Mathware & Soft Computing 2 (1995) 157-165

Retractive MV-Algebras*

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Given algebras $\mathbf{A}$ and $\mathbf{B}$ of the same type, a homomorphism $\rho: \mathbf{A} \to \mathbf{B}$ is called *retractive* provided that there is a homomorphism $\delta: \mathbf{B} \to \mathbf{A}$ such that $\rho \circ \delta = id_B$. Note that a retractive homomorphism must be surjective. A congruence relation θ on $\mathbf{A}$ is called *retractive* when the canonical projection $\pi_\theta: \mathbf{A} \to \mathbf{A}/\theta$ is retractive. An algebra $\mathbf{A}$ is *retractive* provided all its congruence relations are retractive.

In groups, congruence relations can be identified with normal subgroups. A normal subgroup N of a group G is retractive if and only if *G splits over N* (see [8, §15.1]). Then it follows that N is retractive if and only if it is complemented in the lattice of all subgroups of $\mathbf{G}$. An analogous result holds for Boolean algebras: *An ideal I of a Boolean Algebra $\mathbf{B}$ is retractive if and only if the subalgebra generated by I is complemented in the lattice of all subalgebras of* $\mathbf{B}$ (see [1]).

This paper is a first approach to the theory of retractive MV-algebras.

*This work has been made during the stay of the first author in the "Centre de Recerca Matemàtica de l'Institut d'Estudis Catalans" and the Department de Lògica, Història i Fil. de la Ciència of the University of Barcelona (Spain). The second author is partially supported by Grant PB94-0920 of D.G.I.C.Y.T. of Spain.

We assume familiarity with the theory of MV-algebras, as developed in [3, 4, 9] (see also [10, 6, 5]). For the reader convenience, in Section 1 we include a few results on MV-algebras that are used in the remainder of the paper, and we consider the relation between retractivity and complementation in the lattice of subalgebras of an MV-algebra. We obtain that an ideal I of the MV-algebra $\mathbf{A}$ is retractive if and only if the subalgebra of $\mathbf{A}$ generated by I is complemented in the lattice of subalgebras of $\mathbf{A}$. In Section 2, we investigate the retractivity of direct products of MV-algebras and we characterize the finite retractive MV-algebras.

The authors wish to express their gratitude to the Referee and to S. Sessa for improving the original version of Theorem 1.2 and by suggesting the proof presented here.

1 MV-algebras. Retractive ideals.

An **MV-algebra** is an algebra $\mathbf{A} = (A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following equations:

MV1. $(x \oplus y) \oplus z \approx x \oplus (y \oplus z)$

MV2. $x \oplus y \approx y \oplus x$

MV3. $x \oplus 0 \approx x$

MV4. $\neg(\neg x) \approx x$

MV5. $x \oplus \neg 0 \approx \neg 0$

MV6. $\neg(\neg x \oplus y) \oplus y \approx \neg(x \oplus \neg y) \oplus x$

By taking $y = \neg 0$ in MV6, we deduce:

MV7. $x \oplus \neg x \approx \neg 0$.

Therefore, if we set $1 = \neg 0$ and $x \odot y = \neg(\neg x \oplus \neg y)$, then $(A, \oplus, \odot, \neg, 0, 1)$ satisfies all the axioms given in [9, Lemma 2.6.], and hence the above definition of MV-algebras is equivalent to Chang's definition [3] (see also [5, 6]).

In the language of MV-algebras we consider the following terms:

$$x \vee y =_{def} (x \odot \neg y) \oplus y, \quad x \wedge y =_{def} (x \oplus \neg y) \odot y.$$

For each MV-algebra $\mathbf{A}$, the reduct $\mathbf{L}(\mathbf{A}) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, with least element 0 and greatest element 1. The corresponding order relation, which we call the **natural order** of $\mathbf{A}$, is given by $x \leq y$ if and only if $\neg x \oplus y = 1$ (or equivalently, $x \odot \neg y = 0$). An MV-algebra such that its natural order is total is called an **MV-chain**.

Let $\mathbf{A}$ be an MV-algebra. A subset I of A is called **ideal** provided that:

(I1) $0 \in I$,

(I2) $a \in I$ and $b \in I$ imply $a \oplus b \in I$,

(I3) $a \leq b$ and $b \in I$ imply $b \in I$.

Let $\mathcal{I}(\mathbf{A})$ and $Con(\mathbf{A})$ denote, respectively, the set of ideals of $\mathbf{A}$ and the set of congruence relations on $\mathbf{A}$. The correspondence:

$$\theta \mapsto J(\theta) = 0/\theta = \{a \in A \mid \langle a, 0\rangle \in \theta\}$$

establishes an order-isomorphism J from $Con(\mathbf{A})$ onto $\mathcal{I}(\mathbf{A})$, with both sets ordered by inclusion. The inverse of J is given by:

$$J^{-1}(I) = \{\langle a, b\rangle \in A^2 : (x \odot \neg y) \oplus (y \odot \neg x) \in I\},$$

(see [3, 5, 6]). For each ideal I of $\mathbf{A}$, we write $\mathbf{A}/I$ in place of $\mathbf{A}/J^{-1}(I)$, and we denote the equivalence class of an element $a \in A$, by a/I.

An ideal I of an MV-algebra $\mathbf{A}$ is called **retractive** if the associated congruence relation $J^{-1}(I)$ is retractive.

Let $\mathbf{A}$ be an MV-algebra. The universes of the subalgebras of $\mathbf{A}$, ordered by inclusion, form a lattice with smallest element $\{0, 1\}$ and greatest element A, that we denote by $\mathbf{Sub}(\mathbf{A})$. Given an ideal I of $\mathbf{A}$, we represent by $\langle I\rangle$ the universe of the subalgebra generated by I, i.e., $\langle I\rangle = I \cup \neg I$, where $\neg I = \{\neg x \mid x \in I\}$. In order to state the main result of this section we recall a result given in [7].

Lemma 1.1 *[7, Lemma 1.6] Let* $\mathbf{A}$ *be an MV-algebra and* I *an ideal of* $\mathbf{A}$*. Then for any* $a, b \in A$ *the following are equivalent:*

i) $\langle a, b\rangle \in J^{-1}(I)$

ii) $a = (b \odot h) \odot \neg k$*, for some* $h, k \in I$. □

Remark: To show that in the above lemma i) implies ii) it suffices to take $k = \neg a \odot b$ and $h = a \odot \neg b$ (see [7]).

Theorem 1.2 *An ideal* I *in an MV-algebra* $\mathbf{A}$ *is retractive if and only if* $\langle I\rangle$ *is complemented in the lattice* $\mathbf{Sub}(\mathbf{A})$*.*

Proof: Let I be a retractive ideal of $\mathbf{A}$, and let δ_I be the embedding associated to the retractive projection π_I. Clearly $\delta_I(A/I) \cap \langle I\rangle = \{0, 1\}$. Let $a \in A$, since $\langle a, \delta_I(a/I)\rangle \in J^{-1}(I)$, by Lemma 1.1, $a = (\delta_I(a/I) \oplus h) \odot \neg k$ for some $h, k \in I$. Hence $a \in \delta_I(A/I) \bigvee \langle I\rangle$. Thus $A = \delta_I(A/I) \bigvee \langle I\rangle$ and $\delta_I(A/I)$ is the complement of $\langle I\rangle$ in $\mathbf{Sub}(\mathbf{A})$. Conversely, assume that I is an ideal in $\mathbf{A}$ such that $\langle I\rangle$ is complemented in $\mathbf{Sub}(\mathbf{A})$. Let S be a complement of $\langle I\rangle$. Then

(1) $S \cap \langle I\rangle = \{0, 1\}$, and (2) $S \bigvee \langle I\rangle = A$.

Condition (1) implies that the restriction $\pi_I \restriction_S$ is a one-to-one homomorphism from $\mathbf{S}$ to $\mathbf{A}/I$. By condition (2), for each $a \in A$, there is a term in the language of MV-algebras, say $p(x_1, \ldots, x_m, y_1, \ldots, y_n)$, such that

$$a = p^{\mathbf{A}}(a_1, \ldots, a_m, b_1, \ldots, b_n)$$

for some $a_1, \ldots, a_m \in S$ and some $b_1, \ldots, b_n \in \langle I\rangle$. Then

$$a/I = p^{\mathbf{A}/I}(a_1/I, \ldots, a_m/I, b_1/I, \ldots, b_n/I).$$

Since $b_i/I = 0/I$ or $1/I$, we have that there is n-tuple $(t_1, \ldots, t_n)$ of 0's and 1's such that $a/I = p^{\mathbf{A}}(a_1, \ldots, a_m, t_1, \ldots, t_n)/I$, and since $(a_1, \ldots, a_m, t_1, \ldots, t_n) \in S^{n+m}$, we have that $p^{\mathbf{A}}(a_1, \ldots, a_m, t_1, \ldots, t_n) \in S$. Therefore the restriction $\pi_I \restriction_S$ is an isomorphism from $\mathbf{S}$ onto $\mathbf{A}/I$, and we can take $\delta = (\pi_I \restriction_S)^{-1}$. □

For any MV-algebra $\mathbf{A}$, $\mathbf{B}(\mathbf{A})$ denotes the Boolean algebra of all complemented elements in $\mathbf{L}(\mathbf{A})$. Since for any $a \in A$ and $b \in B(\mathbf{A})$, $a \oplus b = a \vee b$ and $a \odot b = a \wedge b$, $\mathbf{B}(\mathbf{A})$ is a subalgebra of $\mathbf{A}$ (see [3, 10, 6]) in which $\neg b$ is the complement of b. Then we have:

Theorem 1.3 *If* $\mathbf{A}$ *is a retractive MV-algebra, then* $\mathbf{B}(\mathbf{A})$ *is retractive Boolean algebra.*

Proof: Let I be a non trivial ideal of $\mathbf{B}(\mathbf{A})$. By 1.2, to prove that I is retractive it suffices to show that $\langle I \rangle$ is complemented in $\mathbf{Sub}(\mathbf{B}(\mathbf{A}))$. Denote by $(I]$ the ideal generated by I in $\mathbf{A}$ and let $\rho \colon \mathbf{A} \to \mathbf{A}/(I]$ the natural projection. Since $\mathbf{A}$ is retractive, there is a monomorphism $\delta \colon \mathbf{A}/(I] \to \mathbf{A}$ such that $\rho \circ \delta = id_{A/(I]}$. It is easy to check that $a \in B(\mathbf{A})$ implies $\delta(a/(I]) \in B(\delta(\mathbf{A}/(I]))$, hence $\mathbf{B}(\delta(\mathbf{A}/(I])) \subseteq \mathbf{B}(A)$. Therefore

$$I \cap B(\delta(\mathbf{A}/(I])) \subseteq (I] \cap \delta(\mathbf{A}/(I]) = \{0,1\}.$$

On the other hand, for any $a \in I$, $\langle a, \delta(a/(I]) \rangle \in J^{-1}((I])$. By the remark following Lemma 1.1, there are $k, h \in (I] \cap B(\mathbf{A}) = I$ such that $a = (\delta(a/(I]) \oplus h) \odot \neg k$. Hence

$$B(\mathbf{A}) = I \bigvee\nolimits_{\mathbf{Sub}(\mathbf{B}(\mathbf{A}))} B(\delta(\mathbf{A}/(I])).$$

Thus $B(\delta(\mathbf{A}/(I]))$ is the complement of $\langle I \rangle$ in $\mathbf{Sub}(\mathbf{B}(\mathbf{A}))$. □

2 Products of retractive MV-algebras

We say that the algebras $\mathbf{A}$ and $\mathbf{B}$ are **compatible** if there are homomorphisms $\varphi \colon \mathbf{A} \to \mathbf{B}$ and $\psi \colon \mathbf{B} \to \mathbf{A}$.

In the next lemma we collect some immediate consequences the definition of a retractive algebra.

Lemma 2.1 *The following properties hold for each retractive algebra* $\mathbf{A}$*:*

(i) *Each non trivial homomorphic image of* $\mathbf{A}$ *is a retractive algebra.*

(ii) *Any two non trivial homomorphic images of* $\mathbf{A}$ *are compatible.*

(iii) *If* $\mathbf{A}$ *is a subdirect product of a family* $\{\mathbf{A}_i\}_{i \in I}$ *of non trivial algebras, then all the algebras* $\mathbf{A}_i$ *are retractive and pairwise compatible.* □

Let $[\mathbf{0},\mathbf{1}] = \langle [0,1], \oplus, \neg, 0\rangle$, where $[0,1]$ denotes the unit segment of the real line, and the operations $\oplus$ and $\neg$ are defined by the prescriptions $a \oplus b = \min\{1, a+b\}$ and $\neg a = 1 - a$. It is well known that $[\mathbf{0},\mathbf{1}]$ and its subalgebras are the only simple MV-algebras. Then it follows that *the identity is the only endomorphism of a simple MV-algebra.*

Lemma 2.2 *The following properties hold for each retractive MV-algebra* $\mathbf{A}$*:*

(i) *If* J, K *are maximal ideals of* $\mathbf{A}$*, then* $\mathbf{A}/J \cong \mathbf{A}/K$*.*

(ii) *If* $\mathbf{A}$ *is a subdirect product of a family* $\{\mathbf{A}_i\}_{i\in I}$ *of simple MV-algebras, then for all* $i, j \in I$*,* $\mathbf{A}_i \cong \mathbf{A}_j$*.*

Proof: (i) Since $\mathbf{A}/J$ and $\mathbf{A}/K$ are homomorphic images of $\mathbf{A}$, by item (ii) in Lemma 2.1, they are compatible. Now the result follows from the fact that the identity is the only endomorphism of a simple MV-algebra.
(ii) follows from (iii) in Lemma 2.1 and (i). □

Lemma 2.3 *If* $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *are MV-algebras, then each ideal* J *of* $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ *is of the form* $J = J_1 \times \cdots \times J_n$*, where, for each* $i = 1, \ldots, n$*,* J_i *is an ideal of* $\mathbf{A}_i$*.*

Proof: Let J be an ideal of $\mathbf{A}$, and for each $i = 1, \ldots, n$, let $J_i = \pi_i(J)$, where π_i denotes the projection onto A_i. It is plain that J_i is an ideal of $\mathbf{A}_i$ and that $J \subseteq J_1 \times \cdots \times J_n$. Suppose that $a = (a_1, \ldots, a_n) \in J_1 \times \cdots \times J_n$. This implies that $b_1 = (a_1, 0, \ldots, 0)$, $b_2 = (0, a_2, 0, \ldots, 0)$, $\ldots$, $b_n = (0, \ldots, 0, a_n)$ are in J, and then $a = b_1 \oplus \cdots \oplus b_n \in J$. Therefore $J = J_1 \times \cdots \times J_n$. □

Theorem 2.4 *If* $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *denote non trivial MV-algebras, then* $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ *is retractive if and only if all the* $\mathbf{A}_i$ *are retractive and pairwise compatible.*

Proof: The *only if* part is a particular case of item (iii) in Lemma 2.1. To prove the *if* part, suppose that for $i = 1, \ldots, n$, the natural projection $\varrho_i \colon \mathbf{A}_i \to \mathbf{A}_i/J_i$ has a right inverse $\delta_j \colon \mathbf{A}_i/J_i \to \mathbf{A}_i$, and that for each $1 \leq i, j \leq n$, there is a homomorphism $\varphi_{ij}: \mathbf{A}_i \to \mathbf{A}_j$. Let J be a proper ideal of $\mathbf{A}$. By Lemma 2.3, $J = J_1 \times \cdots \times J_n$, where, for each $i = 1, \ldots, n$, J_i is an ideal of $\mathbf{A}_i$. Since J is a proper ideal of $\mathbf{A}$, the set $T = \{i \in \{1, \ldots, n\} | J_i \neq A_i\}$ is non empty, say $T = \{i_1, \ldots, i_k\}$. Then $\mathbf{A}/J \cong \mathbf{A}_{i_1}/J_{i_1} \times \cdots \times \mathbf{A}_{i_k}/J_{i_k}$, and we can identify the natural projection ϱ_J with the mapping $(a_1, \ldots, a_n) \mapsto (\varrho_{i_1}(a_{i_1}), \ldots, \varrho_{i_k}(a_{i_k}))$. The homomorphism $\delta: \mathbf{A}_{i_1}/J_{i_1} \times \cdots \times \mathbf{A}_{i_k}/J_{i_k} \to \mathbf{A}$ given by $\delta(a_1/J_{i_1}, \ldots a_k/J_{i_k}) = (b_1, \ldots, b_n)$ where, for each $j = 1, \ldots, n$, b_j is given by the prescription:

$$b_j = \begin{cases} \delta_{i_r}(a_r/J_{i_r}) & \text{if } j = i_r \in T \\ \varphi_{i_1 j}(a_1) & \text{if } j \notin T \end{cases}$$

is a right inverse of ϱ_J. Therefore, J is a retractive ideal of $\mathbf{A}$. □

Corollary 2.5 *For each retractive MV-algebra* $\mathbf{A}$ *and each integer* $n \geq 1$, $\mathbf{A}^n$ *is a retractive MV-algebra.* □

We denote by $\mathbf{L_{n+1}}$ the subalgebra of $[\mathbf{0,1}]$, whose universe is the subset L_{n+1} of $[0, 1]$ formed by the rational fractions with denominator $n + 1$. These algebras are the only finite MV-chains.

Since each simple algebra is retractive, the above corollary implies that *each finite power of a simple MV-algebra is retractive.* Hence, by taking into account that each finite MV-algebra is a direct product of simple MV-algebras (see [10, 6]) and item (ii) in Lemma 2.2, we obtain the following characterization of retractive finite MV- algebras.

Theorem 2.6 *A finite MV-algebra* $\mathbf{A}$ *is retractive if and only if there are natural numbers* $n > 1$ *and* $r > 0$ *such that* $\mathbf{A} \cong \mathbf{L}_n^r$. □

Remark: Since $\mathbf{L}_3^2$ contains a copy of $\mathbf{L}_2 \times \mathbf{L}_3$, we see that *a subalgebra of a retractive MV-algebra may be non retractive.*

Our next task is to show that the restriction to finite families in Theorem 2.4 is indeed necessary.

Theorem 2.7 *The direct product of an infinite family of non trivial MV-algebras is not retractive.*

Proof: Let $\{\mathbf{A}_i\}_{i\in I}$ be a family of non trivial MV- algebras. Since $\mathbf{B}(\prod_{i\in I}\mathbf{A}_i) = \prod_{i\in I}\mathbf{B}(\mathbf{A}_i)$, it follows from Theorem 1.3 that to prove the theorem it suffices to prove the following:

Claim: The direct product of an infinite family of non trivial Boolean algebras is not retractive.

To prove the claim, let $\{\mathbf{B}_i\}_{i\in I}$ be a family of non trivial Boolean algebras, and $\mathbf{B} = \prod_{i\in I}\mathbf{B}_i$. If I is an infinite set, then I contains an infinite denumerable subset, say $D = \{i_1, \ldots, i_n, i_{n+1}, \ldots\}$. For each $n \geq 1$, let b_n be the element of $\mathbf{B}$ defined by the prescription:

$$b_n(i) = \begin{cases} 1 & \text{if } i = i_n \\ 0 & \text{otherwise} \end{cases}$$

and b_0 be the element defined by

$$b_0(i) = \begin{cases} 1 & \text{if } i \in I \setminus D \\ 0 & \text{if } i \in D \end{cases}$$

For each $n \geq 0$, let U_n be an ultrafilter of $\mathbf{B}$ which contains b_n, and let ω be the set of natural numbers. Then the correspondence $x \mapsto \{n \in \omega \mid x \in U_n\}$ defines a homomorphism from $\mathbf{B}$ onto $\mathcal{P}(\omega)$, the Boolean algebra of all subsets of ω (see the proof of [2, Corollary 2.17, p. 407]). Since the ideal of $\mathcal{P}(\omega)$ formed by the finite subsets of ω is not retractive (see [2, Proposition 2.16, p. 407]), we have that $\mathbf{B}$ has a non retractive homomorphic image, and then, by item (i) in Lemma 2.1, $\mathbf{B}$ is not retractive. □

Corollary 2.8 *The direct product of a family* $\{\mathbf{A}_i\}_{i\in I}$ *of non trivial MV-algebras is retractive if and only if the index set I is finite and all the algebras* $\mathbf{A}_i$ *are retractive and pairwise compatible.* □

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