Mathware & Soft Computing 2 (1995) 167-180

Enriched MV–Algebras

Ulrich Höhle
Fachbereich 7 Mathematik, Bergische Universität
Gaußstraße 20, D-42097 Wuppertal, Germany

Abstract

This paper introduces the structure of enriched MV–algebras and studies on this basis various relations between σ–complete MV–algebras and T–tribes.

1 Introduction

In recent years much research work has been devoted to MV–algebras and T- -tribes ([2], [3], [4], [10], [11], [5], [9]). The purpose of this paper is to continue these studies and to explore various relations between *sigma*–complete MV–algebras and T–tribes. In order to fix a common framework we introduce the structure of an *enriched MV–algebra* – this is an MV–algebra provided with an additional, monoidal structure which in a certain sense is compa tible with the underlying MV–algebra. It is not difficult to see that any MV–algebra can be viewed as an enriched MV–algebra in at least two different ways (cf. Proposition 3.1). Moreover, t–norms (cf. [13]) satisfying M.J. Frank's functional equation (cf. [6]) give rise to enriched MV–algebras. In this context we present a purely algebraic proof of Butnariu's and Klement's theorem ([5]) that every T_s–tribe is a σ–complete MV–algebra ($0 < s \leq \infty$). Further we gi ve a sufficient condition under which the MacNeille completion of an *enriched, σ–complete MV–algebra* has *square roots* ([7]). As an immediate consequence we obtain that the

MacNeille completion of any *Prod*–tribe is a complete MV–algebra having square roots (cf. section 6 in [7]).

We start with a preliminary section which recalls some fundamental properties of MV–algebras from the view point of residuated lattices.

2 Preliminary remarks

An integral, commutative, residuated ℓ–monoid is a triple $(L, \leq, *)$ satisfying the following axioms (cf. [1], [7])

- $(L, \leq)$ is a lattice with universal bounds.
- $(L, *)$ is a commutative monoid.
- The universal upper bound 1 is the unity of $(L, *)$.
- There exists a binary operation $\rightarrow : L \times L \longmapsto L$ which is *right adjoint* to $*$ – i.e. the equivalence

$$\alpha * \beta \leq \gamma \iff \alpha \leq \beta \rightarrow \gamma \qquad \text{(AD)}$$

  holds for all $\alpha, \beta, \gamma \in L$.

Because of the antisymmetry of $\leq$, the binary operation $\rightarrow$ is uniquely determined by (AD). In any integral, commutative, residuated ℓ–monoid the relations

$$(\alpha \rightarrow (\beta \rightarrow \gamma) = (\alpha * \beta) \rightarrow \gamma = \beta \rightarrow (\alpha \rightarrow \gamma)$$

$$\alpha \rightarrow \beta = 1 \qquad \iff \qquad \alpha \leq \beta$$

hold true.
An MV–algebra is an integral, commutative, residuated ℓ–monoid provided with the additional, important axiom:

$$\text{(MV)} \qquad ((\alpha \rightarrow \beta) \rightarrow \beta) = \alpha \vee \beta \qquad \text{for all } \alpha, \beta \in L.$$

The following list comprehends the most basic properties of MV–algebras:

- $(\alpha \rightarrow 0) \rightarrow 0 = \alpha$ where 0 is the universal lower bound in L.

- $(\alpha \to 0) \vee \beta = \alpha \to (\alpha * \beta)$.
- $\alpha * (\alpha \to \beta) = \alpha \wedge \beta$. (Divisibility)
- $(\alpha * (\beta \to 0)) \to 0 = \alpha \to \beta$.
- $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$. (Algebraic strong de Morgan law)

An MV–algebra $(L, \leq, *)$ is *σ–complete* (resp. *complete*) iff the underlying lattice $(L, \leq)$ is σ–complete (resp. complete). In any σ–complete MV–algebra the subsequent relations are valid (cf. [3], [4]):

$$\begin{aligned}
(\bigvee_{n\in\mathbb{N}} \alpha_n) \to 0 &= \bigwedge_{n\in\mathbb{N}} (\alpha \to 0) \\
(\bigwedge_{n\in\mathbb{N}} \alpha_n) \to 0 &= \bigvee_{n\in\mathbb{N}} (\alpha \to 0) \\
\alpha * (\bigvee_{n\in\mathbb{N}} \beta_n) &= \bigvee_{n\in\mathbb{N}} (\alpha * \beta_n) \\
\alpha * (\bigwedge_{n\in\mathbb{N}} \beta_n) &= \bigwedge_{n\in\mathbb{N}} (\alpha * \beta_n)
\end{aligned}$$

Further we denote by α^n the n–th power of α w.r.t. $*$. Then in any σ–complete MV–algebra,

$$e_\alpha = \bigwedge_{n\in\mathbb{N}} \alpha^n$$

exists and is *idempotent* w.r.t. $*$.

3 Enriched MV–algebras

A quadruple $(L, \leq, *, \otimes)$ is called an *enriched MV–algebra* if and only if the following conditions are satisfied:

(EMV1) $(L, \leq, *)$ is an MV–algebra.

(EMV2) $(L, \leq, \otimes)$ is a commutative, partially ordered monoid (cf. [1]).

(EMV3) 1 (= upper universal bound) is the unity w.r.t. $\otimes$.

(EMV4) $\alpha * \beta = (\alpha \otimes \beta) * (((\alpha \to 0) \otimes (\beta \to 0)) \to 0)$.

Because of the algebraic strong de Morgan law, every MV–algebra can be viewed twofold as an enriched MV–algebra in the following sense:

Proposition 3.1 *Let* $(L, \leq, *)$ *be an* MV*–algebra. Then* $(L, \leq, *, \wedge)$ *and* $(L, \leq, *, *)$ *are enriched* MV*–algebras.*

Proof. From $(\alpha \to (\beta \to 0)) \vee ((\beta \to 0) \to \alpha) = 1$, we infer

$$((\alpha * \beta) \to 0) \vee (((\alpha \to 0) * (\beta \to 0)) \to 0) = 1;$$

hence the equation $\alpha * \beta = (\alpha * \beta) * (((\alpha \to 0) * (\beta \to 0)) \to 0)$ follows; i.e. $(L, \leq, *, *)$ is an enriched MV–algebra. Applying again the algebraic, strong de Morga n law (resp. the divisibility) we observe that $(L, \leq)$ is a distributive lattice and the inequality $\alpha * \beta \leq (\alpha * \alpha) \vee (\beta * \beta)$ holds. Hence we obtain

$$\begin{aligned} \alpha * \beta &= ((\alpha * \beta) \wedge (\alpha * \alpha)) \vee ((\alpha * \beta) \wedge (\beta * \beta)) \\ &= (\alpha * (\alpha \wedge \beta)) \vee (\beta * (\beta \wedge \alpha)) = (\alpha \wedge \beta) * (\alpha \vee \beta); \end{aligned}$$

i.e. $(L, \leq, *, \wedge)$ is an enriched MV–algebra. ■

Proposition 3.2 *Let* $(L, \leq, *, \otimes)$ *be an enriched* MV*–algebra, and* e *be an idempotent element of* L *w.r.t.* $*$*. Then the relation*

$$e * \alpha = e \otimes \alpha = e \wedge \alpha$$

holds for all $\alpha \in L$*.*

Proof. Since the universal upper bound 1 is the unity in $(L, \leq, *)$ as well as in $(L, \leq, \otimes)$, we derive from (EMV4):

$$\alpha * \beta \quad \leq \quad \alpha \otimes \beta \quad \leq \quad \alpha \wedge \beta.$$

On the other hand, if e is idempotent w.r.t. $*$, we conclude from the divisibility of $(L, \leq, *)$

$$e \wedge \alpha = e * (e \to \alpha) = e * e * (e \to \alpha) \quad \leq \quad e * \alpha;$$

hence the assertion follows. ■

Let us consider the real unit interval $[0,1]$ provided with Łukasiewicz' arithmetic conjunction T_m – i.e.

$$T_m(\alpha,\beta) \;=\; \max(\alpha+\beta-1\;,\;0)$$

Then it is well known that $([0,1],\leq,T_m)$ is a complete MV–algebra. Moreover let T be an arbitrary t–norm on $[0,1]$ – i.e. T is a binary operation on $[0,1]$ satisfying the following conditions:

- $([0,1],\leq,T)$ is a commutative, partially ordered monoid (cf. [1]).
- 1 (resp. 0) is the unity (resp. zero element) w.r.t. T.

Proposition 3.3 (M.J. Frank's functional equation) *Let T be a t–norm. Then the following assertions are equivalent:*

(i) $([0,1],\leq,T_m,T)$ is an enriched MV–algebra.

(ii) $\alpha\,+\,\beta \;=\; T(\alpha,\beta)\,+\,1-T(1-\alpha,1-\beta)$ for all $\alpha,\beta\,\in[0,1]$.

Proof. The implication (ii) $\Longrightarrow$ (i) is obvious. Therefore let us assume that (i) holds. In order to verify (ii) we distinguish the following cases:

Case 1 Let $1\;<\;\alpha+\beta$; then we infer from (EMV4) and the definition of T_m:

$$0 \quad < \quad \alpha+\beta-1 \;=\; \max(T(\alpha,\beta)\,+\,1-T(1-\alpha,1-\beta)-\,1\,,\,0).$$

Case 2 Let $\alpha+\beta\;<\;1$; by analogy to Case 1 we obtain

$$0 \quad < \quad 1-\alpha\,-\,\beta \;=\; T(1-\alpha,1-\beta)\,+\,1-T(\alpha,\beta)\,-\,1.$$

Case 3 Let $\alpha\;=\;1-\beta$; then (ii) holds by definition.
Summing up the assertion (ii) follows from the previous cases 1– 3 .■

Proposition 3.4 *Let $(L,\leq,*)$ be an MV–algebra and $(L,\leq,\otimes)$ be a commutative, partially ordered monoid. Further the universal upper bound is the unity w.r.t. $\otimes$. If $\otimes$ satisfies the following condition:*

(D) $\beta = ((\alpha \otimes \beta) \to 0) \to ((\alpha \to 0) \otimes \beta)$ for all $\alpha, \beta \in L$,

then the quadruple $(L, \leq, *, \otimes)$ *is an enriched* MV*–algebra.*

Proof. Combining (D) with (MV) and the divisibility of MV–algebras, we obtain:

$$(\beta \to 0) \to ((\beta \to 0) \otimes \alpha) =$$
$$[(((\alpha \to 0) \otimes (\beta \to 0)) \to 0) \to ((\beta \to 0) \otimes \alpha)] \to ((\beta \to 0) \otimes \alpha) =$$
$$[((\alpha \to 0) \otimes (\beta \to 0)) \to 0] \bigvee [(\beta \to 0) \otimes \alpha],$$

$$\alpha * [((\beta \to 0) \otimes \alpha) \to 0] =$$
$$[((\beta \to 0) \otimes \alpha) \to 0] * [((\beta \to 0) \otimes \alpha) \to 0) \to (\alpha \otimes \beta)] =$$
$$[((\beta \to 0) \otimes \alpha) \to 0] \wedge (\alpha \otimes \beta).$$

Since the universal upper bound is the unity of $\otimes$, the relations

$$\begin{aligned} ((\alpha \to 0) \otimes (\beta \to 0)) * ((\beta \to 0) \otimes \alpha) &= 0 \\ ((\beta \to 0) \otimes \alpha) * (\alpha \otimes \beta) &= 0 \end{aligned}$$

follow immediately. Hence we have established that

$$\begin{aligned} (\beta \to 0) \to ((\beta \to 0) \otimes \alpha) &= ((\alpha \to 0) \otimes (\beta \to 0)) \to 0 \\ \alpha * (((\beta \to 0) \otimes \alpha) \to 0) &= \alpha \otimes \beta \end{aligned}$$

Taking into account $\beta \leq ((\beta \to 0) \otimes \alpha) \to 0$ we deduce from the divisibility of MV–algebras:

$$\alpha * \beta =$$
$$\alpha * (((\beta \to 0) \otimes \alpha) \to 0) * [(((\beta \to 0) \otimes \alpha) \to 0) \to ((\beta \to 0) \to 0)] =$$
$$(\alpha \otimes \beta) * (((\alpha \to 0) \otimes (\beta \to 0)) \to 0);$$

hence (EMV4) is verified. ■

Examples 3.5 (a) Let $\mathbb{B}$ be a *Boolean algebra.* Then we can view $\mathbb{B}$ as an MV–algebra $(\mathbb{B}, \leq *)$ in which every element $\alpha \in \mathbb{B}$ is *idempotent*

w.r.t. $*$ – i.e. $* = \wedge$. Because of $(\alpha \to 0) \vee \beta = \alpha \to \beta$, the quadruple $(\mathbb{B}, \leq, \wedge, \wedge)$ is an enriched MV–algebra satisfying Axiom (D).
(b) Let T_m be Lukasiewicz' arithmetic conjunction on $[0,1]$ and $Prod$ be the usual multiplication. Since $Prod$ is distributive over the usual addition, the quadruple $([0,1], \leq, T_m, Prod)$ is an enriched MV–algebra satisfying Axiom (D).

■

A t–norm T is called strict Archimedean if and only if T is continuous on the square $[0,1] \times [0,1]$ and fulfills the additional property

$$0 \quad < \quad T(\alpha,\alpha) \quad < \quad \alpha \qquad \text{for all } \alpha \in]0,1[\quad .$$

Proposition 3.6 *For any strict Archimedean t–norm T there exists a binary operation $*$ on $[0,1]$ such that $([0,1], \leq, *, T)$ is an enriched MV–algebra satisfying axiom (D).*

Proof. Since T is strict and Archimedean, we can apply the theorem of Mostert and Shields (cf. [12]) (resp. of Ling [8]) and obtain that $([0,1], \leq, T)$ is order isomorphic to $([0,1], \leq, \cdot)$, where $\cdot$ denotes the usual multiplication on $[0,1]$. In this context the order isomorphism $h : [0,1] \longmapsto [0,1]$ is called the *multiplicative generator* of T – i .e.

$$T(\alpha,\beta) \;=\; h^{-1}(h(\alpha) \cdot h(\beta))$$

Now we introduce a further binary operation $*$ on $[0,1]$ as follows

$$\alpha * \beta \;=\; h^{-1}(T_m(h(\alpha), g(\beta))) \qquad \alpha,\beta \in [0,1],$$

where T_m denotes Lukasiewicz' arithmetic conjunction defined supra. Then $([0,1], \leq, *)$ is a complete MV–algebra, and the right adjoint operation $\to$ is given by

$$\alpha \to \beta \;=\; h^{-1}(\min(1 - h(\alpha) + h(\beta)\,,\, 1)).$$

Moreover the quadruple $([0,1], \leq, T_m, Prod)$ is an enriched MV–algebra provided with Property (D) (cf. 3.5(b)). Since $([0,1], \leq, *, T)$ is order isomorphic to $([0,1], \leq, T_m, Prod)$, the assertion follows. ■

Proposition 3.7 *Let $(L, \leq, *, \otimes)$ be an enriched MV–algebra satisfying (D). Then for every element $\alpha \in L$ with $\alpha \otimes (\alpha \wedge (\alpha \to 0)) \neq 0$, there exists an element $\beta \in L$ provided wit h the following properties:*

(i) $\beta \leq \alpha, \quad \beta \neq 0.$

(ii) $((\beta \to 0) * (\beta \to 0)) \to 0 \quad \leq \quad \alpha.$

Proof. Let us assume $0 \neq \alpha \otimes (\alpha \wedge (\alpha \to 0))$. We define

$$\gamma = \alpha \otimes (\alpha \to 0), \qquad \beta = \gamma \wedge (\alpha * (\gamma \to 0))$$

and derive from (D) and the divisibility of MV–algebras the following relation

$$\alpha * (\gamma \to 0) =$$

$$[(\alpha \otimes (\alpha \to 0)) \to 0] * [((\alpha \otimes (\alpha \to 0)) \to 0) \to (\alpha \otimes \alpha)] = \alpha \otimes \alpha.$$

Thus the inequality

$$\alpha \otimes (\alpha \wedge (\alpha \to 0)) \quad \leq \quad \beta \quad (= (\alpha \otimes (\alpha \to 0)) \wedge (\alpha \otimes \alpha) \quad)$$

follows; in particular $\beta \neq 0$. Now we are in the position to proceed in the same way as we do in the proof of Lemma 6.5 in [7], and we obtain that β fulfills the desired properties. ■

4 Enriched, σ–complete MV–algebras

A quadruple $(L, \leq, *, \otimes)$ is called an *enriched, σ–complete* (resp. *complete*) *MV–algebra* if and only if $(L, \leq, *, \otimes)$ is an enriched MV–algebra satisfying the additional conditions:

(EMV5) $(L, \leq)$ is a σ–complete (resp. complete) lattice.

(EMV6) For every $\alpha \in L$, the element $d_\alpha \overset{\text{def}}{=} \bigwedge_{n \in \mathbb{N}} \underbrace{\alpha \otimes \ldots \otimes \alpha}_{n \ times}$ is *idempotent* w.r.t. $*$.

Referring to the last statement of Section 2 we make the trivial observation that any σ–complete (resp. complete) MV–algebra $(L, \leq, *)$ can be viewed as an enriched σ–complete (resp. complete) MV–algebra $(L, \leq, *, *)$.

Theorem 4.1 *Let $(L, \leq, *, \otimes)$ be an enriched, σ–complete MV–algebra and $\mathcal{T}$ be a subset of L. If $\mathcal{T}$ satisfies the following conditions:*

(i) $\alpha, \beta \in \mathcal{T} \implies \alpha \otimes \beta \in \mathcal{T},\ \alpha \to 0 \in \mathcal{T}.$

(ii) $(\alpha_n)_{n \in \mathbb{N}} \in \mathcal{T}^{\mathbb{N}}$ with $\alpha_{n+1} \leq \alpha_n \implies \bigwedge_{n \in \mathbb{N}} \alpha_n \in \mathcal{T}.$

Then $\mathcal{T}$ is closed w.r.t. $*$. In particular $\mathcal{T}$ is an enriched, σ–complete MV–subalgebra of $(L, \leq, *, \otimes)$.

Proof. We choose $\alpha, \beta \in \mathcal{T}$, put $\varkappa = \alpha * \beta$ and define two sequences $(\alpha_n)_{n \in \mathbb{N}}$ and $(\beta_n)_{n \in \mathbb{N}}$ as follows:

$$\begin{aligned} \alpha_1 &= \alpha,\ \beta_1 = \beta,\ \alpha_{n+1} = \alpha_n \otimes \beta_n\,, \\ \beta_{n+1} &= ((\alpha_n \to 0) \otimes (\beta_n \to 0)) \to 0\,. \end{aligned}$$

Obviously $(\alpha_n)_{n \in \mathbb{N}}$ is nonincreasing and $(\beta_n)_{n \in \mathbb{N}}$ is nondecreasing. By induction, we infer from (EMV4):

$$\varkappa = \alpha_n * \beta_n \quad \forall\, n \in \mathbb{N};$$

hence the relation

$$\varkappa \leq \alpha_{n+1} \quad , \quad \beta_n \to 0 \leq \underbrace{(\varkappa \to 0) \otimes \ldots \otimes (\varkappa \to 0)}_{n \ times} \qquad \forall\, n \in \mathbb{N}$$

follows from the construction of $(\alpha_n)_{n \in \mathbb{N}}$ and $(\beta_n)_{n \in \mathbb{N}}$. Further we define two elements $\gamma, \delta \in L$ by

$$\gamma = \bigwedge_{m \in \mathbb{N}} (\varkappa \to 0)^m, \quad \delta = \bigwedge_{m \in \mathbb{N}} (\bigvee_{n \in \mathbb{N}} \beta_n)^m,$$

where the m–th power is taken w.r.t. $\otimes$. Since $(L, \leq, *, \otimes)$ is an enriched σ–complete MV–algebra, we conclude from (EMV6) that γ

and δ are idempotent w.r.t. $*$. Because of $\beta_n \to 0 \ \leq \ (\varkappa \to 0)^n$ we obtain:

$$\varkappa \quad \leq \quad \bigvee_{m \in \mathbb{N}} ((\varkappa \to 0)^m \to 0) \, = \, \gamma \to 0 \quad \leq \quad \delta.$$

After these preparations we put

$$\lambda = \quad \bigwedge_{n \in \mathbb{N}} (\delta \otimes \alpha_{n+1})$$

and infer from (i) and(ii) that λ is an element of $\mathcal{T}$. In order to verify the assertion of the theorem, it is now sufficient to show that λ coincides with $\varkappa$ ($= \alpha * \beta$). Since δ is idempotent w. r.t. $*$, the inequality $\varkappa \leq \lambda$ follows from $\varkappa \leq \delta \wedge \alpha_{n+1}$ and Proposition 3.2. In order to establish $\lambda \leq \varkappa$ we again apply Proposition 3.2 and obtain from (EMV4):

$$\begin{aligned} \lambda * (\varkappa \to 0) &= \bigwedge_{n \in \mathbb{N}} \delta * \alpha_{n+1} * (\varkappa \to 0) \\ &= \bigwedge_{n \in \mathbb{N}} \delta * (\alpha_n \otimes \beta_n) * ((\alpha_n * \beta_n) \to 0) \\ &= \bigwedge_{n \in \mathbb{N}} \delta * (\alpha_n \otimes \beta_n) * ((\alpha_n \otimes \beta_n) \to ((\alpha_n \to 0) \otimes (\beta_n \to 0))) \\ &\leq \bigwedge_{n \in \mathbb{N}} \delta * ((\alpha_n \to 0) \otimes (\beta_n \to 0)) \\ &\leq \delta * \bigwedge_{n \in \mathbb{N}} (\beta_n \to 0) \quad \leq \quad (\bigvee_{n \in \mathbb{N}} \beta_n) * (\bigwedge_{n \in \mathbb{N}} (\beta_n \to 0)) = 0; \end{aligned}$$

i.e. $\lambda \leq \varkappa$. ■

Remark 4.2 Let T be a strict Archimedean t–norm satisfying Frank's functional equation:

$$\alpha + \beta = T(\alpha, \beta) + 1 - T(1 - \alpha, 1 - \beta).$$

Because of Proposition 3.3. the quadruple $([0,1], \leq, T_m, T)$ is an enriched MV–algebra. Since T is *strict Arichimedean*, we obtain that $([0,1], \leq, T_m, T)$ is even an *enriched, complete MV–algebra.* Further let X be a non empty set. T hen the structure of $([0,1], \leq, T_m, T)$ can be extended pointwise to $[0,1]^X$ by

$$f \preccurlyeq g \iff f(x) \leq g(x) \quad \forall x \in X$$
$$\mathbf{T_m}(f,g)(x) = T_m(f(x), g(x)) , \qquad \mathbf{T}(f,g)(x) = T(f(x), g(x))$$

Obviously $([0,1]^X, \preccurlyeq, \mathbf{T_m}, \mathbf{T})$ is again an enriched, complete MV–algebra. In this setting Theorem 4.1 was first established by D. Butnariu and E.P. Klement (cf. Theorem 1.5 in [5]). In particular, a non empty subset $\mathcal{T}$ of $[0,1]^X$ is called a *T–tribe* iff $\mathcal{T}$ satisfies the conditions (i) and (ii) in Theorem 4.1. ∎

Referring to [7], an MV–algebra $(L, \leq, *)$ is said to have *square roots* iff there exists a (unary) operation $S : L \longmapsto L$ equipped with the following properties (cf. Section 2 in [7]):

- $S(\alpha) * S(\alpha) = \alpha \qquad \forall \alpha \in L.$
- $\beta * \beta \leq \alpha \implies \beta \leq S(\alpha).$

Theorem 4.3 *Let $(L, \leq, *, \otimes)$ be an enriched, σ–complete MV–algebra provided with Axiom (D). Then the MacNeille completion of $(L, \leq, *)$ is a complete MV–algebra with square roots (cf. section 6 in [7]).*

Proof. (a) First we verify the following assertion:

> *For every $\alpha \in L$ with $\alpha \neq 0$ there exists $\beta \in L$ with $\beta \neq 0$ such that $((\beta \to 0) * (\beta \to 0)) \to 0 \leq \alpha$.*

In the case of $\alpha \otimes (\alpha \wedge (\alpha \to 0)) \neq 0$ the previous assertion follows from Proposition 3.7. Therefore let us assume $\alpha \otimes (\alpha \wedge (\alpha \to 0)) = 0$. Then Axiom (D) impli es

$$\alpha = \alpha \otimes (\alpha \vee (\alpha \to 0));$$

hence the relation

$$\alpha = \alpha \otimes (\alpha \vee (\alpha \to 0))^n$$

follows for all $n \in \mathbb{N}$, where the n–th power is taken w.r.t. $\otimes$. If we put $d = \bigwedge_{n \in \mathbb{N}} (\alpha \vee (\alpha \to 0))^n$, then we conclude from (EMV6) that d is idempotent w.r.t. $*$. Ref erring to Proposition 3.2, we obtain:

$$\begin{aligned} \alpha \;&=\; \bigwedge_{n \in \mathbb{N}} (\alpha \otimes (\alpha \vee (\alpha \to 0))^n) \quad \leq \quad \alpha \wedge d \quad \leq \quad \alpha * d \\ &\leq\; \alpha * (\alpha \vee (\alpha \to 0)) \;=\; \alpha * \alpha; \end{aligned}$$

i.e. α is idempotent w.r.t $*$; hence $\alpha \to 0$ is also idempotent w.r.t. $*$. In particular the equation $((\alpha \to 0) * (\alpha \to 0)) \to 0 \;=\; \alpha$ holds, and therewith the asserti on is verified.

(b) Because of the σ–completeness of $(L, \leq, *)$ the triple $(L, \leq, *)$ is a semi–simple MV–algebra (cf. Lemma 6.2 in [7]); hence the MacNeille completion $(L^\sharp, \leq^\sharp, *^\sharp)$ of $(L, \leq, *)$ is again an MV–algebra (cf. Theorem 6.3. in [7]). In order to verifiy the assertion of the theorem, it is sufficient to show that every element $\alpha^\sharp$ of the MacNeille completion of $(L, \leq, *)$ is a square w.r.t. $*^\sharp$ (cf. Theorem 5.3 in [7]). Let us choose an element $\alpha^\sharp \in L^\sharp$ provided with the following property:

(**) If $\lambda^\sharp \;\leq^\sharp\; \alpha^\sharp$, then there exists an idempotent element $\iota \in L^\sharp$ w.r.t. $*^\sharp$ s.t. $\lambda^\sharp \;=\; \alpha^\sharp \wedge \iota$.

We put $e \;=\; \bigwedge_{n \in \mathbb{N}} (\alpha^\sharp)^n$ (where the n–th power is taken w.r.t. $*^\sharp$); then e is idempotent w.r.t. $*^\sharp$ (cf. Lemma 6.1 in [7]).

Referring to Theorem 6.4 in [7], it is sufficient to show that $\alpha^\sharp$ is idempotent w.r.t. $*^\sharp$. Let us assume the contrary – i.e. $\beta^\sharp \overset{\text{def}}{=} \alpha^\sharp *^\sharp (e \to 0) \;\neq\; 0$. By definition of the MacNeille completion, there exists $\beta \in L$ with $\beta \neq 0$ and $\beta \leq^\sharp \beta^\sharp$. Now we apply the assertion in the previous part (a) and choose $\gamma \in L$ such that

$$\gamma \;\neq\; 0, \qquad ((\gamma \to 0) * (\gamma \to 0)) \to 0 \quad \leq \quad \beta \quad \leq^\sharp \quad \alpha^\sharp.$$

Because of property (**), there exist idempotent elements $\iota_1, \iota_2 \in L^\sharp$ such that

$$\gamma \;=\; \alpha^\sharp \wedge \iota_1 \quad , \qquad ((\gamma \to 0) * (\gamma \to 0)) \to 0 \;=\; \alpha^\sharp \wedge \iota_2.$$

Therefore we obtain:

$$(\gamma \to 0) * (\gamma \to 0) \quad = \quad (\alpha^\sharp \to 0) \vee (\iota_2 \to 0)$$

$$= \quad ((\alpha^\sharp \to 0) *^\sharp (\alpha^\sharp \to 0)) \vee (\iota_1 \to 0);$$

i.e. γ ($= \alpha^\sharp \wedge \iota_1 \wedge \iota_2$) is idempotent w.r.t. $*$. Now we derive from the definition of $\beta^\sharp$

$$\gamma \quad \leq \quad \bigwedge_{n \in \mathbb{N}} (\beta^\sharp)^n \;=\; (\bigwedge_{n \in \mathbb{N}} (\alpha^\sharp)^n) * (e \to 0) \;=\; 0 \quad ;$$

i.e. $\gamma = 0$ which is a contradiction to $\gamma \neq 0$. Hence $\alpha^\sharp$ is idempotent w.r.t. $*^\sharp$. ■

Corollary 4.4 *Let Prod be the usual multiplication on* $[0,1]$*. Then the MacNeille completion of any Prod–tribe (cf. Remark 4.2) is a complete MV–algebra with square roots.*

Proof. Since *Prod* is a strict t–norm, we conclude, by Proposition 3.3 and Theorem 4.1 (cf. Remark 4.2), that every *Prod*–tribe is an enriched, σ–complete MV–algebra, where the algebraic operations $*$ and $\otimes$ are defined as follows

$$(f * g)(x) \;=\; T_m(f(x), g(x)) \quad , \qquad (f \otimes g)(x) \;=\; Prod(f(x), g(x)).$$

In particular Axiom (D) is satisfied (cf. Example 3.4(b)). Therefore the assertion follows from Theorem 4.3. ■

We close this section with the remark that a complete characterization of complete MV–algebras with square roots is available (cf. Remark 6.11 in [7]).

References

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publications, Third edition (Amer. Math. Soc., RI, 1973).

[2] L.P. Belluce, *Semisimple algebras of infinite valued logic and bold fuzzy set theory*, Canad. J. Math. **38** (1986), 1356–1379.

[3] ———, *Semisimple and complete MV–algebras*, Algebra Universalis **29** (1992), 1–9.

[4] ———, α–Complete MV–algebras, in: "Non–classical Logics and Their Applications to Fuzzy Subsets: A Handbook of the Mathematical Foundations of Fuzzy Set Theory", (U. Hőhle and E.P. Klement, Eds.), (Kluwer, Acad. Publ., Dordrecht, 1995), 7–21.

[5] D. Butnariu and E.P. Klement, *Triangular norm–based measures and their Markov kernel representation*, J. Math. Anal. Appl. **162** (1991), 111–143.

[6] M.J. Frank, *On the simultaneous associativity of* $F(x,y)$ *and* $x+y-F(x,y)$, Aequationes Math. **19** (1979), 194–226.

[7] U. Hőhle, *Commutative, residuated l–monoids*, in: "Non–classical Logics and Their Applications to Fuzzy Subsets: A Handbook of the Mathematical Foundations of Fuzzy Set Theory", (U. Hőhle and E.P. Klement,Eds.), (Kluwer, Acad. Publ., Dordrecht, 1995), 53–106.

[8] C.H. Ling, *Representation of associative functions*, Publ. Math. Debrecen **12** (1965), 189–212.

[9] R. Mesiar and M. Navara, *T_s–Tribes and T_s–measures*, J. Math. Anal. Appl.,to appear.

[10] D. Mundici, *Interpretation of AF C*–algebras in Lukasiewicz sentential logic*, Functional Analysis **65** (1986), 15–63.

[11] D. Mundici and M. Pasquetto, *A proof of the completeness of the infinite–valued calculus of Lukasiewicz with one variable* in "Non–Classical Logics and Their Applications to Fuzzy Subsets: A Handbook of the Mathematical Foundations of Fuzzy Set Theory", (U. Hőhle and E.P. Klement, Eds.), (Kluwer, Acad. Publ., Dordrecht, 1995),107–123.

[12] A.B. Paalman–de Miranda, *Topological semigroups*, Math. Centrum Amsterdam (Amsterdam 1964).

[13] B. Schweizer and A. Sklar, *Probabilistic metric spaces* (North Holland, New York, 1983).