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Intuitionistic fuzzy relations. (Part II) Effect of Atanassov's operators on the properties of the intuitionistic fuzzy relations

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Abstract

In this paper we study the effect of Atanassov's operator on the properties of properties reflexive, symmetric, antisymmetric, perfect antisymmetric and transitive intuitionistic fuzzy relations. We finish the paper analysing the partial enclosure of the intuitionistic fuzzy relations and its effect on the conservation of the transitive property through Atanassov's operator.

Keywords: Intuitionistic fuzzy relation; fuzzy relation; Atanassov's operators; reflexivity; simmetry; antisymmetry; antisymmetry perfect; partially included relation.

1 Introduction

In the first part of this paper we have introduced the intuitionistic fuzzy relations, as well as the properties of the intuitionistic fuzzy relations in a set. We have also studied the composition of intuitionistic fuzzy relations, and we have analysed the different properties of this composition according to the choice of t-norms and t-conorms made, concluding that the composition of intuitionistic fuzzy relations satisfies the biggest number of properties when we take $\alpha = \vee$, β t-norm, $\lambda = \wedge$ and ρ t-conorm. We will work with this choice of t-norms and t-conorms throughout the paper, unless we indicate it in a different way.

In this paper we study the conditions with which the reflexive, symmetrical, antisymmetrical, perfect antisymmetrical and transitive intuitionistic properties are kept when applying K. Atanassov's operator to an intuitionistic fuzzy relation. In 1986, K. Atanassov established different ways of changing an intuitionistic fuzzy set into a fuzzy set and defined the folowing operator:

If $E \in \text{IFSs}(X)$ then

$$\begin{aligned} D_p(E) &= \{< x, \mu_E(x) + p \cdot \pi_E(x), \\ &\quad 1 - \mu_E(x) - p \cdot \pi_E(x) > | x \in X\} \end{aligned}$$

with $p \in [0,1]$. Obviously $D_p(E) \in \text{FSs}$.

A study of the propierties of this operator, (we will call it *Atanassov's operator*), is made in ([2], [7], [8]).

The paper consists of three items. In the first one, we study the existent relation among $D_0(R)$, $D_1(R)$ and R, as well as the invariability of Δ and ∇ through D_p.

In the second part, we analyse the conservation of the reflexivity, symmetry, antisymmetry and perfect antisymmetry through D_p.

Last part is dedicated to analyse the conservation of transitive property. We start by presenting some examples which show that, generally, this property is not kept by the operator mentioned before. As our objective is to find the most general conditions with which we can assure that the transitive property is kept, we start by studing the existent relation between $D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)$ and $D_p(R)$. Afterwards, we characterize the transitivity of R through the transitivity of $D_0(R)$ and $D_1(R)$. Next, we present the relations existent between $D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)$ and $D_p(R) \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} D_p(R)$. We finally see that a way of assuring the conservation of the transitive property consists in imposing the condition of partial encosure on the starting intuitionistic fuzzy relation. In this way we guarantee that if R is transitive, then $D_p(R)$ is transitive for every p of [0,1].

2 Operator D_p applied to an IFR

Proposition 1 *Let R be an element of IFR$(X \times X)$ and D_p Atanassov's operator. Then*

i) $D_0(R) \leq R \leq D_1(R)$.

ii) *If* $p \in [0,1]$, *then* $R \preceq D_p(R)$.

iii) $D_p(\Delta) = \Delta$, $D_p(\nabla) = \nabla$ $\forall p \in [0,1]$.

iv) $(D_p(R_c))_c = D_{1-p}(R)$ $\forall p \in [0,1]$.

Proof. i) $\mu_{D_0(R)}(x,y) = \mu_R(x,y) + 0 \cdot \pi_R(x,y) =$
$= \mu_R(x,y) \leq \mu_R(x,y) + 1 \cdot \pi_R(x,y) = 1 - \nu_R(x,y) = \mu_{D_1(R)}(x,y)$.
$1 - \mu_{D_0(R)}(x,y) = 1 - \mu_R(x,y) \geq \nu_R(x,y) = \nu_{D_1(R)}(x,y)$ for every $(x,y) \in X \times X$.

ii) Evidently $\mu_R(x,y) \leq \mu_R(x,y) + p \cdot \pi_R(x,y)$; let's see that

$$\nu_R(x,y) \leq \nu_{D_p(R)}(x,y).$$

$$\begin{aligned} \nu_{D_P(R)}(x,y) &= 1 - \mu_R(x,y) - p \cdot \pi_R(x,y) = \\ &= (1-p) - (1-p) \cdot \mu_R(x,y) + p \cdot \nu_R(x,y) = \\ &= (1-p) \cdot (1 - \mu_R(x,y)) + p \cdot \nu_R(x,y) \geq \\ &\geq (1-p) \cdot \nu_R(x,y) + p \cdot \nu_R(x,y) = \nu_R(x,y). \end{aligned}$$

iii) For the relations Δ and ∇ the corresponding π_Δ and π_∇ are always equal to zero and, therefore, the equalities we are going to demostrate are evident.

iv)

$$D_p(R_c) = \{< (x,y), \nu_R(x,y) + p\pi(x,y), 1 - \nu_R(x,y) - p\pi(x,y) >$$

$$
\begin{aligned}
& > \quad |x,y \in X\} \\
(D_p(R_c))_c &= \{<(x,y), 1-\nu_R(x,y)-p\pi(x,y), \nu_R(x,y)+p\pi(x,y)> \\
& > \quad |x,y \in X\} \\
D_{1-p}(R) &= \{<(x,y), \mu_R(x,y)+(1-p)\pi(x,y), \\
& \quad 1-\mu_R(x,y)-(1-p)\pi(x,y) > |x,y \in X\} = \\
&= \{<(x,y), 1-\nu_R(x,y)-p\pi(x,y), \nu_R(x,y)+p\pi(x,y)> \\
& > \quad |\ x,y \in X\} = (D_p(R_c))_c. \quad \square
\end{aligned}
$$

3 Effect of D_p on the reflexive, symmetry, antisymmetrical, perfect antisymmetrical and antisymmetrical intuitionistic fuzzy properties.

Theorem 1 *Let R be an element of IFR$(X \times X)$.*

i) If R is reflexive, $D_p(R)$ is reflexive fuzzy for every $p \in [0,1]$.

ii) If R is symmetrical intuitionistic, then $D_p(R)$ is symmetrical fuzzy for every $p \in [0,1]$.

iii) If R is antisymmetrical intuitionistic, then $D_p(R)$ is antisymmetrical intuitionistic for every $p \in [0,1]$.

Proof. i) R is reflexive. We have for every x in X

$$
\begin{aligned}
\mu_R(x,x) &= 1,\ \nu_R(x,x) = 0\,\forall x \in X, \text{ then} \\
\mu_{D_p(R)}(x,x) &= (1-p)\cdot\mu_R(x,x) + p\cdot(1-\nu_R(x,x)) = 1\ \forall p \in [0,1].
\end{aligned}
$$

ii) R is symmetrical intuitionistic, then $\forall (x,y) \in X \times X$

$$\mu_R(x,y) = \mu_R(y,x),\ \nu_R(x,y) = \nu_R(y,x), \text{ then}$$

$$\mu_{D_p(R)}(x,y) = (1-p)\cdot\mu_R(x,y) + p\cdot(1-\nu_R(x,y)) =$$

$$= (1-p)\cdot \mu_R(y,x) + p\cdot (1-\nu_R(y,x)) = \mu_{D_p(R)}(y,x).$$

iii) Let's take $(x,y) \in X \times X$ and $x \neq y$, then

$$\begin{aligned}\mu_{D_p(R)}(x,y) &= \mu_R(x,y) + p\pi_R(x,y) \neq \mu_R(y,x) + p\pi_R(y,x) = \\ &= \mu_{D_p(R)}(y,x). \quad \square\end{aligned}$$

With regard to the antisymmetry of $R \in \text{IFR}(X \times X)$, $D_0(R)$ and $D_1(R)$, we get the

Theorem 2 *$R \in IFR(X \times X)$*

a) R is perfect antisymmetrical intuitionistic.

b) $D_1(R)$ is perfect antisymmetrical fuzzy.

c) $D_p(R)$ is perfect antisymmetrical fuzzy for every $p \in [0,1]$.

Points a), b) and c) are equivalent.

Proof. a) $\Rightarrow$ b) If for every $(x,y) \in X \times X$ with $x \neq y$, $\mu_{D_1(R)}(x,y) > 0$ is verified, then

$$\mu_R(x,y) + \pi_R(x,y) > 0$$

i) if $\mu_R(x,y) > 0$, then $\mu_{D_1(R)}(y,x) = 0$
ii) if $\mu_R(x,y) = 0$, then $\pi_R(x,y) > 0$ and, therefore $\nu_R(x,y) < 1$, so $\mu_{D_1}(y,x) = 0$.

b) $\Rightarrow$ c) Let's take $(x,y) \in X \times X$ with $x \neq y$ and $\mu_{D_(R)}(x,y) > 0$, then

$$\begin{aligned}\mu_{D_p(R)}(x,y) &= \mu_R(x,y) + p\cdot \pi_R(x,y) = \\ &= (1-p)\mu_R(x,y) + p(1-\nu_R(x,y)) > 0\end{aligned}$$

it is evident that $p \neq 1$, because if p would be $= 1$, the proof would have already finished. It can happen:

i) $1-\nu_R(x,y) > 0$, as $D_1(R)$ is perfect antisymmetrical intuitionistic, we have $\mu_R(y,x) = 0$ and $\nu_R(y,x) = 1$.

ii) We know, because of the condition of intuitionism, that $1 - \nu_R(x,y) = 0$ and $\mu_R(x,y) > 0$ cannot occur at the same time.

Finally, just say that if $p = 0$, then $\mu_R(x,y) > 0$, then $\mu_{D_p(R)}(x,y) > 0$, so $\mu_{D_p(R)}(y,x) = 0$, from where

$$\mu_R(y,x) + p\pi_R(y,x) = 0, \text{ then } \begin{cases} \mu_R(y,x) = 0 \\ \text{and} \\ \nu_R(y,x) = 1 \end{cases}$$

ii) Let's have $x \neq y$ with $\mu_R(x,y) = 0$ and $\nu_R(x,y) < 1$, then $0 < \pi_R(x,y) < 1$, therefore

$$\begin{aligned} \mu_{D_p(R)}(x,y) &= \mu_R(x,y) + p\pi_R(x,y) > 0 \text{ then} \\ \mu_R(y,x) &= 0 \text{ and } \nu_R(y,x) = 1. \quad \Box \end{aligned}$$

4 Conservation of the transitive property through D_p. Partial closure

Notice that the previous theorem do not analyse the transitive property. Next example shows that, in general, transitivity is not kept.

Let's take $X = \{x, y, z\}$ and $R \in \text{IFR}(X \times X)$ transitive, that is, $R \geq R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R \Rightarrow \mu_R \geq \mu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}$ and $\nu_R \leq \nu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R}$, on the following way:

$$\mu_R = \begin{pmatrix} & x & y & z \\ x & 0.2 & 0 & 0.9 \\ y & 0.1 & 0.7 & 0.3 \\ z & 0.1 & 0 & 0.9 \end{pmatrix} \qquad \nu_R = \begin{pmatrix} & x & y & z \\ x & 0.4 & 0.6 & 0 \\ y & 0.7 & 0.2 & 0.6 \\ z & 0.8 & 0.8 & 0.1 \end{pmatrix}$$

therefore

$$\pi_R = \begin{pmatrix} & x & y & z \\ x & 0.4 & 0.4 & 0.1 \\ y & 0.2 & 0.1 & 0.1 \\ z & 0.1 & 0.2 & 0 \end{pmatrix}$$

the composition $R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R$ is given by

$$\mu_{R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R} = \begin{pmatrix} & x & y & z \\ x & 0.2 & 0 & 0.9 \\ y & 0.1 & 0.7 & 0.3 \\ z & 0.1 & 0 & 0.9 \end{pmatrix} \qquad \nu_{R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R} = \begin{pmatrix} & x & y & z \\ x & 0.4 & 0.6 & 0 \\ y & 0.7 & 0.2 & 0.6 \\ z & 0.8 & 0.8 & 0.1 \end{pmatrix}$$

let's notice that $\mu_R(x,y) = \mu_{R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R}(x,y)$ and $\nu_{R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R}(x,y)$ $\forall (x,y)$.

For the composition we have got that

$$\pi_{R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R} = \begin{pmatrix} & x & y & z \\ x & 0.4 & 0.4 & 0.1 \\ y & 0.2 & 0.1 & 0.1 \\ z & 0.1 & 0.2 & 0 \end{pmatrix}$$

Among all the possible values of $p \in [0,1]$, we take $p = 0.6$ and calculate $\mu_{D_{0.6}(R)}$ and $\mu_{D_{0.6}(R) \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} D_{0.6}(R)}$

$$\mu_{D_{0.6}(R)} = \begin{pmatrix} & x & y & z \\ x & 0.44 & 0.24 & 0.96 \\ y & 0.22 & 0.76 & 0.36 \\ z & 0.16 & 0.12 & 0.90 \end{pmatrix}$$

$$\mu_{D_{0.6}(R) \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} D_{0.6}(R)} = \begin{pmatrix} & x & y & z \\ x & 0.44 & 0.24 & 0.96 \\ y & 0.22 & 0.76 & 0.36 \\ z & 0.16 & 0.12 & 0.90 \end{pmatrix}$$

getting, from one hand $0.96 \geq 0.90$ and, from the other hand, $0.12 \leq 0.16$, so that $D_{0.6}(R)$ and $D_{0.6}(R) \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} D_{0.6}(R)$ are not comparable.

We want to study the most general conditions in which we can assure that the transitive property is kept through K. Atanassov's operator. In order to do that, we device this item into four subitems.

4.1 Relations between $D_p(R)$ and $D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)$

Now we propose to relate $D_p(R)$ with $D_p(R \overset{\vee,\beta}{\underset{\wedge,rho}{\circ}} R)$, through their membership functions, in order to use these results in the posterior study of transitivity.

Let R be an element of IFR$(X \times X)$ and let's take the following subsets A, B and C of $X \times X$

$$\begin{aligned} A &= \left\{(x,y) | \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) > \pi_R(x,y)\right\} \\ B &= \left\{(x,y) | \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) = \pi_R(x,y)\right\} \\ C &= \left\{(x,y) | \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) < \pi_R(x,y)\right\}. \end{aligned}$$

Theorem 3 *If* $(x,y) \in B$, *then*

$$\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y) \, \forall p$$

if and only if

$$\mu_{(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_R(x,y).$$

Proof. $\Rightarrow$) As $\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) = \pi_R(x,y)$ and for every p

$$\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y),$$

is fulfilled, then

$$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) + p\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_R(x,y) + p\pi_R(x,y)$$

therefore

$$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_R(x,y).$$

$\Leftarrow$) as $\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) = \pi_R(x,y)$ and $\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y)$ we get

$$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) + p\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y) + p\pi_R(x,y),$$

then

$$\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)\,\forall p. \quad \square$$

Corollary 1 *If $B = X \times X$, then*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \quad \forall p \text{ if and only if } \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y)$$

for every $(x,y) \in X \times X$.

The study made for the elements $(x,y) \in X \times X$ such that $\pi_R(x,y) = \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}}(x,y)$ is resumed in the following table:

$(x,y) \in B$	
$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y)$	$\forall p$ $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$
$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) > \mu_R(x,y)$	$\nexists p \mid$ $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$

Theorem 4 *If $B = X \times X$, then for every p we have*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \text{ ir and only if } R \text{ is transitive.}$$

Proof. $\Rightarrow$) $D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R)$, then

$$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) + p\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_R(x,y) + p\pi_R(x,y)$$

therefore

$$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_R(x,y).$$

$$\pi_R(x,) = \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \ \forall (x,y) \in X \times X \text{ and } \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y),$$

then $\nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \geq \nu_R(x,y)$, therefore $R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R \leq R$.

$\Leftarrow$) R transitive, then $R \geq R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R$ and, therefore $D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R)$ for every p. □

Theorem 5 *If* $(x,y) \in A$ *and* $p \in [0,1]$, *then*

$$\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y) \text{ if and only if } \begin{cases} \mu_R(x,y) \geq \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \\ \text{and} \\ p \leq \frac{\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}{\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y)}. \end{cases}$$

Proof. As $(x,y) \in A$, we have

$$\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y) =$$
$$\left(\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)\right) + \left(\nu_R(x,y) - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)\right) > 0.$$

$\Rightarrow$) We suppose that $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$, then

$$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) + p\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y) + p\pi_R(x,y),$$

therefore

$$\begin{cases} \mu_R(x,y) > \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y) \text{ if } p \neq 0 \\ \mu_R(x,y) = \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y) \text{ if } p = 0 \end{cases}$$

by means of hypothesis we know that $\mu_{D_p(R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$, that is to say,

$$0 \leq p \left(\pi_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R)}(x,y) - \pi_R(x,y) \right) \leq \mu_R(x,y) - \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y),$$

then

$$p \leq \frac{\mu_R(x,y) - \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y)}{\pi_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y) - \pi_R(x,y)}.$$

$\Leftarrow$) $\mu_R(x,y) \geq \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y)$ and $p \leq \frac{\mu_R(x,y) - \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y)}{\pi_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y) - \pi_R(x,y)}$, then

$$\mu_R(x,y) - \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y) \geq p \left(\pi_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y) - \pi_R(x,y) \right)$$

inequality is also verified if $\mu_R(x,y) = \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y)$ because in such way $p = 0$. In conclusion, $\mu_{D_p(\underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$. $\square$

With regard to the sign of $\frac{\mu_R(x,y) - \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y)}{\pi_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y) - \pi_R(x,y)}$ we can do the following considerations:

a) if $\nu_R(x,y) - \nu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y) \geq 0$, then $\frac{\mu_R(x,y) - \mu_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y)}{\pi_{R \underset{\wedge,\rho}{\overset{\vee,\beta}{\circ}} R}(x,y) - \pi_R(x,y)} \leq 1$

b) if $\nu_R(x,y) - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) < 0$, then $\frac{\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}{\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y)} > 1$, therefore, for every $p \in [0,1]$, $p \leq \frac{\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}{\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y)}$, is fulfilled, that is to say

$$\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y).$$

The following table shows the valus that p must take in order to fulfil that $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$ when $(x,y) \in A$.

$a)$	$\mu_R(x,y) \geq \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)$ and $\nu_R(x,y) - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \geq 0$	$\forall p \in \left[0, \frac{\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}{\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y)}\right]$ $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$
$b)$	$\mu_R(x,y) > \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)$ and $\nu_R(x,y) - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) < 0$	$\forall p \in [0,1]$ $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$
$c)$	$\mu_R(x,y) < \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)$	$\nexists p \mid$ $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$

Theorem 6 *If $A = X \times X$ and $p \in [0,1]$ is fixed, then*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \Leftrightarrow \forall (x,y) \in X \times X \begin{cases} \mu_R(x,y) \geq \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \\ p \leq \operatorname*{Inf}_{(x,y)\in A} \frac{\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}{\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y)} \end{cases}$$

Proof. It is evident because of the previous Theorem. □

Corollary 2 *If $A = X \times X$, then*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \forall p \in [0,1] \text{ if and only if } \frac{\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}{\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y)} \geq 1.$$

Proof. It is a direct consequence of the two previous theorems. □

Theorem 7 *If $A = X \times X$, then*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \, \forall p \text{ if and only if } R \text{ is transitive.}$$

Proof. $\Rightarrow$) Through the Corollary 2, we have that

$$\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \geq \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y) > 0, \text{ then}$$
$$\mu_R(x,y) \geq \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y).$$
$$\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \geq \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y) =$$
$$= (\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)) + (\nu_R(x,y) - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)), \text{ therefore}$$
$$\nu_R(x,y) \leq \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y).$$

$\Leftarrow$) It is evident because of the monotony of D_p. □

Theorem 8 *If* $(x,y)\in C$ *and* $p\in[0,1]$ *is fixed, then*

$$\mu_{D_p(R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R)}(x,y)\leq\mu_{D_p(R)}(x,y)\ \text{ if and only if } p\geq\frac{\mu_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)-\mu_R(x,y)}{\pi_R(x,y)(-\pi_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)}.$$

Proof. As $(x,y)\in C$, then $\pi_R(x,y)-\pi_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)$ is bigger than zero, therefore

$$\pi_R(x,y)-\pi_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)=$$
$$=\left(\mu_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)-\mu_R(x,y)\right)+\left(\nu_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)-\nu_R(x,y)\right)>0$$

$\Rightarrow$)

$$\mu_{D_p(R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R)}(x,y)\leq\mu_{D_p(R)}(x,y),\ \text{therefore}$$
$$\mu_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)+p\pi_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)\leq\mu_R(x,y)+p\pi_R(x,y),\ \text{then}$$

i) If $\mu_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)>\mu_R(x,y)$, then

$$p\geq\frac{\mu_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)-\mu_R(x,y)}{\pi_R(x,y)-\pi_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)}$$

ii) If $\mu_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)\leq\mu_R(x,y)$, then $\frac{\mu_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)-\mu_R(x,y)}{\pi_R(x,y)-\pi_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)}\leq 0$, therefore

$$p\geq\frac{\mu_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)-\mu_R(x,y)}{\pi_R(x,y)-\pi_{R\overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}}R}(x,y)}$$

$\Leftarrow$) $p \geq \frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}$ as $\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) > 0$ and $p \in [0,1]$, we have

$$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y) \leq p\left(\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)\right), \text{ then}$$
$$\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y). \quad \square$$

The cases in which $(x,y) \in C$ satisfy $\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) > 0$, that is

$$\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) =$$
$$= \left(\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)\right) + \left(\nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \nu_R(x,y)\right) > 0$$

being possible the following situations

a) $\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y) \geq 0$ and $\nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \nu_R(x,y) \geq 0$.

In this case we have

$$\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \geq \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)$$

therefore

$$\frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)} \leq 1, \text{ then } \forall p \in \left[\frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}, 1\right]$$

$$\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y).$$

Notice that if $\frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)} = 1$, then $\nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) = \nu_R(x,y)$, from where $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$, being $p = 1$ the single value that fulfils the inequality of the Theorem.

b) $\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y) > 0$ and $\nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \nu_R(x,y) < 0$ so

$$\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) < \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y),$$

then

$$\frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)} > 1,$$

therefore

$$\nexists p \in [0,1] \mid \mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y).$$

c) $\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y) \leq 0$ and $\nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \nu_R(x,y) > 0$, then

$$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y) \leq p\left(\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)\right), \text{ then}$$

$$\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y \quad \forall p.$$

The following table shows a review of the three last studied posibilities.

a)	$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y) \geq 0$ and $\nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \nu_R(x,y) \geq 0$	$\forall p \in \left[\frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}, 1 \right]$ $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$
b)	$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y) > 0$ and $\nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) < \nu_R(x,y)$	$\nexists p \in [0,1]$ $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$
c)	$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y) \leq 0$ and $\nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \nu_R(x,y) > 0$	$\forall p$ $\mu_{D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R)}(x,y) \leq \mu_{D_p(R)}(x,y)$

Theorem 9 *If $C = X \times X$ and $p \in [0,1]$ is fixed, then*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \text{ if and only if } p \geq \underset{(x,y)\in C}{\operatorname{Sup}} \frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}.$$

Proof. It is evident through the previous Theorem. □

Corollary 3 *If $C = X \times X$, then*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \quad \forall p \text{ if and only if}$$

$$\operatorname*{Sup}_{(x,y)\in C} \frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)} \leq 0 \textit{ if and only if}$$
$$\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y) \leq 0\, \forall (x,y).$$

Proof. It is a consequence of the two last Theorems. □

Theorem 10 *If $C = X \times X$, then*

$$D_(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R)\, \forall p \textit{ if and only if } R \textit{ is transitive.}$$

Proof. $\Rightarrow$) For every (x,y) of C, we have

$$\pi_R(x,y) > \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)$$
$$\mu_R(x,y) + \nu_R(x,y) < \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) + \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)$$

and as the previous Corollary $\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y)$, the result is

$$\nu_R(x,y) \leq \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y).$$

$\Leftarrow$) It is a consequence of the monotony of D_p. □

The two following Theorems take previous partial results and take the partition $X \times X = A \cup B \cup C$ into account.

Theorem 11 *If $\in IFR(X \times X)$ and $p \in [0,1]$ is fixed, then*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \textit{ if and only if}$$

$$\begin{cases} \forall (x,y) \in B \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y) \\ \forall (x,y) \in A \begin{cases} \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y) \\ p \leq \text{Inf} \frac{\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}{\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y)} \end{cases} \\ \forall (x,y) \in C \; p \geq \text{Sup} \frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}. \end{cases}$$

Proof. It is enough to consider the Theorems 3, 6 and 9. □

Theorem 12 *Let's take $R \in IFR(X \times X)$.*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \forall p \text{ if and only if}$$

$$\begin{cases} \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y) \; \forall (x,y) \in B \cup C \\ \frac{\mu_R(x,y) - \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)}{\pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \pi_R(x,y)} \geq 1 \; \forall (x,y) \in A. \end{cases}$$

Proof. It is a consequence of the Corollaries 1, 2 and 3. □

Theorem 13 *Let's take $R \in IFR(X \times X)$*

$$R \text{ is transitive if and only if } D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \; \forall p.$$

Proof. It is enough to remember the Theorem 4, 7 and 10. □

Corollary 4 *Let's have $R \in IFR(X \times X)$ and $R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R \preceq R$, then*

$$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R) \forall p \in [0,1] \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y) \, \forall (x,y) \\ and \frac{\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) - \mu_R(x,y)}{\pi_R(x,y) - \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)} \; \forall (x,y) \in A. \end{cases}$$

Proof. If $R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R \preceq R$, then

$$\begin{cases} \mu_R(x,y) \geq \mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \\ \nu_R(x,y) \geq \nu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \end{cases}, \text{ therefore } \pi_R(x,y) \leq \pi_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y)$$

for every $(x,y) \in X \times X$, therefore $X \times X = A \cup B$ and through the Theorem 3 and the Corollary 2, the Corollary is proved. □

Corollary 5 *Let's take $R \in IFR(X \times X)$ and $R \preceq R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R$, then*

$D_p(R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R) \leq D_p(R)$ $\forall p \in [0,1]$ *if and only if* $\mu_{R \overset{\vee,\beta}{\underset{\wedge,\rho}{\circ}} R}(x,y) \leq \mu_R(x,y)$

for every $(x,y) \in X \times X$.

Proof. It is analogous to the one made in the previous Corollary. □

4.2 Characterization of the transitivity with D_0 and D_1.

In order to relate the transitivity of R with the transitivities of $D_0(R)$ and $D_1(R)$ and subsequently of $D_p(R)$ we need the

Lemma 1 *For every $R \in IFR(X \times X)$*

i) $D_0(R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R) = D_0(R) \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} D_0(R)$

ii) $D_1(R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R) = D_1(R) \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} D_1(R)$

are fulfilled.

Proof. i) $\mu_{D_0(R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R)}(x,y) = \mu_{(R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R)}(x,y)$

$$\begin{aligned}\mu_{D_0(R) \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} D_0(R)}(x,y) &= \bigvee_x \left\{\beta[\mu_{D_0(R)}(x,z), \mu_{D_0(R)}(z,y)]\right\} = \\ &= \bigvee_x \{\beta[\mu_R(x,z), \mu_R(z,y)]\} = \\ &= \mu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y).\end{aligned}$$

ii) $\mu_{D_1(R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R)}(x,y) = 1 - \nu_{(R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R)}(x,y)$

$$\begin{aligned}\mu_{D_1(R) \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} D_1(R)}(x,y) &= \bigvee_x \left\{\beta[\mu_{D_1(R)}(x,z), \mu_{D_1(R)}(z,y)]\right\} = \\ &= \bigvee_x \{\beta[1-\nu_R(x,z), 1-\nu_R(z,y)]\} = \\ &= \bigvee_x \{1-\beta^*[\nu_R(x,z), 1-\nu_R(z,y)]\} = \\ &= 1 - \bigwedge_x \{\beta^*[\nu_R(x,z), \nu_R(z,y)]\} = \\ &= 1 - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y). \quad \square\end{aligned}$$

Theorem 14 *Let's take $R \in IFR(X \times X)$, R is transitive if and only if $D_0(R)$ and $D_1(R)$ are transitive fuzzy.*

Proof. $\Rightarrow$) Being R transitive, then

$$R \geq R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R, \text{ therefore}$$

$$D_0(R) \geq D_0(R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R) = D_0(R) \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} D_0(R)$$

$$D_1(R) \geq D_1(R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R) = D_1(R) \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} D_1(R).$$

$\Leftarrow$) $\mu_R(x,y) = \mu_{D_0(R)}(x,y) \geq \mu_{D_0(R) \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} D_0(R)}(x,y) = \mu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y).$

$$\begin{aligned} \nu_R(x,y) &= 1 - \mu_{D_1(R)}(x,y) \leq \\ &\leq 1 - \mu_{D_1(R) \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} D_1(R)}(x,y) = \nu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y). \quad \square \end{aligned}$$

Next Theorem establishes a strong condition, with which an intuitionistic fuzzy relation R is transitive and $D_p(R)$ is also transitive for every $p \in [0,1]$.

Theorem 15 *Let R be an element of IFR$(X \times X)$, if*

$$\mu_R(x,y) \geq 1 - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y)$$

for every $(x,y) \in X \times X$, then

i) R is transitive intuitionistic

ii) $D_p(R)$ is transitive fuzzy for every $p \in [0,1]$.

Proof. i) $\mu_R(x,y) \geq 1 - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y)$, then

$$\begin{aligned} &1 - \nu_R(x,y) \geq \mu_R(x,y) \geq 1 - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y) \geq \mu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y) \Rightarrow \\ &\mu_R(x,y) \leq 1 - \nu_R(x,y),\ \mu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y) \leq 1 - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y) \\ &\mu_R(x,y) \geq \mu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y),\ 1 - \nu_R(x,y) \geq 1 - \nu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y), \end{aligned}$$

therefore

$$\begin{aligned} &\mu_R(x,y) \geq \mu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y), \nu_R(x,y) \leq \nu_{R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R}(x,y) \\ &\forall (x,y) \in X \times X \Rightarrow R \geq R \overset{\vee,\beta}{\underset{\wedge,\beta^*}{\circ}} R. \end{aligned}$$

ii) As $\mu_R(x,y) \geq 1 - \nu_{plaf}(x,y)\ \forall (x,y) \in X \times X$ we have

$$D_0(R) \geq D_1(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_1(R).$$

We know through the item i) that R is transitive, then through the Theorem 14, we have that $D_1(R) \geq D_1(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_1(R)$. Using D_p properties and the monotony of the composition, the result is

$$D_0(R) \leq D_p(R) \leq D_1(R) \text{ therefore}$$
$$D_0(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_0(R) \leq D_p(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_p(R) \leq D_1(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_1(R)$$

through the hypotheses of the Theorem, we get

$$D_0(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_0(R) \leq D_p(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_p(R) \leq D_1(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_1(R) \leq$$
$$\leq D_0(R) \leq D_p(R) \leq D_1(R), \text{ then}$$
$$D_p(R) \geq D_p(R) \underset{\wedge,\beta^*}{\overset{\vee,\beta}{\circ}} D_p(R) \quad \forall p \in [0,1]. \quad \square$$

4.3 Partially included relations. Relation between $D_p(R) \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} D_p(R)$ and $D_p(R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R)$.

The relations analysed in the previous Theorem constitute an excessively particular group of the set of all the intuitionistic fuzzy relations, because, for example, if $R \in \mathrm{IFR}(X \times X)$, is reflexive and satisfy the conditions of the Theorem, then R is fuzzy. Our objective is to find the most general conditions with which we can assure that, if R is transitive intuitionistic, then $D_p(R)$ is transitive fuzzy.

The composition that we will consider the following Theorems is $R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R)$, that is, $\beta = \wedge$ and $\rho = \vee$.

Theorem 16 *If $R \in IFR(X \times X)$ is partially included, then*

$$D_p(R) \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} D_p(R) \leq D_p(R \underset{\wedge,\vee}{\overset{\vee,\wedge}{\circ}} R)$$

for every $p \in [0,1]$.

Proof. Because of we being partially included, it is verified that

$$\begin{array}{l}[(1-p)\cdot\mu_R(x,z)+p\cdot(1-\nu_R(x,z))]\wedge\\ \wedge[(1-p)\cdot\mu_R(z,y)+p\cdot(1-\nu_R(z,y))]=\\ =(1-p)\,[\mu_R(x,z)\wedge\mu_R(z,y)]+p\,[1-\nu_R(x,z))\wedge(1-\nu_R(z,y))]\,,\end{array}$$

therefore

$$\begin{array}{rcl}\mu_{D_p(R)\overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}}D_p(R)}(x,y) & = & \bigvee_x\Big\{\Big[(1-p)\cdot\mu_R(x,z)+p\cdot(1-\nu_R(z,y)\Big]=\\ && \wedge\Big[(1-p)\cdot\mu_R(z,y)+p\cdot(1-\nu_R(z,y))\Big]\Big\}=\\ &=& \bigvee_x\Big\{\Big[(1-p)\cdot\Big[\mu_R(x,z)\wedge\mu_R(z,y)\Big]+\\ && +p\cdot\Big[(1-\nu_R(x,z))\wedge(1-\nu_R(z,y))\Big]\Big\}\le\\ &\le& (1-p)\cdot\bigvee_x\Big\{\mu_R(x,z)\wedge\mu_R(z,y)\Big\}+\\ && +p\cdot\bigvee_x\Big\{(1-\nu_R(x,z))\wedge(1-\nu_R(z,y))\Big\}=\\ &=& \mu_{D_p(R\overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}}R)}(x,y),\end{array}$$

so

$$D_p(R)\overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}}D_p(R)\le D_p(R\overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}}R)$$

for every $p\in[0,1]$. □

Next example shows the existence of relations $R\in \mathrm{IFR}(X\times X)$, which fulfil the condition $D_p(R)\overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}}D_p(R)\le D_p(R\overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}}R)$ for every $p\in[0,1]$ and they are not partially included.

Let's take $X=\{x,y\}$ and $R\in \mathrm{IFR}(X\times X)$ given by

$$\mu_R=\begin{pmatrix} & x & y\\ x & 0.2 & 0\\ y & 0.1 & 0.7\end{pmatrix}\quad \nu_R=\begin{pmatrix} & x & y\\ x & 0.4 & 0.6\\ y & 0.7 & 0.2\end{pmatrix},$$

from where the result is

$$\mu_{R\overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}}R}=\begin{pmatrix} & x & y\\ x & 0.2 & 0\\ y & 0.1 & 0.7\end{pmatrix}\quad \nu_{R\overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}}R}=\begin{pmatrix} & x & y\\ x & 0.4 & 0.6\\ y & 0.7 & 0.2\end{pmatrix}.$$

For every $p \in [0,1]$ we have

$$\mu_{D_p(R)} = \begin{pmatrix} & x & y \\ x & 0.2+0.4p & 0+0.4p \\ y & 0.1+0.2p & 0.7+0.1p \end{pmatrix}.$$

therefore

$$\mu_{D_p(R) \vee D_p(R)} = \begin{pmatrix} & x & y \\ x & 0.2+0.4p & 0+0.4p \\ y & 0.1+0.2p & 0.7+0.1p \end{pmatrix}.$$

$$\mu_{D_pR \vee D_pR} = \begin{pmatrix} & x & y \\ x & 0.2+0.4p & 0+0.4p \\ y & 0.1+0.2p & 0.7+0.1p \end{pmatrix}.$$

so

$$D_p(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_p(R) = D_p(R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R) \quad \forall p \in [0,1]$$

and, however, R is not partially included, because

$$\begin{array}{l} \mu_R(x,y) - \mu_R(y,x) = -0.1 \\ \nu_R(y,x) - \nu_R(x,y) = 0.1 \end{array}$$

The following relations present some of the properties studied in this paper.

A) Transitive and partially included.

Let's take $X = \{x, y\}$ and $R \in \text{IFR}(X \times X)$ given by

$$\mu_R \begin{pmatrix} & x & y \\ x & 1 & 0.205 \\ y & 0.326 & 1 \end{pmatrix}. \quad \nu_R \begin{pmatrix} & x & y \\ x & 0 & 0.016 \\ y & 0.016 & 0 \end{pmatrix}.$$

B) Partially included and non-transitive.

Let's take $X = \{x, y, x\}$ and $R \in \text{IFR}(X \times X)$ such that

$$\mu_R \begin{pmatrix} & x & y & z \\ x & 0.177 & 0.177 & 0.298 \\ y & 0.177 & 0.177 & 0.177 \\ z & 0.177 & 0.177 & 0.177 \end{pmatrix}. \quad \nu_R \begin{pmatrix} & x & y & z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0.101 \end{pmatrix}.$$

C) Transitive and not partially included.
Let's take $X = \{x, y, x\}$ and $R \in \text{IFR}(X \times X)$

$$\mu_R \begin{pmatrix} & x & y & z \\ x & 1 & 0.089 & 0.089 \\ y & 0.541 & 1 & 0.807 \\ z & 0.541 & 0.475 & 1 \end{pmatrix} . \quad \nu_R \begin{pmatrix} & x & y & z \\ x & 0 & 0.176 & 0.176 \\ y & 0.271 & 0 & 0.146 \\ z & 0.271 & 0.086 & 0 \end{pmatrix} .$$

D) It is not transitive and not partially included.
Let's take $X = \{x, y, x\}$ and $R \in \text{IFR}(X \times X)$

$$\mu_R \begin{pmatrix} & x & y & z \\ x & 0 & 0.377 & 0.793 \\ y & 0 & 0.190 & 0.069 \\ z & 0 & 0.005 & 0.307 \end{pmatrix} . \quad \nu_R \begin{pmatrix} & x & y & z \\ x & 0 & 0.320 & 0.063 \\ y & 0 & 0.634 & 0.517 \\ z & 0 & 0.858 & 0.080 \end{pmatrix} .$$

4.4 Transitive and partially included relations. Conservation of transitivity.

Notice that we take $\beta = \wedge$ and $\rho = \vee$.

Theorem 17 *Let's take $R \in IFR(X \times X)$ partially included.*
R is transitive if and only if $D_p(R)$ is transitive fuzzy for every $p \in [0, 1]$.

Proof. Through the Theorem 16, we know that if R is partially included, then $D_p(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_p(R) \leq D_p(R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R)$, besides, R is transitive if and only if $D_p(R) \geq D_p(R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R)$ for every $p \in [0, 1]$, so

$$D_p(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_p(R) \leq D_p(R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R) \leq D_p(R) \, \forall p \in [0, 1]. \quad \square$$

The previous condition of R being partially included is not superfluous, because next example shows a relation R transitive and not partially included with $D_p(R)$ transitive for every $p \in [0, 1]$.

$$\begin{pmatrix} & x & y \\ x & 1 & 0 \\ y & 0.1 & 1 \end{pmatrix} \quad \nu_R = \begin{pmatrix} & x & y \\ x & 0 & 0.6 \\ y & 0.7 & 0 \end{pmatrix} ,$$

it is easily proved that R is transitive and not partially included, because

$$\mu_R(x,y) - \mu_R(y,x) = -0.1$$
$$\nu_R(x,y) - \mu_R(y,x) = 0.1,$$

besides

$$\mu_{D_p(R)} = \begin{pmatrix} & x & y \\ x & 1 & 0+0.4p \\ y & 0.1+0.2p & 1 \end{pmatrix}$$
$$\nu_{D_p(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_p(R)} = \begin{pmatrix} & x & y \\ x & 1 & 0+0.4p \\ y & 0.1+0.2p & 1 \end{pmatrix}.$$

therefore D_p is transitive for every $p \in [0,1]$.

Now we are going to present, by means of simple verified examples, non- transitive intuitionistic fuzzy relations R, such that $D_p(R)$ is transitive fuzzy for some concrete values of $p \in [0,1]$.

A) Intuitionistic fuzzy relation R that is neither transitive nor partially included, and however, $D_{0.29}(R)$ is transitive fuzzy

$$\mu_R = \begin{pmatrix} & x & y \\ x & 0 & 0 \\ y & 0.047 & 0.081 \end{pmatrix} \quad \nu_R = \begin{pmatrix} & x & y \\ x & 0 & 0 \\ y & 0.1 & 0.2 \end{pmatrix},$$

it is not transitive because

$$\mu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R} = \begin{pmatrix} & x & y \\ x & 0 & 0 \\ y & 0.047 & 0.081 \end{pmatrix} \quad \nu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R} = \begin{pmatrix} & x & y \\ x & 0 & 0 \\ y & 0.1 & 0.1 \end{pmatrix},$$

it is not partially included, so that

$$\mu_R(x,y) - \mu_R(y,y) = -0.081$$
$$\nu_R(y,y) - \nu_R(x,y) = 0.2$$

and $D_{0.29}(R)$ is transitive, because $D_{0.29}(R) = D_{0.29}(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_{0.29}(R)$.

$$\mu_{D_{0.29}(R)} = \begin{pmatrix} & x & y \\ x & 0.290 & 0.290 \\ y & 0.294 & 0.290 \end{pmatrix},$$

$$\nu_{D_{0.29}(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_{0.29}(R)} = \begin{pmatrix} & x & y \\ x & 0.29 & 0.29 \\ y & 0.29 & 0.29 \end{pmatrix}.$$

B) Next relation $R \in \text{IFR}(X \times X)$, is partially included, non- transitive and it is such that $D_0(R)$ is transitive

$$\mu_R = \begin{pmatrix} & x & y \\ x & 0.121 & 0.121 \\ y & 0.121 & 0.121 \end{pmatrix} \quad \nu_R = \begin{pmatrix} & x & y \\ x & 0.006 & 0.006 \\ y & 0.006 & 0.107 \end{pmatrix},$$

it is evidently partially included, it is not transitive because

$$\mu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R} = \begin{pmatrix} & x & y \\ x & 0.121 & 0.121 \\ y & 0.121 & 0.121 \end{pmatrix} \quad \nu_{R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R} = \begin{pmatrix} & x & y \\ x & 0.006 & 0.006 \\ y & 0.006 & 0.006 \end{pmatrix},$$

besides

$$\mu_{D_0(R)} = \begin{pmatrix} & x & y \\ x & 0.121 & 0.121 \\ y & 0.121 & 0.121 \end{pmatrix}$$

$$\nu_{D_0(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_0(R)} = \begin{pmatrix} & x & y \\ x & 0.121 & 0.121 \\ y & 0.121 & 0.121 \end{pmatrix}.$$

Theorem 18 *If $R \in IFR(X \times X)$ is reflexive, transitive and partially included, then*

$$D_p(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_p(R) = D_p(R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R) = D_p(R)$$

for every $p \in [0, 1]$.

Proof. Through the Theorem 17, we have

$$D_p(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_p(R) \leq D_p(R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R) \leq D_p(R)$$

and as R is reflexive, then $D_p(R)$ is reflexive (Theorem 1), that is to say

$$D_p(R) \leq D_p(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_p(R),$$

so

$$D_p(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_p(R) = D_p(R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R) = D_p(R). \quad \square$$

Lemma 2 *For every $R \in IFR(X \times X)$, it is verified that*

i) $D_0(\hat{R}) = D_0(R) \vee D_0^2(R) \vee D_0^3(R) \vee \ldots \vee D_0^n(R)$

i) $D_1(\hat{R}) = D_1(R) \vee D_1^2(R) \vee D_1^3(R) \vee \ldots \vee D_1^n(R)$

Proof. We will use the following notaion in the whole proof

$$R^2 = R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R,\ R^3 = R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R,\ D_0^2(R) = D_0(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_0(R),\ \text{etc...}$$

i) $\hat{R} = R \vee R^2 \vee R^3 \vee \ldots \vee R^n$, from where

$$\begin{aligned} \mu_{\hat{R}}(x,y) &= \mu_{R\vee R^2\vee R^3\vee\ldots\vee R^n}(x,y) = \\ &= \mu_R(x,y) \vee \mu_{R^2}(x,y) \vee \mu_{R^3}(x,y) \vee \ldots \vee \mu_{R^n}(x,y) \end{aligned}$$

for every $(x,y) \in X \times X$, through the Lemma 1 we know that

$$\mu_{D_0(R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R)}(x,y) = \mu_{D_0(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_0(R)}(x,y),$$

then

$$\begin{aligned} \mu_{D_0(\hat{R})}(x,y) &= \mu_{\hat{R}}(x,y) = \\ &= \mu_R(x,y) \vee \mu_{R^2}(x,y) \vee \mu_{R^3}(x,y) \vee \ldots \vee \mu_{R^n}(x,y) = \\ &= \mu_{D_0(R)}(x,y) \vee \mu_{D_0 \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_0}(x,y)\ldots \end{aligned}$$

therefore

$$D_0(\hat{R}) = D_0(R) \vee D_0^2(R) \vee \ldots \vee D_0^n(R)$$

ii) we know that

$$\begin{aligned} \nu_{\hat{R}}(x,y) &= \nu_{R\wedge R^2\wedge R^3\wedge\ldots\wedge R^n}(x,y) = \\ &= \nu_R(x,y) \wedge \nu_{R^2}(x,y) \wedge \nu_{R^3}(x,y) \wedge \ldots \wedge \nu_{R^n}(x,y) \end{aligned}$$

for every $(x, y) \in X \times X$, through the Lemma 1, we have

$$\mu_{D_1(R \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} R)}(x, y) = \mu_{D_1(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_1(R)}(x, y),$$

then

$$\begin{aligned} \mu_{D_0(\hat{R})}(x, y) &= 1 - \nu_{\hat{R}}(x, y) = \\ &= 1 - \wedge(\nu_R(x, y), \nu_{R^2}(x, y), \nu_{R^3}(x, y) \ldots) = \\ &= (1 - \nu_R(x, y)) \vee (1 - \nu_{R^2}(x, y)) \vee \ldots \vee (1 - \nu_{R^n}(x, y)) = \\ &= \mu_{D_1(R)} \vee \mu_{D_1^2(R)} \vee \ldots \vee \mu_{D_1^n}(R), \end{aligned}$$

therefore

$$D_1(\hat{R}) = D_1(R) \vee D_1^2(R) \vee D_1^3(R) \vee \ldots \vee D_1^n(R). \quad \square$$

Theorem 19 *For every $R \in IFR(X \times X)$, it is verifed that*

i) $D_0(\hat{R}) = \widehat{D_0(R)}$

ii) $D_1(\hat{R}) = \widehat{D_1(R)}$

Proof. It is a consequence of the Lemmas 1 and 2. $\square$

Theorem 20 *If for every $R \in IFR(X \times X)$, is reflexive, then*

$$\widehat{D_p(R)} = D_p(R) \overset{\widehat{\vee,\wedge}}{\underset{\wedge,\vee}{\circ}} D_p(R).$$

Proof. We sill use the same notation that in the Lemma 2

$$\begin{aligned} D_p(R) &\leq D_p(R) \overset{\vee,\wedge}{\underset{\wedge,\vee}{\circ}} D_p(R) \Rightarrow \widehat{D_p(R)} < \widehat{D_p^2(R)} \Rightarrow \\ D_p(R) &\leq D_p^2(R) \leq D_p^3(R) \leq D_p^4(R) \leq D_p^5(R) \leq D_p^6(R) \leq \ldots \end{aligned}$$

calculating $\widehat{D_p(R)}$ and $\widehat{D_p^2(R)}$ and imposing that $\widehat{D_p(R)} \leq \widehat{D_p^2(R)}$, we get

$$\begin{aligned} & D_p(R) \vee D_p^2(R) \vee D_p^3(R) \vee D_p^4(R) \vee D_p^5(R) \cdots = \\ & = D_p^2(R) \vee D_p^4(R) \vee D_p^6(R) \vee \cdots \quad \square \end{aligned}$$

5 Conclusions

From the study made on the intuitionistic fuzzy relations we can say that these ones generalize the fuzzy relations.

However it is necessary to remark that certain properties, such as intuitionistic antisymmetry doesn't recover the fuzzy antisymmetry given by A. Kaufmann when you apply the definition given by us on fuzzy relations.

We have also seen that Atanassov's operators support in a natural way properties as reflexivity, symmetry, intuitionistic antisymmetry, etc. Nevertheless for the intuitionistic transitive property to be supported by these operators we have had to study (and define) the partially included relations. Finally, it is worth to remark that all fuzzy relation is partially included; which is not verified for the intuitionistic relations.

6 Future works

Nowadays we are using the theory developed in these two papers in the study of the Equations of intuitionistic fuzzy relations from the algebraic point of vies as from the algorithmic point of view.

We have also started the study of the Intuitionistic Logic and its possible application in knowledge Engineering, Natural language, etc...

Finally we have to say that these intuitionistic fuzzy relations are being applied by K. Atanassov and ourselves on Graphos Theory and Neural Network.

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