $I \cap J = \{0\}$, then $J \subset I^{\perp} \subset B(A)$. Let $x \in J \cap \{A - B_1(A)\}$. Then $x = 1$, which means that $J = A$, contradicting $I \cap J = \{0\}$. Hence $I \cap J \neq \{0\}$, that is, I is weakly essential.

Definition 2.14 *Suppose that I and J are ideals of A. We say that $A = I \oplus J$ if $I \cap J = \{0\}$, and for each $x \in A$, we can find unique elements $a \in I$, $b \in J$ such that $x = a + b$.*

Lemma 2.15 *Suppose that I and J are ideals of A and $I \cap J = \{0\}$. If $a + b = c + d$ where $a, c \in I$ and $b, d \in J$, then $a = c$ and $b = d$.*

Proof. We have $a * (c + d) = a * (a + b) = 0$. Thus $(a * c) * d = 0$. Hence $a * c \leq d$. This means that $a * c \in J$. But since $a \in I$ and $a * c \leq a$, we have that $a * c \in I$. Hence $a * c \in I \cap J = \{0\}$, that is, $a \leq c$. Similarly $c \leq a$ and hence $a = c$. In a similar way, we can show that $b = d$.

Let $e \in B(A)$ and let $J = \{x | x \leq e\}$. Then $J^{\perp} = \{x | x \leq \bar{e}\}$ and J, $J^{\perp}$ are ideals of A. We have the following result (see [10]).

Theorem 2.16 $A = J \oplus J^{\perp}$.

Proof. Clearly $J \cap J^{\perp} = \{0\}$. Let $x \in A$. Then $x = xe + x\bar{e}$ and $xe \in J$, $x\bar{e} \in J^{\perp}$.

Remark. While we can always express A as a direct sum in this way for any idempotent e, it is the goal of this section to show that we can express A as a direct sum of a linearly ordered ideal and a prime ideal, and as a direct sum of other types of ideals with stronger properties.

Theorem 2.17 *Suppose that I is linearly ordered and $I \cap$ Rad $(A) = \{0\}$. Then $A = I \oplus I^{\perp}$, and there exists a unique $e \in B(A)$ such that $I = \{x | x \leq e\}$ and $I^{\perp} = \{x | x \leq \bar{e}\}$.*

Proof. By Corollary 2.8, we have a unique $e \in B(A)$ with $I = \{x | x \leq e\}$ and $I^{\perp} = \{x | x \leq \bar{e}\}$. By Theorem 2.16, we have $A = I \oplus I^{\perp}$.

Corollary 2.18 *Suppose that I is maximal. Then $A = I^{\perp} \oplus I^{\perp\perp}$ and there exists a unique $e \in B(A)$ with $I^{\perp} = \{x | x \leq e\}$ and $I^{\perp\perp} = \{x | x \leq \bar{e}\}$.*

Proof. Clearly $I^{\perp} \cap$ Rad $(A) = \{0\}$. Now apply Theorem 2.17 to $I^{\perp}$.

Remark. Let $e \in B(A)$. Then $I = \{x | x \leq e\}$ is an MV-algebra with the same zero as A, with e as its largest element, and with the complement $\widetilde{x} = e\bar{x}$. The multiplication and addition in I are as in A (see [2] or [7] page 576).

Theorem 2.19 *Suppose that I is linearly ordered and $I \cap$ Rad $(A) = \{0\}$. If J is an ideal of the MV-algebra $I^{\perp} = \{x | x \leq \bar{e}\}$ where $e \in B(A)$, then it is an ideal of A. Also* Rad $(I^{\perp}) = \bar{e}$ Rad (A), *and* $B(I^{\perp}) = eB(A)$.

Proof. Clearly $0 \in J$. Suppose that $x * y,\ y \in J$ where $x, y \in A$. Then since by Theorem 2.17, $I = \{x | x \leq e\}$ and $I^{\perp} = \{x | x \leq \bar{e}\}$ for a unique $e \in B(A)$, we have $y \leq \bar{e}$. Thus $ye = 0$. We can write $x = xe + x\bar{e}$. Then $x * y = (xe + x\bar{e})\bar{y} = xe\bar{y} + x\bar{e}\bar{y}$ since $(xe) \wedge (x\bar{e}) \wedge \bar{y} \leq e \wedge \bar{e} = 0$. Since $x * y \in J \subset I^{\perp}$ we have $x * y \leq \bar{e}$, that is, $x\bar{y}e = 0$. This means that $xe \leq y$. Hence $xe \in I \cap I^{\perp} = \{0\}$. Thus $x = x\bar{e} \in I^{\perp}$. We now have that $x, y \in I^{\perp}$ and $x * y,\ y \in J$, where J is an ideal of $I^{\perp}$. This means that $x \in J$, proving that J is an ideal of A. Now let $x \in$ Rad (A). Then $nx \leq \bar{x}$ for all integers $n \geq 0$. But $n(\bar{e}x) = (\bar{e}x) + \ldots + (\bar{e}x) = \bar{e}(x + \ldots + x) = \bar{e}(nx)$. This means that $n(\bar{e}x) = \bar{e}(nx) \leq \bar{e}\bar{x} = \bar{e}e + \bar{e}\bar{x} = \bar{e}(e + \bar{x}) = \bar{e}\ \overline{\bar{e}x} = \widetilde{\bar{e}x}$ for all integers $n \geq 0$. This means that $\bar{e}x \in$ Rad $(I^{\perp})$. Thus $\bar{e}$ Rad $(A) \subset$ Rad $(I^{\perp})$. Conversely, if $x \in$ Rad $(I^{\perp})$, then $x \in I^{\perp}$ and $nx \leq \tilde{x} = \bar{e}\bar{x}$ for all integers $n \geq 0$. But $x = x\bar{e}$ since $x \in I^{\perp}$. This means that $nx \leq \bar{e}\bar{x} \leq \bar{x}$, that is, $x \in$ Rad (A), and $x = \bar{e}x \in \bar{e}$ Rad (A). Thus Rad $(I^{\perp}) \subset \bar{e}$ Rad (A) and hence Rad $(I^{\perp}) = \bar{e}$ Rad (A). Finally, let $f \in B(I^{\perp})$. Then $f \in B(A)$. But since $f \in I^{\perp}$ we have $f \leq \bar{e}$, that is, $fe = 0$. Also $f = f(e + \bar{e}) = fe + f\bar{e} = f\bar{e}$. Thus $f \in \bar{e}B(A)$, that is, $B(I^{\perp}) \subset \bar{e}B(A)$. Conversely, if $f \in B(A)$, then $\bar{e}f \in I^{\perp}$ and $\bar{e}f \in B(I^{\perp})$. This means that $\bar{e}B(A) \subset B(I^{\perp})$, proving that $B(I^{\perp}) = \bar{e}B(A)$.

Theorem 2.20 *Let J be an ideal of $I^{\perp}$. Then $J \cap$* Rad $(I^{\perp}) = \{0\}$ *if and only if $J \cap$* Rad $(A) = \{0\}$.

Proof. Suppose that $J \cap$ Rad $(I^{\perp}) = \{0\}$. Let $x \in J \cap$ Rad (A). Then $x \in J \subset I^{\perp}$ and $x \in$ Rad (A). This means that $x = \bar{e}x \in \bar{e}$ Rad $(A) =$ Rad $(I^{\perp})$. Thus $x \in J \cap$ Rad $(I^{\perp}) = \{0\}$. On the other hand, suppose that $J \cap$ Rad $(A) = \{0\}$ and $x \in J \cap$ Rad $(I^{\perp})$. Then $x \in J$ and $x \in$ Rad $(I^{\perp}) = \bar{e}$ Rad (A). Hence $x = \bar{e}y$ for some $y \in$ Rad (A). This means that $x \leq y$ and hence $x \in$ Rad (A). Then $x \in J \cap$ Rad $(A) = \{0\}$.

Lemma 2.21 *Let J be an ideal of $I^{\perp}$. Then $J^{\perp} \cap I^{\perp} = \{x \in I^{\perp} | x \wedge y = 0$ for all $y \in J\}$ = the orthogonal complement of J in $I^{\perp}$, where $J^{\perp} = \{x \in A | x \wedge y = 0$ for all $y \in J\}$.*

Proof. Clearly $\{x \in I^{\perp} | x \wedge y = 0 \text{ for all } y \in J\} \subset J^{\perp} \cap I^{\perp}$. Conversely, let $x \in J^{\perp} \cap I^{\perp}$. Then $x \in I^{\perp}$ and $x \wedge y = 0$ for all $y \in J$. Thus $x \in \{z \in I^{\perp} | z \wedge y = 0 \text{ for all } y \in J\}$.

We can now iterate the construction described in Theorem 2.17. Suppose that I_1 is a linearly ordered ideal of A such that $I_1 \cap$ Rad $(A) = \{0\}$. Then there exists a unique $e_1 \in B(A)$ such that $I_1 = \{x \in A | x \leq e_1\}$, $I_1^{\perp} = \{x \in A | x \leq \bar{e}_1\}$ and $A = I_1 \oplus I_1^{\perp}$. Now suppose that I_2 is a linearly ordered ideal of $I_1^{\perp}$ such that $I_2 \cap$ Rad $(I_1^{\perp}) = \{0\}$, and hence I_2 is a linearly ordered ideal of A such that $I_2 \cap$ Rad $(A) = \{0\}$. Then there exists a unique $e_2 \in B(I_1^{\perp}) = \bar{e}_1 B(A)$ and hence a unique $e_2' \in B(A)$ such that $e_2 = \bar{e}e_2'$, $I_2 = \{x \in I_1^{\perp} | x \leq e_2\}$, $I_2^{\perp} = \{x \in I_1^{\perp} | x \leq \bar{e}_2\}$ and $I_1^{\perp} = I_2 \oplus I_2^{\perp}$. Here $I_2^{\perp} \cap I_1^{\perp} = \{x \in I_1^{\perp} | x \leq \tilde{e}_2 = \bar{e}_1 \bar{e}_2\}$. Then $I_1^{\perp} = I_2 \oplus (I_2^{\perp} \cap I_1^{\perp})$. Thus $A = I_1 \oplus I_1^{\perp} = I_1 \oplus I_2 \oplus (I_2^{\perp} \cap I_1^{\perp})$. Here $e_2 \leq \bar{e}_1$, that is, $e_1 e_2 = 0$. Hence $\bar{e}_1 + \bar{e}_2 = 1$. We can continue this process as long as we can find non-zero linearly ordered ideals which have zero intersection with Rad (A). Eventually, we obtain $A = I_1 \oplus I_2 \oplus \ldots \oplus I_n \oplus J$ where $I_1, \ldots I_n$, $n \leq \infty$, are linearly ordered ideals of A with $I_j \cap$ Rad $(A) = \{0\}$, and J contains no non-zero linearly ordered ideals with zero intersection with Rad (A). In case A is semisimple, that is, Rad $(A) = \{0\}$, we end when we reach J with no non-zero linearly ordered ideals. We observe that it was shown in [7] that atoms generate linearly ordered ideals. Hence, as long as there are atoms, we will obtain non-zero linearly orded ideals.

3 Atoms, The Radical and The Socle

We first establish a distributive law for $\wedge$ over $+$ under certain conditions that we shall need in later computations.

Theorem 3.1 *For all $a, b, c \in A$, we have $a \wedge (b+c) \leq (a \wedge b) + (a \wedge c)$.*

Proof. We have $[a \wedge (b + c)]\bar{b} \leq (b + c)\bar{b} = \bar{b} \wedge c \leq c$, and clearly $[a \wedge (b + c)] \leq a$, so that $[a \wedge (b + c)]\bar{a} = 0$. Then $a \wedge (b + c) =$

$[a \wedge (b+c)] \vee (a \wedge b) = \{[a \wedge (b+c)](\bar{a} \vee \bar{b})\} + (a \wedge b) = \{([a \wedge (b+c)]\bar{a}) \vee ([a \wedge (b+c)]\bar{b}\} + (a \wedge b) \leq (a \wedge c) + (a \wedge b)$ because we also have $a \wedge (b+c)\bar{b} \leq a \wedge (b+c) \leq a$.

Corollary 3.2 *Suppose that $a, x_1, \ldots, x_n$ are elements of A satisfying $a \wedge x_i \wedge x_j = 0$ for all $1 \leq i,\ j \leq n$. Then $a \wedge (x_1 + \ldots + x_n) = (a \wedge x_1) + (a \wedge x_2) + \ldots + (a \wedge x_n)$.*

Proof. An easy induction proves that $a \wedge (x_1 + \ldots + x_n) \leq (a \wedge x_1) + \ldots + (a \wedge x_n)$. Since $a \wedge (x_1 + \ldots + x_n) \geq (a \wedge x_1) \vee \ldots \vee (a \wedge x_n)$ and $(a \wedge x_1) \wedge \ldots \wedge (a \wedge x_n) = 0$, we have $a \wedge (x_1 + \ldots + x_n) \geq (a \wedge x_1) + \ldots + (a \wedge x_n)$.

We now improve Theorem 5.26 of [7] by removing the condition that the algebra be linearly ordered. We shall prove the result in general for commutative BCK-algebras. We thank M. Palasinski for useful comments on Lemma 3.3 and Theorem 3.4.

Lemma 3.3 *Let A be a commutative BCK-algebra and let $I \neq \{0\}$ be an ideal in A. If $x, y, z \in I$ satisfy $z < x < y$ then $y * x < y * z$.*

Proof. We have $y*x \leq y*z$. If $y*x = y*z$, then $y*(y*x) = y*(y*z)$, that is, $y \wedge x = y \wedge z$. This means that $x = z$, a contradiction. Thus $y * x < y * z$.

Theorem 3.4 *If I is an ideal of A such that $I \cap At(A) = \emptyset$, and if $x < y \in I$, then we can find $z \in I$ such that $x < z < y$, that is, I is densely ordered.*

Proof. If $x = 0$, then $0 < y \in I$ and hence $y \notin At(A)$. This means that we can find z (which will necessarily be in I) such that $0 < z < y$. Hence we may assume that $0 < x < y$. Then $y*x < y$. Let $u = x \wedge (y*x)$. Then $u \leq y * x < y$. If $u = 0$, then $0 < y * x < y$ since $x < y$. Hence $y * x \in I$ and hence $y * x \notin At(A)$. Thus we can find an element t such that $0 < t < y*x < y$. This means that $y*t < y$, and since $t < y*x < y$ we have $x = y \wedge x = y * (y * x) < y * t < y$. Here obviously $y * t \in I$. On the other hand, if $u > 0$, then since $u \in I$ we have $u \notin At(A)$. Hence we can find an element t such that $0 < t < u < y$. This means that $x = y \wedge x = y * (y * x) \leq y * \{x \wedge (y * x)\} = y * u < y * t < y$. Again obviously $y * t \in I$.

Theorem 3.5 *Suppose that $At(A) \cap B(A) = \emptyset$ and let $I \neq \{0\}$ be an implicative ideal of A. Then either I is essential or $I^{\perp}$ is densely ordered ideal of $B(A)$.*

Proof. We have $I^{\perp} \subset B(A)$. If I is not essential, then $I^{\perp} \neq \{0\}$. We have $I^{\perp} \cap At(A) \subset B(A) \cap At(A) = \emptyset$. Hence by Theorem 3.4, $I^{\perp}$ is a densely ordered ideal of $B(A)$.

Definition 3.6 *Let Max be the set of all maximal ideals of A,* IMax *the set of all ideals of A that are both maximal and implicative,* Imp *the set of all implicative ideals of A, $\mathcal{E}$ the set of all essential ideals of A, and $\mathcal{WE}$ the set of all weakly essential ideals of A. Let $\mathcal{W}$ Soc $(\mathcal{A}) = \cap\mathcal{WE}$ be the weak socle of A.*

In general, $\mathcal{E} \subset \mathcal{WE}$ (see [9] and [12]). Hence $\mathcal{W}$ Soc $(\mathcal{A}) \subset$ Soc $(\mathcal{A})$. We have IMax $\subset$ Max and hence Rad $(A) \subset \cap$ IMax. If $N(A)$ is an ideal, then by [10] Theorem 3.1, we have Max $\subset$ IMax, that is, Max $=$ IMax. Hence Rad $(A) = \cap$ IMax. In this case, we have Rad $(A) =$ Inf $(A) = N(A) = I$ Rad (A) ([10], Theorem 3.1).

Theorem 3.7 *If $At(A) \cap B(A) = \emptyset$ and $N(A) \neq \{0\}$, then* Soc $(A) \subset \cap$ IMax. *If further $N(A)$ is an ideal of A, then* Soc $(A) \subset$ Rad $(A) =$ Inf $(A) = N(A) = I$ Rad (A).

Proof. By Theorem 2.11, IMax $\subset \mathcal{E}$. Hence Soc $(A) \subset \cap$ IMax. The rest of the result follows from the above comments.

Theorem 3.8 *If $N(A) \neq \{0\}$, then $\mathcal{W}$* Soc $(\mathcal{A}) \subset \mathcal{I}$ Rad $(\mathcal{A})$.

Proof. By Theorem 2.13, Imp $\subset \mathcal{WE}$.

We now impose a stronger condition on A.

Theorem 3.9 *Suppose that for each $a \in At(A)$ we have $\langle a \rangle \cap B(A) = \{0\}$. Then $At(A) \subset$* Rad (A).

Proof. Let $a \in At(A)$ and let $n \geq 1$ be an integer. Now, $(na)a \leq a$. If $(na)a = a$, then $a = a * \bar{a}^n$. Here $a \wedge \bar{a}^n = 0$, that is, $\bar{a}^n = \bar{a}^n * a = \bar{a}^{n+1}$. Then $na = (n+1)a$, that is, $na \in B(A)$. Thus $na \in \langle a \rangle \cap B(A) = \{0\}$. This means that $a = 0$, a contradiction. Hence we must have $(na)a = 0$, that is, $na \leq \bar{a}$. This means that $a \in$ Rad A.

Lemma 3.10 *Suppose that $A \neq \{0,1\}$ and that $At(A) \cap B(A) = \emptyset$. If I is weakly essential then $At(A) \subset I$ and $I^{\perp}$ is densely ordered.*

Proof. Let $a \in At(A)$. Then $a \in A - B_1(A)$ and hence $\langle a \rangle \cap \{A - B_1(A)\} \neq \emptyset$. This means that $I \cap \langle a \rangle \neq \{0\}$. Thus $na \in I$ for some integer $n \geq 1$, and hence $a \in I$. We now have $I^{\perp} \cap At(A) \subset I^{\perp} \cap I = \{0\}$. Then by Theorem 3.4, $I^{\perp}$ is densely ordered.

We now obtain a stronger result than Lemma 2.1.

Theorem 3.11 *If I is weakly essential, then $I^{\perp} \subset B(A)$. Further, if $At(A) \cap B(A) = \emptyset$, then $I^{\perp}$ is densely ordered.*

Proof. Let $x \in I^{\perp}$. If $x \notin B(A)$, then $x \in A - B_1(A)$. Hence $\langle x \rangle \cap \{A - B_1(A)\} \neq \emptyset$. Since I is weakly essential, it follows that $I \cap \langle x \rangle \neq \{0\}$. But $I \cap \langle x \rangle \subset I \cap I^{\perp} = \{0\}$, a contradiction. Hence $x \in B(A)$ that is, $I^{\perp} \subset B(A)$. If $At(A) \cap B(A) = \emptyset$, then $I^{\perp} \cap At(A) = \emptyset$ and hence $I^{\perp}$ is densely ordered by Theorem 3.4.

Corollary 3.12 *Suppose that $A \neq \{0,1\}$ and that $B(A) = \{0,1\}$. Then I is weakly essential if and only if it is essential.*

Proof. In one direction, it is obvious. Suppose that I is weakly essential. Since $B(A) = \{0,1\}$, we have $At(A) \cap B(A) = \emptyset$. The result then follows by Theorem 3.11.

Remark. If $I \neq \{0\}$ is implicative, then by Theorem 2.13, it is weakly essential. Then by Theorem 3.11, $I^{\perp} \subset B(A)$. Thus Theorems 2.13 and 3.11 imply Lemma 2.1.

Lemma 3.13 *If $I^{\perp} \subset B(A)$, then $I^{\perp\perp}$ is implicative. Thus, if I is weakly essential, then $I^{\perp\perp}$ is implicative.*

Proof. Let $x \in$ Inf (A) and let $e \in I^{\perp}$. Then $x \wedge e \leq x \leq \bar{x}$ since $x^2 = 0$. Thus $x \wedge e \leq \bar{x} \vee \bar{e}$ and hence $0 = (x \wedge e) * (\bar{x} \vee \bar{e}) = (x \wedge e)^2$. This means that $x \wedge e =$ Inf (A). But $x \wedge e \in I^{\perp} \subset B(A)$ and Inf $(A) \cap B(A) = \{0\}$. Hence $x \wedge e = 0$, and hence $x \in I^{\perp\perp}$. Thus Inf $(A) \subset I^{\perp\perp}$, proving that $I^{\perp\perp}$ is implicative. The rest of the result follows from Theorem 3.11.

Theorem 3.14 *Suppose that $At(A) \cap B(A) = \emptyset$. If I is weakly essential and prime, then I is essential.*

Proof. We have $I^\perp \subset B(A)$. Suppose that $I^\perp \neq \{0\}$. Let e, f be nonzero elements of $I^\perp$. Since $e \wedge \bar{e} = 0$, then either $e \in I$ or $\bar{e} \in I$. If $e \in I$, then $e \in I \cap I^\perp = \{0\}$, a contradiction. Thus $\bar{e} \in I$. Similarly, $\bar{f} \in I$. Now $f = f \wedge 1 = f \wedge (e \vee \bar{e}) = (f \wedge e) \vee (f \wedge \bar{e})$. But $f \wedge \bar{e} \in I^\perp \cap I = \{0\}$. Thus $f = f \wedge e$, that is, $f \leq e$. Similarly, $e \leq f$ and hence $e = f$. Thus $I^\perp = \{0, e\}$ for some $0 \neq e \in B(A)$. This means that $e \notin At(A)$ and hence $I^\perp \cap At(A) = \emptyset$. By Theorem 3.4, it follows that $I^\perp$ is densely ordered. This is obviously impossible. Hence $I^\perp = \{0\}$.

Lemma 3.15 *If I is maximal, then $I^\perp \subset B(A) \cup N(A)$.*

Proof. Let $x \in I^\perp$. If $x \notin B(A)$, then $x \wedge \bar{x} \neq 0$. Hence $\bar{x} \notin I$. It follows then that $x^n \in I$ for some integer $n \geq 1$. Thus $x^n \in I \cap I^\perp = \{0\}$, that is, $x \in N(A)$.

Remark. It follows that if $N(A) = \{0\}$ and I is maximal, then if I is not essential, we have that I is implicative. This follows from Lemmas 3.13 and 3.15.

Theorem 3.16 *Suppose that $I \neq \{0\}$ is maximal. Then for each $0 \neq x \in I^\perp$, there exists an integer $n \geq 1$ such that $nx \in B(A)$, and $nx \in At(B(A))$ if $I^\perp \neq \{0\}$. Further, if P is any prime ideal of A and M is a maximal ideal of A such that $P \subset M$, then $I^\perp \cap P = I^\perp \cap M$.*

Proof. Let $x \in I^\perp$. Suppose that $x \neq 0$. Then $x \neq 0, 1$ since $I \neq \{0\}$. Now, $x \notin I$ and hence $\bar{x}^n \in I$ for some integer $n \geq 1$. Hence $\bar{x}^n \wedge x = 0$, that is, $\bar{x}^n = \bar{x}^{n+1}$. This means that $\bar{x}^n = \bar{x}^{2n}$, that is, $\bar{x}^n \in B(A)$. Hence $nx \in B(A)$. Now suppose that $e \in B(A)$ and $0 \leq e < nx$. Then $\bar{x}^n < \bar{e}$. But $\bar{e} \in B(A)$ and $\{y \in A | y \leq \bar{e}\}$ is an ideal of A properly containing $\{y \in A | y \leq \bar{x}^n\}$. Since I is prime, we have that $I^\perp$ is linearly ordered. If $I^\perp \neq \{0\}$, then since $nx \in B(A) \cap I^\perp$, we have $I^\perp = \{y \in A | y \leq nx\}$ and hence $I = \{y \in A | y \leq \bar{x}^n\}$. Since I is maximal, it follows that $\bar{e} = 1$ and hence $e = 0$, proving that $nx \in At(B(A))$. Now suppose that P is a prime ideal of A and M is a maximal ideal of A such that $P \subset M$. Then $I^\perp \cap P \subset I^\perp \cap M$.

Let $x \in I^{\perp} \cap M$. If $x = 0$, then $x \in I^{\perp} \cap P$. If $x \neq 0$, then by the earlier result we have $nx \in B(A)$ for some integer $n \geq 1$. Hence $0 = (nx) \wedge \bar{x}^n \in P$. If $\bar{x}^n \in P$ then $\bar{x}^n \in M$, and $nx \in M$ gives a contradiction $1 = nx + \bar{x}^n \in M$. Hence $nx \in P$, which means that $x \in I^{\perp} \cap P$.

Lemma 3.17 *If $\{I_\alpha\}_{\alpha \in J}$ is a family of ideals A, then $\cap\{I_\alpha^{\perp} | \alpha \in J\} = \langle \cup\{I_\alpha | \alpha \in J\}\rangle^{\perp}$.*

Proof. Let $x \in \cap\{I_\alpha^{\perp} | \alpha \in J\}$. Then $x \in I_\alpha^{\perp}$ for each $\alpha \in J$. Let $y \in \langle \cup\{I_\alpha | \alpha \in J\}\rangle$. Then $y \leq y_1 + \ldots + y_n$ where each $y_i \in I_{\alpha_i}$. If $n = 1$, then $y \in I_{\alpha_1}$ and hence $x \wedge y = 0$. If $n \geq 2$, we have $x \wedge y_i \wedge y_j = 0$ for all $1 \leq i,\ j \leq n$. Then by Corollary 3.2, we have $x \wedge y \leq x \wedge (y_1 + \ldots + y_n) = (x \wedge y_1) + \ldots + (x \wedge y_n) = 0$. Thus we have $x \in \langle \cup\{I_\alpha | \alpha \in J\}\rangle^{\perp}$. Conversely, since $I_\alpha \subset \cup\{I_\alpha | \alpha \in J\}$ for all $\alpha \in J$, we have $I_\alpha \subset \langle \cup\{I_\alpha | \alpha \in J\}\rangle$ for each $\alpha \in J$, and hence $\langle \cup\{I_\alpha | \alpha \in J\}\rangle^{\perp} \subset \cap\{I_\alpha^{\perp} | \alpha \in J\}$.

Theorem 3.18 *Suppose that I is maximal and $At(A) - I \neq \emptyset$. Then $I = I^{\perp\perp} = \langle a | a \in At(A) - I\rangle^{\perp} = \langle a\rangle^{\perp}$ for any $a \in At(A) - I$.*

Proof. Let $a \in At(A) - I$. Since $At(A) \subset I \cup I^{\perp}$, it follows that $a \in I^{\perp}$. Hence $\langle a\rangle \subset I^{\perp}$. This means that $I \subset I^{\perp\perp} \subset \langle a\rangle^{\perp}$. Therefore $I = I^{\perp\perp} = \langle a\rangle^{\perp}$ since $\langle a\rangle^{\perp} \neq A$ and I is maximal. Then by Lemma 3.17, we have $I = I^{\perp\perp} = \cap\{\langle a\rangle^{\perp} | a \in At(A) - I\} = \langle At(A) - I\rangle^{\perp}$.

Corollary 3.19 *Suppose that $At(A) \neq \emptyset$ and I is maximal and $At(A) \cap I = \emptyset$. Then I is densely ordered and $I = I^{\perp\perp} = \langle At(A)\rangle^{\perp} = \langle a\rangle^{\perp}$ for any $a \in At(A)$, and hence $A = I^{\perp} \oplus I^{\perp\perp}$ where $I^{\perp}$ is atomic and $I^{\perp\perp}$ is atom free.*

Proof. The fact that I is densely ordered follows from Theorem 3.4. Let $a \in At(A)$. Then $a \in I^{\perp}$ and the proof proceeds as in that of Theorem 3.18 to show that $I = I^{\perp\perp} = \langle At(A)\rangle^{\perp} = \langle a\rangle^{\perp}$ for any $a \in At(A)$. Hence by Corollary 2.18, $A = I^{\perp} \oplus I^{\perp\perp}$. Here $I^{\perp\perp} = \langle At(A)\rangle^{\perp}$ is atom free. Also $I^{\perp} = \langle a\rangle^{\perp\perp}$ for any $a \in At(A)$. Clearly, since $\langle a\rangle$ is linearly ordered (see [7] Corollary 4.6), it follows that $\langle a\rangle^{\perp}$ is prime (see [7], Theorem 4.14), and hence $\langle a\rangle^{\perp\perp}$ is linearly ordered by Theorem 2.3. Thus if $x \in I^{\perp} = \langle a\rangle^{\perp\perp}$, then either $a \leq x$ or $x = 0$.

Theorem 3.20 *Suppose that* $At(B(A)) \cap B_1(A) = \emptyset$ *and* $I \neq \{0\}$ *is maximal. Then* I *is essential.*

Proof. Suppose that $I^{\perp} \neq \{0\}$. Then by Theorem 2.10, we have $I = I^{\perp\perp} = \{x \in A | x \leq e\}$ for some $e \in B(A)$. Here $e \neq 0$. We have $\overline{I} = \{x \in A | \bar{e} \leq x\}$ and $A_I = I \cup \overline{I}$. Clearly $\bar{e} \neq 1$, and $\bar{e} \in B(A_I)$. We have $\bar{e} \in At(B(A_I))$. In fact, if $x \in B(A_I)$ and $0 \leq x < \bar{e}$, then since $x \in A_I$, we have $x \leq e$ and hence $x = 0$. Now, $B(A) \subset A_I$ by Lemma 2.5. Hence $B(A) = B(A_I) \subset A_I$ and $At(B(A)) = At(B(A_I))$. Thus $\bar{e} \in At(B(A)) \cap B_1(A) = \emptyset$, a contradiction. Thus I is essential.

Corollary 3.21 *Suppose that* $At(B(A)) \cap B_1(A) = \emptyset$ *and* $\{0\}$ *is not a maximal ideal of* A. *Then* $At(A) \subset \text{Soc}\,(A) \subset \text{Rad}\,(A) \subset \text{Inf}\,(A) \subset N(A) \subset I\,\text{Rad}\,(A)$. *Hence if* A *is semisimple as well, we have* $At(A) = \emptyset$ *and* $\text{Soc}\,(A) = \{0\}$.

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