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Remark on Intuitionistic Fuzzy Logic and Intuitionistic Logic

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Abstract

It is shown that the axioms of the intuitionistic logic can be proved as theorems in the frames of the intuitionistic fuzzy logic.

1 Short Remarks on Intuitionistic Fuzzy Logics

In this note we shall show that the axioms of the intuitionistic logic (IL) (see [1]) can be proved as theorems in the frames of the Intuitionistic Fuzzy Logic (IFL) (see [2-4]).

First, we shall give some definitions related to IFL.

To each proposition (in the classical sense) one can assign its truth value: truth –denoted by 1, or falsity –0. In the case of fuzzy logics this truth value is a real number in the interval [0,1] and can be called "truth degree" of a particular proposition. In the IFL it is added once more value – "falsity degree"– which will be in the interval [0,1] as well.

Thus one assigns to the proposition p two real numbers $\mu(p)$ and $\gamma(p)$ with the following constraint to hold:

$$\mu(p) + \gamma(p) \leq 1.$$

Let $\pi(p) = 1 - \mu(p) - \gamma(p)$.

Let this assignment be provided by an evaluation function V defined in such a way that:

$$V(p) =< \mu(p), \gamma(p) > .$$

The evaluation of the negation $\neg p$ of the proposition p will be defined through:

$$V(\neg p) =< \gamma(p), \mu(p) > .$$

When $\gamma(p) = 1 - \mu(p)$, i.e.

$$V(p) =< \mu(p), 1 - \mu(p) >,$$

for $\neg p$ we get:

$$V(\neg p) =< 1 - \mu(p), \mu(p) >,$$

which agrees with the result for an ordinary fuzzy logic (see e.g., [5,6]).

When the values $V(p)$ and $V(q)$ of the propositions p and q are known, the evaluation function V can be extended also for the operations "&", "$\veebar$" and "$\supset$" through the definition:

$$\begin{array}{rcl} V(p \ \& \ q) & = & < \min(\mu(p), \mu(q)), \max(\gamma(p), \gamma(q)) >, \\ V(p \veebar q) & = & < \max(\mu(p), \mu(q)), \min(\gamma(p), \gamma(q)) >, \\ V(p \supset q) & = & < \max(\gamma(p), \mu(q)), \min(\mu(p), \gamma(q)) > . \end{array}$$

By analogy with the operations over IFSs it will be convenient to define for the propositions $p, q \in S$:

$$\begin{array}{rcl} \neg V(p) & = & V(\neg p), \\ V(p) \wedge V(q) & = & V(p \ \& \ q), \\ V(p) \vee V(q) & = & V(p \veebar q), \\ V(p) \rightarrow V(q) & = & V(p \supset q), \end{array}$$

For the needs of the discussion below we shall define the notion of Intuitionistic Fuzzy Tautology (IFT):

$$A \text{ is an } IFT \text{ iff } \mu(A) \geq \gamma(A)$$

while

$$A \text{ is a standard tautology iff } V(A) = < 1, 0 > .$$

2 On the relation between IFL and IL

Let everywhere A be a given propositional form (c.f. [7]: each proposition is a propositional form; if A is a propositional form then $\neg A$ is a propositional form; if A and B are propositional forms, then $A \,\&\, B$, $A \veebar B$, $A \supset B$ are propositional forms).

Following [1], we shall prove

Theorem 1 *If A, B and C are arbitrary propositional forms then:*

(t 0) $A \supset A$,

(t13) $A \supset (B \supset A)$

(t14) $A \supset (B \supset (A \,\&\, B))$

(t15) $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$

(t16) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

(t17) $A \supset \neg\neg A$

(t19) $\neg(A \,\&\, \neg A)$

(t21) $(\neg A \veebar B) \supset (A \supset B)$

(t22) $\neg(A \veebar B) \supset (\neg A \,\&\, \neg B)$

(t23) $(\neg A \,\&\, \neg B) \supset \neg(A \veebar B)$

(t25) $(\neg A \veebar \neg B) \supset \neg(A \,\&\, B)$

(t26) $(A \supset B) \supset (\neg B \supset \neg A)$

(t27) $(A \supset \neg B) \supset (B \supset \neg A)$

(t65) $\neg\neg\neg A \supset \neg A$

(t66) $\neg A \supset \neg\neg\neg A$

(t67) $\neg\neg(A \supset B) \supset (A \supset \neg\neg B)$

(t68) $(C \supset A) \supset ((C \supset (A \supset B)) \supset (C \supset B))$

are IFTs.

Proof. (t16) Let us assume everywhere below that

$$\begin{array}{rcl} V(A) & = & < a, b > \\ V(B) & = & < c, d > \\ V(C) & = & < e, f > \end{array}$$

$$\begin{array}{rl} & V((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))) \\ = & (< a, b > \to < \max(d, e), \min(c, f) >) \to \\ \to & (< \max(b, c), \min(a, d) > \to < \max(b, e), \min(a, f) >) = \\ = & < \max(b, d, e), \min(a, c, f) > \to \\ \to & < \max(b, e, \min(a, d)), \min(a, f, \max(b, c)) > \\ = & < \max(b, e, \min(a, d), \min(a, c, f)), \min(a, f, \max(b, c), \\ & \max(b, d, e)) > \end{array}$$

and

$$\begin{array}{rcl} \max(b, e, \min(a, d), \min(a, c, f)) & \geq & \max(b, e, \min(a, d)) \geq \\ & \geq & \min(a, \max(b, d, e)) \geq \\ & \geq & \min(a, f, \max(b, c), \max(b, d, e)). \end{array}$$

The validity of the other axioms is checked analogically.

Therefore all axioms of the IL can be proved as theorems in the IFL. On the other hand, the following assertion can be proved in similar way.

Theorem 2 *If A, B and C are arbitrary propositional forms then:*

(t12) $A \veebar \neg A$

(t18) $\neg\neg A \veebar A$

(t20) $(A \supset B) \supset (\neg A \veebar B)$

(t24) $\neg(A \ \& \ B) \supset (\neg A \veebar \neg B)$

(t28) $(\neg A \supset B) \supset (\neg B \supset A)$

(t29) $(\neg A \supset \neg B) \supset (B \supset A)$

(t30) $((A \supset B) \supset A) \supset A)$

are IFTs.

On the other hand these expressions are not IL-axioms (see [1]). Therefore, both types of logics are essentially different.

3 Conclusion

The IFL is a new area of the fuzzy set theory and the standard logic. There are a lot of open problems, related to it. The IFL contains temporal [8] and modal [9-13] aspects. In future the relations between IFL, and some other logics (in this number, temporal, modal and other) must be studied. The IFL must be classified in the frames of all mathematical logic.

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