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Synthesis of membership functions to determine a radius of influence

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Abstract

In this work we study the synthesis of membership functions when they have been calculated from a set of observations according to the definition of (Zhang, 1993). The results obtained have been used to determine the radius of influence of a tramp for *Podarcis lilfordi* (a kind of saurians) in the island of Cabrera (Balearic Islands). The measure of this radius was used in a later work to estimate the density of these saurians in the island.

Keywords: Consensus, membership functions, radius of influence.

0 Introduction

The definition and application of synthesis functions in fuzzy sets appear in several fields (Grabisch, 1995), e.g., in education (Fourali, 1994), knowledge acquisition (Torra, et al., 1995). The study of these functions is often focused to the analysis of their properties (unanimity, independence of irrellevant alternatives, ...) but without considering in detail the way in which these membership functions have been generated, and therefore what they represent. See for example the analysis of synthesis functions in (Dubois, et al., 1991, Bardossy et al., 1993, Torra, 1995).

In this paper we study the combination of membership functions built according to the method introduced in (Zhang, 1993) when the synthesised membership function is also of the same form.

This work begins in section 1 with some preliminaries used in the following sections. In particular, we review the definitions given in (Zhang, 1993). In section 2 the main results about synthesis functions are given. The paper finishes with section 3 where we show the application of the results obtained in section 2 to determine the radius of influence of a tramp.

1 Preliminaries

In (Zhang, 1993) a formal definition of fuzzy sets to describe fuzzy categories is introduced. This definition supposes that for any fuzzy category $\hat{a}$ with reference set X we can partition X into three subsets X_0, X_f and X_1. The first subset consists on all the elements that do not belong to the category $\hat{a}$, X_1 consists on all the elements that without any doubt belong to the category $\hat{a}$ and in X_f there are those elements whose memberships in the category are doubtful. The three elements, according to Zhang, will be respectively referred to as the *0-subset,* the *fringe* and the *1-subset* for $\hat{a}$, and the partition will be called *fringe partition.*

Zhang also supposes (property 2 in (Zhang, 1993)) a partial ordering $\geq_{\hat{a}}$ in the fringe of $\hat{a}$ such that $x \geq_{\hat{a}} y$ means x is at least as qualified as to be a member of $\hat{a}$ as y. In order to define the membership function corresponding to $\hat{a}$ we need a function μ called measure of reference that applied to a set X corresponds to the number of elements that we find in the set X. Putting these elements together, we get the quadruplet

$$s(\hat{a}) := (X, \mu, (X_0, X_f, X_1), \geq_{\hat{a}})$$

Zhang refers to this quadruplet as the Q-structure of $\hat{a}$.

Definition 1 *The partition* (X_0, X_f, X_1) *corresponding to the category* $\hat{a}$ *together with a partial ordering* $\leq_{\hat{a}}$ *and the measure of reference* μ

allow us to build a membership function $\hat{A}_t$. We define, accordingly to (Zhang, 1993), given a Q-structure $(X, \mu, (X_0, X_f, X_1), \geq_{\hat{a}})$ the membership function $\hat{A}_t$ corresponding to the category $\hat{a}$ as:

$$\hat{A}_t(x) := \begin{array}{ll} 0 & \text{if } x \in X_0 \\ [\alpha(x), \alpha^+(x)] & \text{if } x \in X_f \\ 1 & \text{if } x \in X_1 \end{array}$$

where $\alpha(x)$ and $\alpha^+(x)$ are defined as:

$$\begin{aligned} \alpha(x) : &= \mu(\{y \in X_f | x >_{\hat{a}} y\})/\mu(X_f) \\ \alpha^+(x) : &= \mu(\{y \in X_f | x \geq_{\hat{a}} y\})/\mu(X_f) \end{aligned}$$

where $>_{\hat{a}}$ is defined from $\geq_{\hat{a}}$ in the usual way.

When there is a domain with a total ordering in the reference set, e.g. the real axis, and when the monotonicity of the membership function in relation to the ordering is known, then it is possible to build from a set of observations a fuzzy set that approximates the membership function of the fuzzy category $\hat{a}$. After introducing the concept of consistency of a set of observations in relation to a reference set, we define this construction.

Definition 2 *Let X be a completely ordered reference set; let $\geq$ be a relation of total ordering. Let Ξ be a set of observations of elements of X ($\Xi \subset X$) and let $\Xi = (\xi_0, \xi_f, \xi_1)$ be a partition of Ξ where the elements of each ξ_i are elements of the partition X_i ($\xi_i \subset X_i$). These conditions mean that after observing the set of elements Ξ we know which of the observed elements do not belong to the category $\hat{a}$, which belong to $\hat{a}$ with a certain degree and which completely belong to $\hat{a}$. In these conditions, we say that the partition $\Xi = (\xi_0, \xi_f, \xi_1)$ is consistent with the ordering function if and only if the following conditions are satisfied:*

$$\sup\{x : x \in \xi_0\} \leq \inf\{x : x \in \xi_f\}$$
$$\sup\{x : x \in \xi_f\} \leq \inf\{x : x \in \xi_1\}$$

Definition 3 *Let X be a completely ordered reference set, $\geq$ be a relation of total ordering over X. Let Ξ be a set of observations and $\Xi = (\xi_0, \xi_f, \xi_1)$ be a partition consistent with X. Let η be a relative measure function (e.g., the cardinality function). In these conditions the membership function of the category $\hat{a}$ related to the Q-structure $(X, \eta, (\xi_0, \xi_f, \xi_1), \geq)$ is defined as:*

$$\hat{A}(x) := \begin{array}{lll} 0 & \text{if exists } a \in \xi_0 & x \leq a \\ 1 & \text{if exists } a \in \xi_f & a \leq x \\ [\alpha(x), \alpha^+(x)] & \text{otherwise} & \end{array}$$

where $\alpha(x)$ and $\alpha^+(x)$ are defined as:

$$\begin{array}{rcl} \alpha(x) : & = & \eta(\{y \in \xi_f | x > y\})/\eta(\xi_f) \\ \alpha^+(x) : & = & \eta(\{y \in \xi_f | x \geq y\})/\eta(\xi_f) \end{array}$$

This function $\hat{A}$, that depends on the Q-structure $(X, \eta, (\xi_0, \xi_f, \xi_1), \geq)$ is used in the following as an approximation of the function $\hat{A}_t$.

2 Results

We present in this section several results about synthesis of membership functions when they have been built according to definition 3 from Q-structures of the form $(X, \eta, (\xi_0, \xi_f, \xi_1), \geq)$. We begin defining the compatibility of two partitions and the union of two partitions.

Definition 4 *Let Ξ and Ξ' be two sets of observations over the same domain X, and let $\Xi = (\xi_0, \xi_f, \xi_1)$ and $\Xi' = (\xi'_0, \xi'_f, \xi'_1)$ two consistent partitions of Ξ and Ξ' with respect to an ordering function $\geq$. In these conditions we say that the partitions are compatibles if they satisfy:*

$$\begin{array}{rcll} sup\{x : x \in \xi_0\} & < & inf\,\{x : x \in \xi'_f\} & (r1) \\ sup\{x : x \in \xi'_0\} & < & inf\,\{x : x \in \xi_f\} & \\ sup\{x : x \in \xi_f\} & < & inf\,\{x : x \in \xi'_1\} & \\ sup\{x : x \in \xi'_f\} & < & inf\,\{x : x \in \xi_1\} & \end{array}$$

This is, partitions are compatibles when they do not overlap.

Definition 5 *Let $\Xi = (\xi_0, \xi_f, \xi_1)$ and $\Xi' = (\xi'_0, \xi'_f, \xi'_1)$ be two set of compatible observations. Then the union set of observations $\Xi_\cup$ is defined as $\Xi_\cup = (\xi_0 \cup \xi'_0, \xi_f \cup \xi'_f, \xi_1 \cup \xi'_1)$.*

Now we consider the use of arithmetic means as synthesis functions for fuzzy sets. We compare the synthesised membership function obtained with this function and the fuzzy set built from to the union set of observations. We show first a result when the synthesis function is the arithmetic mean without weights, and then when it is the weighted arithmetic mean.

Proposition 1 *Let $\hat{A}$ and $\hat{A}'$ be two membership functions built according to definition 3 from the Q-structures $(X, \eta, \Xi = (\xi_0, \xi_f, \xi_1), \geq)$ and $(X, \eta, \Xi' = (\xi'_0, \xi'_f, \xi'_1), \geq)$. Let Ξ and Ξ' be two sets of compatible observations with an empty intersection of ξ_f and ξ'_f. In these conditions the membership function defined as $A_c(x) = (\hat{A}(x) + \hat{A}'(x))/2$ is equivalent to the membership function $\hat{A}_\cup$ built from the union set of observations $\Xi_\cup$ when the measure of ξ_f equals the measure of ξ'_f.*

Proof. From the equality

$$\hat{A}_c(x) = (\hat{A}_1(x) + \hat{A}_2(x))/2 \tag{1}$$

after developing $\hat{A}_i(x)$ for a certain x in the set $\xi_f \cup \xi'_f$, replacing $\alpha_i(x)$ and $\alpha'_i(x)$ by their definition

$$\begin{aligned} \alpha(x) &= \eta(\{y \in \xi_f | x > y\})/\eta(\xi_f) \\ \alpha'(x) &= \eta(\{y \in \xi'_f | x > y\})/\eta(\xi'_f) \\ \alpha_\cup(x) &= \eta(\{y \in \xi_f \cup \xi'_f | x > y\})/\eta(\xi_f \cup \xi'_f) \end{aligned}$$

and defining for that x in $\xi_f \cup \xi'_f$ the following constants α, β, a and b:

$$\begin{array}{ll} \alpha = \eta(\{y \in \xi_f | x > y\}) & a = \mu(\xi_f) \\ \beta = \eta(\{y \in \xi'_f | x > y\}) & b = \mu(\xi'_f) \end{array}$$

we get the equation:

$$2(\alpha + \beta)/(a + b) = \alpha/a + \beta/b$$

This equation can also be written as:

$$(a - b)(\alpha b - \beta a) = 0.$$

The solutions of this equation are:

$$\begin{aligned} a &= b \\ \alpha/a &= \beta/b \end{aligned}$$

Among these conditions only the first one, that is equivalent to:

$$\eta(\xi_f) = \eta(\xi'_f) \tag{2}$$

is meaningful in the framework of fringe partitions.

The second equation $\alpha/a = \beta/b$, after replacing α, β, a and b by their definitions given above, is equivalent to:

$$\eta(\{y \in \xi_f | x > y\}) = \eta(\{y \in \xi'_f | x > y\})\eta(\xi_f)/\eta(\xi'_f)$$

If we make in this equation $x = x_0$ for a certain $x_0 \in \xi'_f$ we obtain a value for $\eta(\{y \in \xi_f | x_0 > y\})$. If we make now $x = x_1$ with $x_1 > x_0$ in a way that $\eta(\{y \in \xi_f | x_1 > y\})$ equals $\eta(\{y \in \xi_f | x_0 > y\})$, we obtain a value for $\eta(\{y \in \xi'_f | x_1 > y\})$ different to $\eta(\{y \in \xi'_f | x_0 > y\})$. So, we get a contradiction. Therefore, only (2) is a valid solution.

□

In figures 1 and 2 we give two examples of synthesised membership functions. We have used the function η defined as the cardinality function. In the first example we have two membership functions satisfying $\eta(\xi_f) = \eta(\xi'_f)$ together with their corresponding synthesised membership function (the synthesised function with the arithmetic mean, according to proposition 1, equals the one obtained from the union set of observations). In figure 2 we show two fuzzy sets when $\eta(\xi_f) \neq \eta(\xi'_f)$, the synthesised membership functions with the arithmetic mean together with the fuzzy set corresponding to the union set of observations $\Xi_\cup$. In figure 1 the measures of the fringe satisfy $\eta(\xi_1) = \eta(\xi_2) = 3$ while in figure 2 they satisfy $\eta(\xi_1) = 4$, $\eta(\xi_2) = 2$.

Figure 1. Figure 2.

Proposition 2 *Let $\hat{A}$ and $\hat{A}'$ be two membership functions built according to definition 3 from the Q-structures $(X, \eta, \Xi = (\xi_0, \xi_f, \xi_1), \geq)$ and $(X, \eta, \Xi' = (\xi'_0, \xi'_f, \xi'_1), \geq)$. Let Ξ and Ξ' be two sets of compatible observations with an empty intersection of ξ_f and ξ'_f. In these conditions the synthesised membership function defined as $A_c(x) = (p\hat{A}(x) + q\hat{A}'(x))/2$ with $p + q = 1$ is equivalent to the membership function $\hat{A}_\cup$ built from the union set of observations $\Xi_\cup$ if and only if*

$$p = \eta(\xi_f)/(\eta(\xi_f) + \eta(\xi'_f)).$$

In the preceeding propositions we have considered that the intersection of ξ_f and ξ'_f is empty and therefore its measure is zero. We consider next a non empty intersection.

Proposition 3 *Let $\hat{A}$ and $\hat{A}'$ be two membership functions build according to definition 3 from the Q-structures $(X, \eta, \Xi = (\xi_0, \xi_f, \xi_1), \geq)$ and $(X, \eta, \Xi' = (\xi'_0, \xi'_f, \xi'_1), \geq)$. Let Ξ and Ξ' be two sets of compatible observations. In these conditions the synthesised membership function defined as $A_c(x) = (\hat{A}(x) + \hat{A}'(x))/2$ is equivalent to the membership function $\hat{A}_\cup$ built from the union set of observations $\Xi_\cup$ if and only if*

$$(a - b)(\alpha b - \beta a) + (a + c)(\alpha c - \gamma a) + (b + c)(\beta c - \gamma b) = 0$$

where

$$c = \eta(\xi_f \cap \xi'_f) \qquad \gamma = \eta(\{y \in \xi_f \cap \xi'_f | x > y\})$$

$$\begin{array}{llll} a & = \eta(\xi_f) - c & \alpha = \eta(\{y \in \xi_f | x > y\}) - \gamma \\ b & = \eta(\xi'_f) - c & \beta = \eta(\{y \in \xi'_f | x > y\}) - \gamma \end{array}$$

Corollary 1 *When $\eta(\xi_f) = \eta(\xi'_f)$ the equality given above is satisfied if,*

$$\eta(\xi_f \cap \xi'_f) = 0$$

or if,

$$\eta(\xi_f) - \eta(\xi_f \cap \xi'_f) = \eta(\xi'_f) - \eta(\xi_f \cap \xi'_f) = 0$$

That is, or the measure of the intersection is zero (this case corresponds to the one in proposition 1) or we have that sets ξ_f and ξ'_f have the same measure and this measure corresponds to the measure of the intersection set.

3 Application

In this section we apply the results obtained previously to a framework where membership functions are defined over the real axis (so it exists a total ordering) and they are monotonic in relation to this axis. In our case, in order to define the membership function instead of a single set of observations we have two of them. The first one, that corresponds to the so-called positive observations and is represented as Ξ^+, consists on the elements x_i that support the construction of the fuzzy set R. About this element we know that if there is an element $x_i \leq x$ the $R(x_i) \geq R(x)$. On the other hand, the second set, that corresponds to the so-called negative observations (Ξ^-), consists on the elements that support the construction of the fuzzy set negation of R, represented $R_\neg$. In this case the elements x_i satisfy the reversal condition: if there is an element $x_i \leq x$ the $R_\neg(x_i) \leq R_\neg(x)$.

Given both sets Ξ^+ and Ξ^- of positive and negative observations we can build, according to definition 3, two membership functions R_{Ξ^+} and R_{Ξ^-} that approximate the theoretical fuzzy sets R and $R_\neg$. Due to the fact that these concepts, R and $R_\neg$, are complementary, they should satisfy the following equation:

$$R(x) = N(R_\neg(x)) \tag{3}$$

This equality can also be writen in the following way:

$$R = (pR + qN(R_{\neg})) \qquad \text{with } p + q = 1 \tag{4}$$

However, as R_{Ξ^+} and R_{Ξ^-} are only approximations of R and $R_{\neg}$, they, in general, neither satisfy equality (3) nor (4). Still, equation (4) can be used to get a new approximation of the membership function of R. We represent this new approximation by R_s. The membership function of R_s can be defined, using equation (4), in the following way:

$$R_s = (pR_{\Xi^+} + qN(R_{\Xi^-})) \qquad \text{with } p + q = 1 \tag{5}$$

This new fuzzy set, due to the fact that it is defined as the synthesis of R_{Ξ^+} and R_{Ξ^-} would be more accurate and reliable than the initial sets Ξ^+ and Ξ^-. The results described above have been used to determine the radius of influence of a "pitfall" tramp (Tellería, 1986) to capture saurians. This determination has been a part of a procedure (census method) used to estimate the density of a population of *Podarcis lilfordi* in the Cabrera Island (Balearic Islands) (Sáez, 1993a, 1993b).

The census method used, the Petersen one (Krebs, 1989), consisted of (i) a phase of capture and mark of the individuals of the population and (ii) a phase of recapture. The size of the population is estimated as the quotient of individuals captured with marks.

In order to carry out the capture, a set of tramps arranged as a network are installed to cover all the sampling zone (see figure 3).

This method supposes on one hand that the population placed in the sampling plot is closed (i.e., there are not changes due to natality, mortality or migration). On the other hand it is also supposed that each individual occupies a particular area, the vital domain, in which the animal moves and does all its rutinary activities. This area can overlap, in a certain degree, with vital domains corresponding to some other individuals.

However, there are two main difficulties to apply this technique (Skalski, et al., 1992). These difficulties are:

1. To satisfy the assumption of equal capturability of sampled individuals. This is, all the individuals have to be capturable and

Figure 3. *Sampling zone and radius of influence.*

have to be capturable with the same probability. This condition implies a correct disposition of the tramps, so all the individuals that interact with the area comprised by the surface of the tramps network were potencially capturables. This forces to choose the distance between tramps small enough so that no spaces remain without sampling inside of the network. At the same time, in order to get the best results with the less number of observations is adequate to maximise the distance between tramps to cover the maximum of the sampled area.

2. To know the real area where the sampled individuals interact. This is needed to estimate correctly the density and therefore the population. Due to the fact that it could exist some individuals with their vital domains overlapping only partially the area comprised by the network (and therefore capable of being capturated) the real sampled area is greater than the one comprised by the network.

Up to now, none of the works published about marking and recapture in saurians takes into account both problems. However, altough there have been developed several approaches to solve these problems in populations of micromammals (Tellería, 1986; Skalski, et al. 1992), they are not appropriated in populations of high density as is the case of *Podarcis lilfordi*. This is due to the fact that in this case a high effort of sampling is required.

In order to solve both problems, we have defined an experience of marking-reobservation to determine the radius of influence of a tramp, i.e., the distance from which the individuals fall into the tramp. The resultant value has been used as an approximation of the distance to separate the tramp and also of the band width that should be added to the area of the network to determine precisely the sampled area. The experience consisted in the marking of the individuals that were captured in a single tramp in a single day (5th of August of 1993). In the next day 8 radial transects (itineraries) were done. They were approximately equidistants and convergents to the place where the tramp was situated the previous day. Along the course we registered the distance where individuals were observed. We also registered whether the individual had been marked the day before or not.

The radius of influence was calculated from the observations according to the results of the previous sections. These results can be applied because the radius of influence is a fuzzy quantity that can be expressed by means of a decreasing nonmonotonic function when the tramp is in the origin of coordenates. See that if an animal at a certain distance x to the origin (to the tramp) is within the radius of influence then any animal in a smaller distance is also in the radius.

The observations were grouped in two sets:

- Ξ^+ with positive observations, i.e., the marked individuals (they support the concept of influence radius)
- Ξ^- with negative observations, i.e., the nonmarked individuals (they support the complementary concept "noninfluence of the tramp")

In our case, as we suppose that the concept *radius of influence* (R) is the negation *of noninfluence of a tramp* ($R_\neg$), equation (3) should be satisfied. Therefore, their approximations R_{Ξ^+} and R_{Ξ^-}, generated, respectively, from Ξ^+ and Ξ^-, should satisfy (5). In this case, we can apply proposition 2 to R_{Ξ^+} and R_{Ξ^-} to get a better approximation of

Figure 4. *Fringe partitions. Marked individuals* (Ξ^+) *and nonmarked individuals* (Ξ^-).

the concept radius of influence. To do so, we have defined first a fringe partition of the set $X = [0, \infty)$ in the following way:

$$
\begin{aligned}
X_{R,1} &= X_{\neg R,0} = [0, \inf_x\{x \in \Xi^-\}) \\
X_{R,0} &= X_{\neg R,1} = (\sup_x\{x \in \Xi^+\}, \infty) \\
X_{R,f} &= X_{\neg R,f} = (\inf_x\{x \in \Xi^-\}, \sup_x\{x \in \Xi^+\})
\end{aligned}
$$

The partition of X leads to the following fringe partitions of Ξ^+ and Ξ^- (see figure 4):

$$
\begin{aligned}
\Xi^+ &= (\Xi^+ \cap X_{R,0}, \Xi^+ \cap X_{R,f}, \Xi^+ \cap X_{R,1}) \\
\Xi^- &= (\Xi^- \cap X_{R,0}, \Xi^- \cap X_{R,f}, \Xi^- \cap X_{R,1})
\end{aligned}
$$

Now, as these fringe partitions of Ξ^+ and Ξ^- are compatibles, proposition 3 can be applied.

With the observations described above of marked individuals, Ξ^+, and nonmarked individuals, Ξ^-, we built, respectively the membership functions R_{Ξ^+} and R_{Ξ^-} and the synthesised membership function R_s according to (2) with weights p and q satisfying proposition 2. The membership function R_s was used by means of an α-cut to choose a radius of influence. This radius was used in three sesions of capture in three consecutive days (10th, 11th and 12th of August of 1993) by means of the instalation of 12 tramps. The results of the cens can be found in (Sáez, 1993b).

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