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On some constructions of new triangular norms

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Abstract

We discuss the properties of two types of construction of a new t-norm from a given t-norm proposed recently by B. Demant, namely the dilatation and the contraction. In general, the dilatation of a t-norm is an ordinal sum t-norm and the continuity of the outgoing t-norm is preserved. On the other hand, the contraction may violate the continuity as well as the non-continuity of the outgoing t-norm. Several examples are given.
Keywords: contraction, dilatation, ordinal sum, triangular norm.

1 Introduction

Among several constructions of the new t-norms form given ones [2,3,4], we recall two basic constructions arisen from the semigroup interpretation of a triangular norm, see Schweizer and Sklar [3].

Ordinal sum: Let $\left[ ]a_k, b_k[;\ k \in \mathcal{K} \right]$ be a disjoint system of open subintervals of the unit interval [0,1] and let $[T_k;\ k \in \mathcal{K}]$ be a system of given t-norms. For $x, y \in [0,1]$, put

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k \left((x - a_k)/(b_k - a_k), \right. & \\ \left. (y - a_k)/(b_k - a_k) \right) & \text{if } x, y \in [a_k, b_k] \\ & \text{for some } k \in \mathcal{K} \\ \min(x,y) & \text{otherwise} \end{cases}$$

Then T is a t-norm and it is called an ordinal sum with summands $< a_k, b_k, T_k >$, $k \in \mathcal{K}$, briefly $T \sim [< a_k, b_k, T_k >; k \in \mathcal{K}]$.

Semigroup deformation: let $\phi : [0,1] \to [a,1]$, $a \in [0,1[$, be an increasing bijection. Let T be a given t-norm. For $x, y \in [0,1]$ put

$$T_\phi(x,y) = \phi^{(-1)}(T(\phi(x), \phi(y)),$$

where $\phi^{(-1)} : [0,1] \to [0,1]$ is the pseudo-inverse of ϕ, $\phi^{(-1)}(x) = \phi^{(-1)}(\max(a,x))$. Note that if $a = 0$ then T_ϕ is called a ϕ-transformation of T and T and T_ϕ are isomorphic (and hence the properties such as continuity, strictness, etc., are preserved). If $a > 0$ then the deformation T_ϕ preserves the continuity and the Archimedean property (and the nilpotency) but the strictness may be violated. Take, e.g., the product t-norm T_P and let $\phi(x) = 2^{x-1}$, i.e., $a = 1/2$ and $\phi^{-1}(x) = 1 + \log_2 x$. Then

$$(T_P)_\phi(x,y) = 1 + \log_2 \max(1/2, 2^{x-1} \cdot 2^{y-1}) = \max(0, x + y, -1).$$

Hence the ϕ-deformation of the strict product t-norm T_P is the nilpotent Lukasiewicz t-norm T_L, $(T_P)_\phi = T_L$.

Recently, Demant [1] has suggested two new types of t-norm constructions. Let $\phi : [0,1] \to [0,a]$, $a \in]0,1]$, be a given increasing bijection. For $x \in [0,1]$ we define the pseudo-inverse of ϕ by $\phi^{(-1)}(x) = \phi^{-1}(\min(a,x))$. Let T be a given t-norm. For $x, y \in [0,1]$, we define:

Contraction:

$$T^{(\phi)}(x,y) = \begin{cases} \phi^{(-1)}\left(T(\phi(x), \phi(y))\right) & \text{if } \max\,(x,y) < 1 \\ T(x,y) & \text{otherwise} \end{cases};$$

Dilatation:

$$T_{(\phi)}(x,y) = \begin{cases} \phi\left(T(\phi^{(-1)}(x), \phi^{(-1)}(y))\right) & \text{if } T(x,y) < a \\ T(x,y) & \text{otherwise} \end{cases} .$$

Both the contraction and the dilatation of a t-norm T are again t-norms, see [1]. Note that in the case $a = 1$ both the dilatation and the contraction are the usual semigroup transformation of Schweizer and Sklar [3], $T^{(\phi)} = T_{\phi}$ and $T_{(\phi)} = T_{\phi^{-1}}$. Further note that the only t- norms preserved by arbitrary deformation, transformation, and contraction are the limit t-norms T_M and T_W. However, the only t-norm preserved by an arbitrary dilatation is T_M, while $(T_W)_{(\phi)} \neq T_W$ whenever $\phi(1) \neq 1$.

2 Contractions of t-norms

For a given bijection $\phi : [0,1] \to [0,a]$ with $a < 1$ and a given t- norm T, the values of the contraction $T^{(\phi)}$ on the half-open square $[0,1[^2$ depend on the values of T on the half-open square $[0,a[^2$ only (the remainder of the domain is contained in the borders of the unit square where all t-norms coincide). Hence the non-continuity (and the absence of the Archimedean property) of T need not be true for its contraction $T^{(\phi)}$. On the other hand, the continuity of T may be violated by $T^{(\phi)}$, too, while the Archimedean property remains preserved.

Example 1 *i) Let $\phi(x) = a \cdot x$, $a \in]0,1[$, and let $T = T_P$. Then the contraction $T^{(\phi)}$ is defined by*

$$T^{(\phi)}(x,y) = \begin{cases} a \cdot x \cdot y & \text{if } \max(x,y) < 1 \\ x \cdot y & \text{otherwise} \end{cases} .$$

Note that $T^{(\phi)}$ is not continuous although T is continuous. Further, the strictness $T^{(\phi)}(x,y) < T^{(\phi)}(x,z)$ for each $x > 0$, $y < z$, holds true.

ii) Let $\phi(x) = a \cdot x$, $a \in]0,1[$, *and let* $T \sim \Big[< 0, a, T_P >, < a, 1, T_W > \Big]$ *be an ordinal sum t-norm. Then* T *is non- continuous (and non-Archimedean) but* $T^{(\phi)} = T_P$ *is continuous and Archimedean.*

■

For a composition law of two contractions we have the following result.

Proposition 1 *Let* $\phi : [0,1] \to [0,a]$ *and* $\psi : [0,1] \to [0,b]$ *be two bijections with* $a \leq 1$ *and* $b \leq 1$*. Let* T *be a given t-norm. Then*

$$\left[T^{(\phi)}\right]^{(\psi)} = T^{(\phi \circ \psi)},$$

i.e., the ψ*-contraction of a* ϕ*-contraction of* T *is the* $\phi \circ \psi$*-contraction of* T*.*

■

The problem of t-norms invariant under given ϕ-contraction will be partially solved in the next section.

3 Dilatations of t-norms

Non-trivial dilatations (i.e., when $a < 1$) are always ordinal sums with two summands.

Proposition 2 *Let* $\phi : [0,1] \to [0,a]$ *be a given increasing bijection where* $a \in]0,1[$ *and let* T *be a given t-norm. Then the* ϕ*-dilatation of* T *is an ordinal sum with two summands,*

$$T_{(\phi)} \sim \left[< 0, a, T_{\phi/a} >, < a, 1, T_a >\right],$$

where $T_{\phi/a}$ *is the transformation* T *of with respect to the mapping* $\phi/a : [0,1] \to [0,1]$*, while* T_a *is the deformation of* T *with respect to the linear transformation* $\lambda_a : [0,1] \to [a,1]$*,* $\lambda_a(x) = a + (1-a) \cdot x$*, depending only on* T *and* a *(independent of* ϕ *up to the value* $a = \phi(1)$*),* $T_a = T_{\lambda_a}$*,*

$$T_a(x,y) = \Big(\max\big[0, T(a + (1-a) \cdot x, a + (1-a) \cdot y) - a\big]\Big) / (1-a).$$

■

Remark 1 For Archimedean continuous t-norms we have the following result: let f be an additive generator of a given t-norm T [5] and let the left derivative of f in the point 1 be non-trivial, $f'_{-}(1) \in]-\infty, 0[$. Then $\lim_{a \to 1^-} T_a = T_L$, where T_L is the Lukasiewicz t-norm. The proof follows from the fact that if T has an additive generator f then T_a has an additive generator $f(a + (1-a) \cdot x)$.

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It is easy to see that for arbitrary dilatation the t-norm T_M remains stable. Further, for each $a \in]0,1[$ it is $[T_W]_a = T_W$ and hence for arbitrary ϕ it is $\left[T_W\right]_{(\phi)} \sim \left[< 0, a, T_W >, < a, 1, T_W > \right]$. Applying the ϕ-dilatation to T_W infinitely many times we get a new t-norm T_a^* depending only on a and invariant under ϕ-dilatation,

$$T_a^* \sim \left[< a^n, a^{n-1}, T_W >;\ n \in \mathbb{N}\right].$$

A natural question arises: for a given transformation ϕ, are there some other ϕ-dilatation invariant t-norms up to T_M (a continuous t-norm) and T_a^* (a discontinuous t-norm)? It is obvious that each ϕ-dilatation invariant t-norm T^* different from T_M should be an ordinal sum of type

$$T^* \sim \left[< a^n, a^{n-1}, T >; n \in N\right],$$

where T is a t-norm such that $T = T_a = T_{\phi/a}$. Requiring the continuity of T, we have the following result.

Proposition 3 *Let T be a continuous t-norm and let $a \in]0,1[$. Then T_a equals T if and only if T is the member of the extended Yager's family $\left[T_p^y, p \in]0, \infty]\right]$ [6], i.e., $T_\infty^y = T_M$ and for $p \in]0, \infty[$, the t-norm T_p^y is generated by an additive generator f_p, $f_p(x) = (1-x)^p$, $x \in [0,1]$.*

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Note that the proof is based on a modified Cauchy functional equation. Further, let f be an additive generator of a given t-norm T. Then $T_{\phi/a}$ has an additive generator $f \circ (\phi/a)$, see [5], and thus T equals

$T_{\phi/a}$ if and only if f differs from $f \circ (\phi/a)$ only by a multiplicative constant. For nilpotent t-norm T with the normed generator f (this is the case of the Yager's t-norms) this means that ϕ/a is the identity, i.e., $\phi(x) = a \cdot x$. We have just shown the next result.

Proposition 4 *Let $\phi : [0,1] \to [0,a]$, where $a \in]0,1[$, be an increasing bijection. If ϕ is not linear then the only continuous t-norm invariant under ϕ-dilatation is the strongest t-norm T_M. If ϕ is linear, then the only continuous t-norms invariant under ϕ-dilatation are the members of the family $\left[T^*_{a,p};\ p \in]0,\infty]\right]$, where $T^*_{a,p} \sim \left[< a^n, a^{n-1}, T^y_p >;\ n \in N\right]$.* ■

Note that $T^*_\infty = T_M$. Further, it is usual to put $T^y_0 = T_W$ (the left limit member of the Yager family). Then each member of the family $\left[T^*_{a,p};\ p \in [0,\infty]\right]$, where $T^*_{a,0} = T^*_a$, is invariant under the ϕ-dilatation for $\phi(x) = a \cdot x$.

Remark 2 Note that the ϕ-contraction acts as an inverse of the ϕ-dilatation, the opposite being not true, i.e., for arbitrary t-norm T it is $\left[T_{(\phi)}\right]^{(\phi)} = T$. Now, it is obvious that if a given t-norm T is invariant with respect to a given ϕ-dilatation it has to be invariant also with respect to the corresponding ϕ-contraction.

References

[1] Demant, B., Deformationen von t-Normen, ihre Symmetrien und Symmetriebrechungen, *preprint*.

[2] Fodor, J.C., A remark on constructing t-norms, *Fuzzy Sets and Systems* **41** (1991), 195-199.

[3] Schweizer, B. and Sklar, A., Associative functions and statistical triangle inequalities, *Publ. Math. Debrecen* **8** (1961), 169-186.

[4] Schweizer, B. and Sklar, A., Associative functions and abstract semigroups, *Publ. Math. Debrecen* **10** (1963), 69-81.

[5] Schweizer, B. and Sklar, A., *Probabilistic metric spaces,* North-Holland, New York, 1983.

[6] Yager, R.R., On a general class of fuzzy connectives, *Fuzzy Sets and Systems* **4** (1980), 235-242.