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On the Identity of Fuzzy Material Conditionals

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Abstract

Given $\mu,\ \eta : X \to [0,1]$ we study when the equality $I_\mu^T = I_\eta^T$ holds, T being a continuous t-norm, and I_θ^T the elemental preorder:

$$I_\theta^T(y/x) = sup\{z\ ;\ T(\theta(x), z) \leq \theta(y)\}.$$

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1 Introduction.

Given a subset $A \subset X$, $A \neq \emptyset$, the material conditional associated to A is the relation in $X \times X$ given by $\rightarrow_A = (A \times A) \cup (A' \times X)$, which is obviously a preorder. The following theorem gives a characterization of a subset through its associated material conditional [?].

Theorem 1 *If A and B are non-empty subsets of X, then:*

$$\rightarrow_A = \rightarrow_B \text{ if and only if } A = B$$

Proof. The sufficient condition is immediate.

Furthermore, since $\rightarrow_A = (A \times A) \cup (A' \times X) = (X \times X) - (A \times A')$, if $\rightarrow_A = \rightarrow_B$, that is, if $(X \times X) - (A \times A') = (X \times X) - (B \times B')$, then $A \times A' = B \times B'$, which holds if and only if $A = B$. $\square$

Given a fuzzy relation $I : X \times X \to [0,1]$, and a continuous t-norm T, it is known [?] that I is a T-preorder if and only if

$$I = \inf_{\mu \in \mathcal{F}} I_{\mu}^{T},$$

$\mathcal{F}$ being the set of the T-logical states of I, that is

$$\mathcal{F} = \{\mu : X \to [0,1] \ ; \ T(\mu(x), I(y/x)) \leq \mu(y) \ \ \forall x, y\},$$

and I_{μ}^{T} the elemental preorder defined by

$$I_{\mu}^{T}(y/x) = \sup\{z \in [0,1] \ : \ T(\mu(x), z) \leq \mu(y)\}.$$

Choosing $\mu = \varphi_A$, the classic characteristic function of A, is

$$I_{\varphi_A}^{T}(y/x) = \varphi_{\to A} \ ;$$

and then it is natural to consider that the fuzzy elemental preorders are a generalization of the classic material conditionals[1]. We can therefore study whether it is possible to extend theorem 1 to the fuzzy preorders.

2 Equality of fuzzy material conditionals.

If we start to pay attention to the equivalence for the Min t-norm, we obtain that if the preorders I_{μ}^{Min}, I_{η}^{Min} are equal, then μ and η can only be different at the maximum values of μx and ηx.

Theorem 2 $I_{\mu}^{Min} = I_{\eta}^{Min}$ *if and only if*

$$\mu x \neq \eta x \ \Rightarrow \ \mu x \geq \mu y \ \ and \ \ \eta x \geq \eta y \ for \ all \ y.$$

Proof. Let $I_{\mu}^{Min}(y/x) = I_{\eta}^{Min}(y/x)$, for each x, y. We know that:

$$I_{\theta}^{Min}(y/x) = \sup\{z \ ; \ Min(\theta x, z) \leq \theta y\} = \begin{cases} 1, & if \ \ \theta x \leq \theta y \\ \theta y, & if \ \ \theta x > \theta y \ . \end{cases}$$

Let us choose an x such that $\mu x \neq \eta x$, and let us suppose that there exists a y verifying $\mu x < \mu y$. We will obtain $I_{\mu}^{Min}(y/x) = 1 = I_{\eta}^{Min}(y/x)$, and then $\eta x \leq \eta y$.

Furthermore, $I_{\mu}^{Min}(x/y) = \mu x = I_{\eta}^{Min}(x/y) < 1$, and so $I_{\eta}^{Min}(x/y) = \eta x = \mu x$, which gives a contradiction; so, it must be $\mu x \geq \mu y$ for all

[1] The elemental preorders are not the unique generalizations of the material conditional: see for example [?]

y. In a similar way, $\eta x \geq \eta y$ is obtained for all y.

Reciprocally, let us suppose that if $\mu x \neq \eta x$ then $\mu x \geq \mu y$ and $\eta x \geq \eta y$, for all y. We will prove that the equality $I_\mu^{Min} = I_\eta^{Min}$ holds.

- If $\mu x = \eta x$ and $\mu y = \eta y$, clearly $I_\mu^{Min}(y/x) = I_\eta^{Min}(y/x)$.

- If $\mu x = \eta x$ and $\mu y \neq \eta y$, then $\mu y \geq \mu x$, $\eta y \geq \eta x$, and $I_\mu^{Min}(y/x) = I_\eta^{Min}(y/x) = 1$.

- If $\mu x \neq \eta x$ and $\mu y = \eta y$, we obtain $\mu x \geq \mu y$, $\eta x \geq \eta y$, and $I_\mu^{Min}(y/x) = \mu y = \eta y = I_\eta^{Min}(y/x)$.

- Finally, if $\mu x \neq \eta x$ and $\mu y \neq \eta y$, since $\mu x \geq \mu y$, $\eta x \geq \eta y$, and $\mu y \geq \mu x$, $\eta y \geq \eta x$, obviously $\mu x = \mu y$, $\eta x = \eta y$, and $I_\mu^{Min}(y/x) = I_\eta^{Min}(y/x)$. □

Note. If T is Archimedean with additive generator h [?], then from:
$I_\mu^T(y/x) = \sup\{z \ / \ T(\mu x, z) \leq \mu y\} = \sup\{z / \ h^{(-1)}(h(\mu x) + h(z)) \leq \mu y\}$,
it follows:

- If $\mu x \leq \mu y$, then $h(\mu x) \geq h(\mu y)$, and $h(\mu x) + h(z) \geq h(\mu y)$ for all z. So, $h^{(-1)}(h(\mu x) + h(z)) \leq h^{(-1)} h(\mu y) = \mu y$ for all z, and $I_\mu^T(y/x) = 1$.

- If $\mu x > \mu y$, $h(\mu x) < h(\mu y)$ and $h(\mu y) - h(\mu x) \in [0, h(0)]$. Let us see that $I_\mu^T(y/x) = h^{(-1)}(h(\mu y) - h(\mu x))$. In fact, $h^{(-1)}(h(\mu x) + h \circ h^{(-1)}(h(\mu y) - h(\mu x))) = h^{(-1)}(h(\mu x) + h(\mu y) - h(\mu x)) = h^{(-1)}(h(\mu y)) = \mu y$ and for all $z > h^{(-1)}(h(\mu y) - h(\mu x))$, it is $h(z) < h(\mu y) - h(\mu x)$, $h(\mu x) + h(z) < h(\mu y) \leq h(0)$ and $h^{(-1)}(h(\mu x) + h(z)) > h^{(-1)}(h(\mu y)) = \mu y$.

Therefore, we obtain

$$I_\mu^T(y/x) = \begin{cases} 1, & \text{if } \mu x \leq \mu y \\ h^{(-1)}(h(\mu y) - h(\mu x)), & \text{if } \mu x > \mu y. \end{cases}$$

The following theorem asserts that in the case of strict Archimedean t-norms, the preorders associated to μ and η are equal if and only if μ and η have "similar forms".

Theorem 3 *If T is a strict Archimedean t-norm with additive generator h, $I_\mu^T = I_\eta^T$ if and only if there exists $k \in R$ such that for all x $h(\mu x) = k + h(\eta x)$.*

Proof. As T is strict, $h^{(-1)} = h^{-1}$. Let $I_\mu^T = I_\eta^T$, and let us choose some x, y.

Firstly, let us point out that if $\mu x < \mu y$, then $I_\eta^T(x/y) = I_\mu^T(x/y) = h^{(-1)}(h(\mu x) - h(\mu y)) < 1$; so $\eta x < \eta y$. Similarly, if $\eta x < \eta y$, it is $\mu x < \mu y$.

So, if $\mu x < \mu y$, $I_\mu^T(x/y) = h^{-1}(h(\mu x) - h(\mu y)) = I_\eta^T(x/y) = h^{-1}(h(\eta x) - h(\eta y))$, if and only if $h(\mu x) - h(\mu y) = h(\eta x) - h(\eta y)$, if and only if $h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y)$.

Also, if $\mu y < \mu x$, then $h(\mu y) - h(\eta y) = h(\mu x) - h(\eta x)$.

Finally, if $\mu x = \mu y$ then $\eta x = \eta y$, and newly $h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y)$.

Therefore $h(\mu x) - h(\eta x)$ is a constant k for all x, and $h(\mu x) = k + h(\eta x)$ for each x.

Reciprocally, if there exists $k \in R$ such that for all x the equality $h(\mu x) = k + h(\eta x)$ holds, since $h(\mu x) - h(\eta x) = k = h(\mu y) - h(\eta y)$, we obtain $h(\mu x) + h(\eta y) = h(\mu y) + h(\eta x)$; then $\mu x \leq \mu y$ if and only if $h(\mu x) \geq h(\mu y)$, if and only if $h(\eta x) \geq h(\eta y)$, if and only if $\eta x \leq \eta y$.

So, if $\mu x \leq \mu y$, then $\eta x \leq \eta y$ and $I_\mu^T(y/x) = 1 = I_\eta^T(y/x)$; and if $\mu x > \mu y$ then $\eta x > \eta y$, and $I_\mu^T(y/x) = h^{-1}(h(\mu y) - h(\mu x)) = h^{-1}(k + h(\eta y) - k - h(\eta x)) = h^{-1}(h(\eta y) - h(\eta x)) = I_\eta^T(y/x)$. □

In the particular case in which μ and η "have points", that is, they are normalized, we can get:

Corollary 1 *If T is Archimedean and strict, and μ and η are such that there exist x, y with $\mu x = 1$, $\eta y = 1$, then*

$$I_\mu^T = I_\eta^T \text{ if and only if } \mu = \eta.$$

Proof. The sufficient condition is clear.

On the other hand, if $I_\mu^T = I_\eta^T$, there exists $k \in R$ such that for all z $h(\mu z) = k + h(\eta z)$.

In particular, $h(\mu x) = 0 = k + h(\eta x) \leq h(\eta x)$, and necessarily $k \leq 0$.

Analogously, $h(\mu y) = k + h(\eta y) = k + 0$, and then $k \geq 0$.

So $k = 0$, and then $h(\mu z) = h(\eta z)$, and $\mu z = \eta z$ for all z. □

Now, for the case in which μ and η "have not some points", we have

Corollary 2 *If T is Archimedean and strict, and μ and η are such that there exist x, y with $\mu x = 0$, $\eta y = 0$, then*

$$I_\mu^T = I_\eta^T \text{ if and only if } \mu = \eta$$

Proof. If $I_\mu^T = I_\eta^T$, there exists $k \in R$ with $h(\mu z) = k + h(\eta z)$ for all z.

As $\mu x = 0$, it is $h(0) = k + h(\eta x) \leq k + h(0)$, and then $0 \leq k$. And since $\eta y = 0$, $h(\mu y) = k + h(0) \leq h(0)$, and $k \leq 0$.

We get that $k = 0$ and $\mu = \eta$. □

Theorem 4 *If T is a non-strict Archimedean t-norm, with additive generator h, then:*
$I_\mu^T = I_\eta^T$ if and only if there exists $k \in R$ such that for all x it is $h(\mu x) = k + h(\eta x)$.
Furthermore, if that holds, for all x it is $k \leq h(0) - h(\eta x)$.

Proof. Let $I_\mu^T = I_\eta^T$ be, and let us choose any x, y.

- If $\mu x < \mu y$, $h(\mu x) > h(\mu y)$ and $I_\mu^T(x/y) = h^{(-1)}(h(\mu x) - h(\mu y)) = I_\eta^T(x/y) < 1$, and then $\eta x < \eta y$, $h(\eta x) > h(\eta y)$ and $I_\eta^T(x/y) = h^{(-1)}(h(\eta x) - h(\eta y)) = h^{(-1)}(h(\mu x) - h(\mu y))$, which implies (because of $h(\eta x) - h(\eta y) \in [0, h(0)]$ and $h(\mu x) - h(\mu y) \in [0, h(0)]$) that $h(\eta x) - h(\eta y) = h(\mu x) - h(\mu y)$ and $h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y)$.
- In a similar way if $\mu y < \mu x$, then $\eta y < \eta x$ y $h(\mu y) - h(\eta y) = h(\mu x) - h(\eta x)$.
- In the case in which $\mu x = \mu y$, $\eta x = \eta y$ and $h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y)$.

Then for all x, y holds $h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y)$, and there exists $k \in R$ such that for all x $h(\mu x) - h(\eta x) = k$ and $h(\mu x) = k + h(\eta x)$.

In this case, if there exists x with $k > h(0) - h(\eta x)$, we obtain the contradiction $h(\mu x) = k + h(\eta x) > h(0) - h(\eta x) + h(\eta x) = h(0)$. So $k \leq h(0) - h(\eta x)$ for all x.

Reciprocally, let us suppose that there exists $k \in R$ such that for all x $k \leq h(0) - h(\eta x)$ y $h(\mu x) = k + h(\eta x)$.

- If $\mu x \leq \mu y$, $k + h(\eta x) \geq k + h(\eta y)$, $h(\eta x) \geq h(\eta y)$ and $\eta x \leq \eta y$. Therefore, $I_\mu^T(y/x) = I_\eta^T(y/x) = 1$.
- If $\mu x > \mu y$, $k + h(\eta x) < k + h(\eta y)$, $h(\eta x) < h(\eta y)$, and $\eta x > \eta y$.

Then, $I_\mu^T(y/x) = h^{(-1)}(h(\mu y) - h(\mu x)) = h^{(-1)}(k + h(\eta y) - k - h(\eta x)) = h^{(-1)}(h(\eta y) - h(\eta x)) = I_\eta^T(y/x)$. □

Let us point out that newly, if the preorders associated to μ and to η concur, μ and η must have "similar forms", but now, furthermore, their distance is bounded by $h(0) - \sup_x\{h(\eta x)\}$.

Analogously, in the case in which μ and η "have points", and in the case in which its complements "have points", it holds:

Corollary 3 *If T is Archimedean non-strict and μ and η are such that there exist x, y with $\mu x = 1$, $\eta y = 1$, or there exist x, y with $\mu x = 0$, $\eta y = 0$, then*

$$I_\mu^T = I_\eta^T \text{ if and only if } \mu = \eta.$$

Proof. Similar to the case of the Archimedean strict t-norms.

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